ABOUT AN INEQUALITY BY D.M.BĂTINEȚU-GIURGIU-I

By Marin Chirciu – Romania

1) Let $x, y, z > 0$ with the property that $xy + yz + zx = xyz$. Prove that:

$$\frac{x^2 y^2}{z(1 + xy)^2} + \frac{y^2 z^2}{y(1 + yz)^2} + \frac{z^2 x^2}{y(1 + zx)^2} \geq \frac{81}{100}$$

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Solution *We have $xy + yz + xz = xyz \Leftrightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Denoting $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$ we*

have $a + b + c = 1$. We can reformulate the problem:

2) If $a, b, c > 0$ such that $a + b + c = 1$ prove that:

$$\frac{a}{(1 + bc)^2} + \frac{b}{(1 + ca)^2} + \frac{c}{(1 + ab)^2} \geq \frac{81}{100}$$

Proof. *Using Bergström's inequality we obtain:*

$$\sum \frac{a}{(1 + bc)^2} = \sum \frac{a^2}{a(1 + bc)^2} \geq \frac{(a + b + c)^2}{\sum a(1 + bc)^2} = \frac{1}{\sum a(1 + bc)^2}$$

It suffices to prove that $\frac{1}{\sum a(1+bc)^2} \geq \frac{81}{100} \Leftrightarrow \sum a(1 + bc)^2 \leq \frac{100}{81}$

As $a, b, c > 0$ such that $a + b + c = 1$ there is an ΔABC with

$a = \tan\frac{B}{2}\tan\frac{C}{2}, b = \tan\frac{C}{2}\tan\frac{A}{2}, c = \tan\frac{A}{2}\tan\frac{B}{2}$, *and the inequality that we have to prove ca*

be written:

$$\sum \tan\frac{B}{2}\tan\frac{C}{2}\left(1 + \tan\frac{C}{2}\tan\frac{A}{2}\tan\frac{A}{2}\tan\frac{B}{2}\right)^2 \leq \frac{100}{81} \Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(1 + \frac{r}{s}\tan\frac{A}{2}\right)^2 \leq \frac{100}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(1 + 2\frac{r}{s}\tan\frac{A}{2} + \frac{r^2}{s^2}\tan^2\frac{A}{2}\right) \leq \frac{100}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2} + 2\frac{r}{s}\sum \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} + \frac{r^2}{s^2}\sum \tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leq \frac{100}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2} + 2\frac{r}{s} \cdot 3\prod \tan\frac{A}{2} + \frac{r^2}{s^2}\prod \tan\frac{A}{2}\sum \tan\frac{A}{2} \leq \frac{100}{81} \Leftrightarrow$$

$$\Leftrightarrow 1 + 2\frac{r}{s} \cdot 3\frac{r}{s} + \frac{r^2}{s^2} \cdot \frac{r}{s} \cdot \frac{4R + r}{s} \leq \frac{100}{81} \Leftrightarrow 1 + 6\frac{r^2}{s^2} + \frac{r^3}{s^3} \cdot \frac{4R + r}{s} \leq \frac{100}{81} \Leftrightarrow$$

$$\Leftrightarrow \frac{6r^2}{s^2} + \frac{r^3(4R+r)}{s^4} \le \frac{19}{81} \Leftrightarrow 19s^4 \ge 486s^2r^2 + 81r^3(4R+r) \Leftrightarrow$$

$$\Leftrightarrow s^2(19s^2 - 486r^2) \ge 81r^3(4R+r), \text{ which follows from Gerretsen's inequality}$$

$$s^2 \ge 16Rr - 5r^2. \text{ It remains to prove that:}$$

$$(16Rr - 5r^2)[19(16Rr - 5r^2) - 486r^2] \ge 81r^3(4R+r) \Leftrightarrow$$

$$\Leftrightarrow 1216R^2 - 278Rr + 706r^2 \ge 0 \Leftrightarrow (R - 2r)(1216R - 353r) \ge 0, \text{ obviously from}$$

$$\text{Euler's inequality } R \ge 2r.$$

Observations.

O1. Above we have used the known inequalities in triangle:

$$\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1, \prod \tan\frac{A}{2} = \frac{r}{s} \text{ and } \sum \tan\frac{A}{2} = \frac{4R+r}{s}$$

O2. Gerretsen's inequality $s^2 \ge 16Rr - 5r^2$ follows from the remarkable distance in

triangle $GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$. As $GI^2 \ge 0$ it follows $s^2 + 5r^2 - 16Rr \ge 0 \Leftrightarrow$

$$\Leftrightarrow s^2 \ge 16Rr - 5r^2. \text{ Equality holds if } GI^2 = 0 \Leftrightarrow G \equiv 1 \Leftrightarrow \Delta ABC \text{ is equilateral.}$$

$$\text{Equality holds if and only if the triangle is equilateral.}$$

Remark. The problem can be developed.

3) Let be $x, y, z > 0$ having the property that $xy + yz + zx = xyz$ and $n \ge 0$. Prove that:

$$\frac{x^2y^2}{z(n+xy)^2} + \frac{y^2z^2}{y(n+yz)^2} + \frac{z^2x^2}{y(n+zx)^2} \ge \frac{81}{(9n+1)^2}$$

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Solution We have $xy + yz + zx = xyz \Leftrightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$. Denoting $\frac{1}{x} = a, \frac{1}{y} = b, \frac{1}{z} = c$ we

have $a + b + c = 1$. We can reformulate the problem:

4) If $a, b, c > 0$ such that $a + b + c = 1$ and $n \ge 0$ prove that:

$$\frac{a}{(n+bc)^2} + \frac{b}{(n+ca)^2} + \frac{c}{(n+ab)^2} \ge \frac{81}{(9n+1)^2}$$

Proof. Using Bergström's inequality:

$$\sum \frac{a}{(n+bc)^2} = \sum \frac{a^2}{a(n+bc)^2} \ge \frac{(a+b+c)^2}{\sum a(n+bc)^2} = \frac{1}{\sum a(n+bc)^2}$$

It suffices to prove that: $\frac{1}{\sum a(n+bc)^2} \geq \frac{81}{(9n+1)^2} \Leftrightarrow \sum a(n+bc)^2 \leq \frac{(9n+1)^2}{81}$

As $a, b, c > 0$ *such that* $a + b + c = 1$ *there is* ΔABC *with*

$a = \tan\frac{B}{2}\tan\frac{C}{2}, b = \tan\frac{C}{2}\tan\frac{A}{2}, c = \tan\frac{A}{2}\tan\frac{B}{2}$, *and the inequality that we have to prove*

can be written:

$$\sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n + \tan\frac{C}{2}\tan\frac{A}{2} \cdot \tan\frac{A}{2}\tan\frac{B}{2}\right)^2 \leq \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n + \frac{r}{s}\tan\frac{A}{2}\right)^2 \leq \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow \sum \tan\frac{B}{2}\tan\frac{C}{2}\left(n^2 + 2n\frac{r}{s}\tan\frac{A}{2} + \frac{r^2}{s^2}\tan^2\frac{A}{2}\right) \leq \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow n^2 \sum \tan\frac{B}{2}\tan\frac{C}{2} + 2n\frac{r}{s}\sum \tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} + \frac{r^2}{s^2}\sum \tan^2\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2} \leq \frac{(9n+1)^2}{81}$$

$$\Leftrightarrow n^2 \sum \tan\frac{B}{2}\tan\frac{C}{2} + 2n\frac{r}{s} \cdot 3\prod \tan\frac{A}{2} + \frac{r^2}{s^2}\prod \tan\frac{A}{2}\sum \tan\frac{A}{2} \leq \frac{(9n+1)^2}{81} \Leftrightarrow$$

$$\Leftrightarrow n^2 + 2n\frac{r}{s} \cdot 3\frac{r}{s} + \frac{r^2}{s^2} \cdot \frac{r}{s} \cdot \frac{4R+r}{s} \leq \frac{(9n+1)^2}{81} \Leftrightarrow n^2 + 6n\frac{r^2}{s^2} + \frac{r^3}{s^3} \cdot \frac{4R+r}{s} \leq \frac{(9n+1)^2}{81}$$

$$\Leftrightarrow \frac{6nr^2}{s^2} + \frac{r^3(4R+r)}{s^4} \leq \frac{18n+1}{81} \Leftrightarrow (18n+1)s^4 \geq 486ns^2r^2 + 81r^3(4R+r) \Leftrightarrow$$

$\Leftrightarrow s^2[(18n+1)s^2 - 486nr^2] \geq 81r^3(4R+r)$, *which follows from Gerretsen's inequality*

$s^2 \geq 16Rr - 5r^2$. *It remains to prove that:*

$$(16Rr - 5r^2)[(18n+1)(16Rr - 5r^2) - 486nr^2] \geq 81r^3(4R+r) \Leftrightarrow$$

$$\Leftrightarrow 64(18n+1)R^2 - (2664n+121)Rr + (720n-14)r^2 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R - 2r)[64(18n+1)R + (7-360n)r] \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Observations.

O1. Above we've used the known inequalities in triangle:

$\sum \tan\frac{B}{2}\tan\frac{C}{2} = 1, \prod \tan\frac{A}{2} = \frac{r}{s}$ *and* $\sum \tan\frac{A}{2} = \frac{4R+r}{s}$

O2. Gerretsen's inequality $s^2 \geq 16Rr - 5r^2$ *it follows from the remarkable in triangle*

$GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr)$. *As* $GI^2 \geq 0$ *it follows*

$$s^2 + 5r^2 - 16Rr \geq 0 \Leftrightarrow s^2 \geq 16Rr - 5r^2$$

Equality holds if and only if $GI^2 = 0 \Leftrightarrow G \equiv 1 \Leftrightarrow \Delta ABC$ *is equilateral.*

Equality holds if and only if the triangle is equilateral.

Note. *For $n = 1$ we obtain problem 27691 from GM 5/2019, D.M. Bătineţu-Giurgiu –*

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