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# R M M

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# PROBLEMS FOR JUNIORS

**JP.331. In acute $\Delta ABC$ with the lengths $BC = a, CA = b$, $AB = c$ holds:**

$$\frac{a(b+c-a)}{b^2+c^2-a^2}+\frac{b(c+a-b)}{c^2+a^2-b^2}+\frac{c(a+b-c)}{a^2+b^2-c^2}\geq 3$$

*Proposed by Hoang Le Nhat Tung – Vietnam*

**JP.332. If $x_i > 1, \forall i = \overline{1,n}; n \in \mathbb{N}, n \geq 3$ then prove:**

$$\frac{\log x_2}{\log^2(x_1^2x_2)}+\frac{\log x_3}{\log^2(x_1^2x_2^2x_3)}+\ldots+\frac{\log x_n}{\log^2(x_1^2x_2^2\ldots x_{n-1}^2x_n)}\leq$$

$$\leq\frac{\log\sqrt[4]{x_2x_3\ldots x_n}}{\log x_1\cdot\log(x_1x_2x_3\ldots x_n)}$$

*Proposed by Florică Anastase - Romania*

**JP.333. In $\Delta ABC$ the following relationship holds:**

$$\sqrt{r_ar_b}+\sqrt{r_br_c}+\sqrt{r_cr_a}\leq\sqrt{(ab+bc+ca)\left(2+\frac{r}{2R}\right)}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**JP.334. In $\Delta ABC$ the following relationship holds:**

$$\sqrt{\frac{a+b}{a-b+c}}+\sqrt{\frac{b+c}{b-c+a}}+\sqrt{\frac{c+a}{c-a+b}}\leq\frac{3R}{\sqrt{2}r}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**JP.335. If $a, b, c > 0$ such that $ab + bc + ca \leq 3$ then prove:**

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}\geq\frac{15}{4}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**JP.336. For all positive integers prove that:**

$$\frac{\sqrt{2n+1}-1}{2}<\frac{1}{\sqrt{1}+\sqrt{2}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\ldots+\frac{1}{\sqrt{2n-1}+\sqrt{2n}}<\frac{\sqrt{2n}}{2}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**JP.337. If $a_i, b_i \in (0,1); p, q \in \mathbb{N}^*, n \geq 2$ then prove:**

$$\sum_{i=1}^{n} \log a_i \sqrt[n]{\frac{2a_i^{2p} \cdot b_i^{2q}}{a_i^{2p} + b_i^{2q}}} + \sum_{i=1}^{n} \log b_i \sqrt[n]{\frac{2a_i^{2q} \cdot b_i^{2p}}{a_i^{2q} + b_i^{2p}}} \geq (\sqrt{p} + \sqrt{q})^2$$

*Proposed by Florică Anastase - Romania*

**JP.338. In $\Delta ABC, P, Q \in Int(\Delta ABC)$ such that:**

$\beta\overrightarrow{AB} + \gamma\overrightarrow{BP} + \overrightarrow{PC} = 0$ **and** $\overrightarrow{AQ} + \alpha\overrightarrow{QB} + \overrightarrow{BC} = 0, \alpha, \beta, \gamma \in \mathbb{R}, \alpha, \gamma \neq 1$

**Prove that $A, P, Q$ are collinear if and only if $\alpha + \gamma = \beta + 1$**

*Proposed by Florică Anastase - Romania*

**JP.339. Solve for real numbers the system:**

$$\begin{cases} 11(x^4 - y^4) + 4xy(x^2 + y^2) + x = 0 \\ 2(x^4 - y^4) - 22xy(x^2 + y^2) + y = 0 \end{cases}$$

*Proposed by Florică Anastase - Romania*

**JP.340. Prove that:**

$$\sin 10^\circ = \frac{1}{4} - \frac{\sqrt{3}}{4}\tan 10^\circ + \frac{1}{4}\tan^2 10^\circ - \frac{\sqrt{3}}{4}\tan^3 10^\circ$$

*Proposed by Pedro Henrique O. Pantoja - Brazil*

**JP.341. Find all positive integers $n$ such that:**

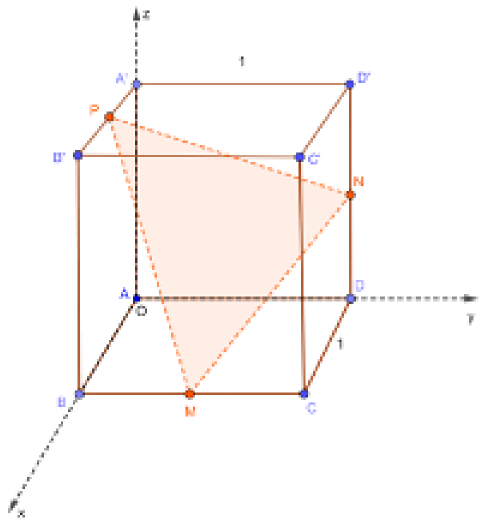
$$N = \frac{2^{2n} - n^2 - 1}{n!}$$

*Proposed by Pedro Henrique O. Pantoja - Brazil*

**JP.342. Let be $ABCDA'B'C'D'$ cube with length side 1 and $M \in BC, N \in DD', P \in A'B'$. Find minimum perimeter of**

$\Delta MNP$.




*Proposed by Florentin Vişescu - Romania*

**JP.343.** In acute $\Delta ABC, g_a$ - Gergonne's cevian, the following relationship holds:

$$\max\{g_a^2 \cdot \cos A, g_b^2 \cdot \cos B, g_c^2 \cdot \cos C\} \geq r^2\left(1+\frac{r}{R}\right)\left(\frac{43}{9}-\frac{8R}{9r}\right)$$

*Proposed by Radu Diaconu - Romania*

**JP.344.** Let $a, b, c$ be positive real numbers such that $ab+bc+ca=3$. Prove that:

$$(3a^5-3a+2b^3+34)(3b^5-3b+2c^3+34)(3c^5-3c+2a^3+34) \geq 6^6$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**JP.345.** If $a, b, c \in \mathbb{C}; |a|=|b|=|c|=3$ then:

$$\sum_{cyc}|a+3|+3\sum_{cyc}|a^2+1|+\sum_{cyc}|a^3+3| \geq 18$$

*Proposed by Daniel Sitaru - Romania*

# PROBLEMS FOR SENIORS

**SP.331.** If $\Delta ABC$ has inradius $r$, circumradius $R$, sides lengths $a=BC, b=AC, c=AB$, and altitudes $h_a, h_b, h_c$ from the vertices $A, B, C$, respectively, then:

$$\frac{9r^2}{R} \leq \frac{c}{b+c}\cdot h_a+\frac{a}{c+a}\cdot h_b+\frac{b}{a+b}\cdot h_c \leq \frac{9R}{4}$$

*Proposed by George Apostolopoulos-Greece*

**SP.332. Let $a, b, c$ be the lengths of the sides of a triangle $ABC$ with inradius $r$ and circumradius $R$. Prove that:**

$$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b} \leq \frac{3\sqrt{6}R}{4r}\sqrt{R^2-2r^2}$$

*Proposed by George Apostolopoulos-Greece*

**SP.333. Let $x, y, z > 0$ be positive real numbers such that $x+y+z=3$. Find the minimum value of expression:**

$$P=\frac{x}{2\sqrt{y}+\sqrt{z}}+\frac{y}{2\sqrt{z}+\sqrt{x}}+\frac{z}{2\sqrt{x}+\sqrt{y}}+\frac{(x+y)(y+z)(z+x)}{16}$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**SP.334. Let $x, y, z$ be positive real numbers such that $x+y+z=1$. Prove that:**

$$(3x^2+1)(3y^2+1)(3z^2+1) \geq 27(xy+z)(yz+x)(zx+y)$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**SP.335. Let $x, y, z > 0$ be positive real numbers such that:**

$$(\sqrt{x^3}+\sqrt{y^3})(\sqrt{y^3}+\sqrt{z^3})(\sqrt{z^3}+\sqrt{x^3})=8$$

**Prove that:**

$$x+y+z \geq \sqrt[3]{xyz(x^2+xy+y^2)(y^2+yz+z^2)(z^2+zx+x^2)}$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**SP.336. Let $x, y, z$ be positive real numbers such that:**

$$(x^6+y^6)(y^6+z^6)(z^6+x^6)=8$$

**Prove that:**

$$(3x^2-4xy+3y^2)(3y^2-4yz+3z^2)(3z^2-4zx+3x^2) \geq 8$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**SP.337. Let $x, y, z > 0$.**
**1) If $xy+yz+zx \leq 3(2\sqrt{3}-3)$ then:**

$$\sqrt{\frac{xy+yz+zx}{3}}+1 \leq \sqrt[3]{(x+1)(y+1)(z+1)}$$

**2) If $xy+yz+zx > 3(2\sqrt{3}-3)$ then:**

$$\sqrt{xy+yz+zx}+1 < \sqrt{(x+1)(y+1)(z+1)}$$

*Proposed by Florentin Vişescu - Romania*

**SP.338.** If $t \in [0, 2\pi); n \in \mathbb{N}$ then:

$$|1+\cos nt+i\sin nt|+|1+\cos 2nt+i\sin 2nt|+|1+\cos 3nt+i\sin 3nt| \geq 2$$

*Proposed by Daniel Sitaru - Romania*

**SP.339.** Solve for real numbers:

$$\sqrt{x^3 - 2x^2 + 2x} + 3\sqrt[3]{x^2 - x + 1} + 2\sqrt[4]{4x - 3x^4} = \frac{x^4 - 3x^3}{2} + 7$$

*Proposed by Hoang Le Nhat Tung - Vietnam*

**SP.340.** Find all pairs of integers $(x, y)$ such that:

$$x^4 - 2x^2 - y^2 - 5y - 3 = 0$$

*Proposed by George Apostolopoulos-Greece*

**SP.341.** Let $a, b, c$ be positive real numbers such that $abc+ab+bc+ca = 4$. Find the maximum value of the expression:

$$T = \frac{1}{\sqrt{2a^5 + b^3 - 2a^2 + 26}} + \frac{1}{\sqrt{2b^5 + c^3 - 2b^2 + 26}} + \frac{1}{\sqrt{2c^5 + a^3 - 2c^2 + 26}}$$

*Proposed by Hoang Le Nhat-Tung - Vietnam*

**SP.342.** Let $a, b, c$ be positive real numbers such that $a + b + c = 1 = 4abc$. Find the minimum value of expression:

$$S = \frac{1}{\sqrt[3]{2a^5 - 2a^3 + b^2 + 26}} + \frac{1}{\sqrt[3]{2b^5 - 2b^3 + c^2 + 26}} + \frac{1}{\sqrt[3]{2c^5 - 2c^3 + a^2 + 26}}$$

*Proposed by Hoang Le Nhat-Tung - Vietnam*

**SP.343.** If $a, b, c \in \mathbb{C}; |a| = |b| = |c| = 5$ then:

$$\sum_{cyc} |a + 5| + 5\sum_{cyc} |a^{10} + 1| + \sum_{cyc} |a^{11} + 5| \geq 30$$

*Proposed by Daniel Sitaru - Romania*

**SP.344.** If $n \in \mathbb{N}, n \geq 2$ prove that:

$$\frac{n}{n+2} + \int_0^1 (\tan^{-1}(x^n))^2 dx \geq 2\int_0^1 \tan^{-1}(x^n)\sqrt[n]{\tan^{-1} x}\,dx$$

*Proposed by Florică Anastase - Romania*

**SP.345.** Prove that in any triangle $ABC$,

$$\left(\frac{b + c - a}{a}\right)^2 + \left(\frac{c + a - b}{b}\right)^2 + \left(\frac{a + b - c}{c}\right)^2 + \frac{8r}{R} \geq 7$$

*Proposed by Nguyen Viet Hung - Vietnam*

# UNDERGRADUATE PROBLEMS

**UP.331. If $a,b,c\in(0,1), n\in\mathbb{N}, n\geq 2$ then prove:**

$$\sum_{cyc}(1-\sqrt[n]{\sin a})\geq\sum_{cyc}\frac{1-\sin a\sin b}{2n+1-\sin a\sin b}$$

*Proposed by Florică Anastase - Romania*

**UP.332. Let $(x_n)_{n\geq1},(y_n)_{n\geq1}$ be sequences of positive real numbers such that:**

$$x_1>1, x_{n+1}=\frac{1+(n-1)x_n^n}{nx_n^{n-1}}; y_1>0, y_{n+1}=\frac{(n+1)n^n y_n}{y_n^n+n^n(n-1)}$$

**Find:**

$$\lim_{n\to\infty}\left(\frac{x_n+y_n}{y_n}\right)^{\frac{\sqrt{n}}{x_n}}$$

*Proposed by Florică Anastase - Romania*

**UP.333. If $x_p=a_p+ib_p, p=\overline{1,4}$ are roots of the equation:**

$$x^4-2(k+1)x^3+2(k+1)^2x^2-2(k^2+1)(k+1)x+(k^2+1)^2=0, k\in\mathbb{R}^*$$

**Then prove:**

$$\sum_{p=1}^{4}\arctan\frac{a_p}{|b_p|}=\pi+2\left(\frac{k-|k|}{k}\right)\arctan k$$

*Proposed by Florentin Vişescu - Romania*

**UP.334. Let be $n\in\mathbb{N}^*$ and $A_n\in M_{8n}(Q)$, such that:**

$$\det(A_n^4+2A_n^2(1-n^2-k)+(1+n^2+k)^2I_{8n})=0, \forall k=\overline{1,2n}$$

**Then find:**

$$\lim_{n\to\infty}\det\left(\frac{1}{n}A_n\right).$$

*Proposed by Florentin Vişescu - Romania*

**UP.335. If $a,b,c\in(0,\frac{\pi}{2}), a+b+c=\pi$ and**

$$I(n)=\sum_{i=1}^{n}\int_i^{i+1}\frac{dx}{(ae^{\tan a}x^2+be^{\tan b}x+ce^{\tan c})(e^c\tan cx^2+e^b\tan bx+e^a\tan a)}$$

**Then find maximum values of the expression:**

$$\Omega=\prod_{k=1}^{2020}I(k)$$

*Proposed by Florică Anastase - Romania*

**UP.336. If $0 < a < b < \frac{\pi}{2}$ then prove:**

$$\frac{3(b-a)\sqrt[3]{4(a+b)}}{\sqrt[3]{4(a+b)-\sin 4(a+b)}} < 3\int_a^b \frac{dx}{\sqrt[3]{1-\cos 4x}} < \cot 2a - \cot 2b + \frac{\pi}{4}$$

*Proposed by Florică Anastase - Romania*

**UP.337. If $0 < a \leq b$ then:**

$$\int_a^b \int_a^b \int_a^b \frac{yzdxdydz}{3x^2+2y^2+z^2} \leq \frac{(b-a)^2(b+a)}{12} \cdot \log\left(\frac{b}{a}\right)$$

*Proposed by Daniel Sitaru - Romania*

**UP.338. Let $a, b, c$ be positive real numbers such that $a^2+b^2+c^2=12$. Prove that:**

$$\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ba}\right)\left(\frac{a^2}{\sqrt{a^3+1}}+\frac{b^2}{\sqrt{b^3+1}}+\frac{c^2}{\sqrt{c^3+1}}\right) \geq 12$$

*Proposed by George Apostolopoulos - Greece*

**UP.339. Prove that for any positive real numbers $a, b, c$:**

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}+\frac{1}{2}(a+b+c) \geq \frac{9(a^2+b^2+c^2)}{2(a+b+c)}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**UP.340. If $0 < a \leq b$; $f : [a, b] \to [1, \infty)$; $f$ continuous, then:**

$$3(b-a)^2\int_a^b f(x)dx \leq 2(b-a)^3 + \left(\int_a^b f(x)dx\right)^3$$

*Proposed by Daniel Sitaru - Romania*

**UP.341. Find:**

$$\Omega = \lim_{n\to\infty}\left(\int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x(1+\cos^n x)}dx\right)$$

*Proposed by Daniel Sitaru - Romania*

**UP.342. Prove that if $0 < a \leq b$ then:**

$$\left(\int_a^b \frac{\log x}{x}dx\right)^2 \geq \left(\int_{\frac{a+b}{2}}^b \frac{\log x}{x}dx + \int_{\sqrt{ab}}^b \frac{\log x}{x}dx\right) \cdot$$

$$\cdot \left(\int_a^{\frac{a+b}{2}} \frac{\log x}{x}dx + \int_a^{\sqrt{ab}} \frac{\log x}{x}dx\right)$$

*Proposed by Daniel Sitaru - Romania*

**UP.343. Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that:**

$$2(a^4 + b^4 + c^4) - (a^3 + b^3 + c^3) \geq 3abc$$

*Proposed by George Apostolopoulos - Greece*

**UP.344. Let $a, b, c$ be non-negative real numbers, no two of which are zero. Prove that:**

$$\frac{a}{a^2 + 2(b+c)^2} + \frac{b}{b^2 + 2(c+a)^2} + \frac{c}{c^2 + 2(a+b)^2} \geq \frac{1}{a+b+c}$$

*Proposed by Nguyen Viet Hung - Vietnam*

**UP.345. Let $a, b, c$ be positive real numbers such that $a^2 + b^2 + c^2 = 3$. Prove that:**

$$(a+b)(b+c)(c+a) - 2abc \leq 6$$

*Proposed by George Apostolopoulos - Greece*

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