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Solve for real numbers:

$$\begin{cases} x, y, z > 0 \\ \dfrac{3x+3y}{y+2z} + \dfrac{3y+3z}{x+2z} + \dfrac{3x+9z}{x+y+z} = 8 \\ x^x + y^y + z^z = 3 \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution 1 by George Florin Şerban-Romania, Solution 2 by Florică Anastase-Romania

Solution 1 by George Florin Şerban-Romania

$$8 = \frac{3x+3y}{y+2z} + \frac{3y+3z}{x+2z} + \frac{3x+9z}{x+y+z} =$$

$$= \frac{3x}{y+2z} + \frac{3y}{x+2z} + \frac{3z}{x+y+z} + \frac{3y}{y+2z} + \frac{3z}{x+2z} + \frac{9z}{x+y+z} =$$

$$= \frac{(3x)^2}{3xy+6xz} + \frac{(3y)^2}{3y^2+6yz} + \frac{(3y)^2}{3xy+6yz} + \frac{(3z)^2}{3xz+6z^2} + \frac{(3x)^2}{3x^2+3xy+3xz} +$$

$$+ \frac{(3z)^2}{3xz+3yz+3z^2} + \frac{(3z)^2}{3xz+3yz+3z^2} + \frac{(3z)^2}{3xz+3yz+3z^2} \overset{Bergstrom}{\geq}$$

$$\geq \frac{(6x+6y+12z)^2}{3x^2+3y^2+15z^2+9xy+21xz+21yz};\ x, y, z > 0$$

$$8 \geq \frac{(6x+6y+12z)^2}{3x^2+3y^2+15z^2+9xy+21xz+21yz} \Leftrightarrow$$

$(z-y)^2+(z-x)^2\le 0 \Leftrightarrow x=y=z$ and from $x^x+y^y+z^z=3$ we get:

$$3x^x=3 \Rightarrow x^x=1$$

Therefore,

$$(x,y,z)=(1,1,1)$$

Solution 2 by Florică Anastase-Romania

Case I: $x,y,z\ge 1$

$$x^x+y^y+z^z=3 \Rightarrow \frac{x^x+y^y+z^z}{3}=1$$

From AGM inequality:

$$\left(\frac{x^x+y^y+z^z}{3}\right)^3 \ge x^xy^yz^z \Leftrightarrow x^xy^yz^z\le 1 \Leftrightarrow$$

$$xlogx+ylogy+zlogz\le 0;(1)$$

Now from $x^xy^yz^z\ge x^zy^xz^y \Leftrightarrow$

$$x^zy^xz^y\le 1 \Leftrightarrow zlogx+xlogy+ylogz\le 0;(2)$$

From $(1),(2)$ we have:

$$(x-z)logx+(y-x)logy+(z-y)logz\le 0;(3)$$

Using Abel identity:

$$am+bn+cp=a(m-n)+(a+b)(n-p)+(a+b+c)p$$

Let: $x\ge y\ge z$ and

$$(a,b,c)=(x-z,y-x,z-y);(m,n,p)=(logx,logy,logz),$$

$$(x-z)logx+(y-x)logy+(z-y)logz=$$

$$=(x-z)log\frac{x}{y}+(y-z)log\frac{y}{z}\ge 0;(4)$$

From $(3),(4)$ we get: $(x-z)logx+(y-x)logy+(z-y)logz=0$

$\Leftrightarrow x=y=z$ and from $\frac{3x+3y}{y+2z}+\frac{3y+3z}{x+2z}+\frac{3x+9z}{x+y+z}=8$ we get:

$$x=y=z=1.$$

Case II: $1>x,y,z>0$***-analogous.***