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Let P_n denote the n −th Pell number defined by $P_{n+1} = 2P_n + P_{n-1}, P_0 = 0, P_1 = 1$. Further, let T_n denote the n −th triunghiular number, that is $T_n = \binom{n+1}{2}$. Show that:

$$\sum_{n=0}^{\infty} 4T_n \cdot \frac{P_n}{3^{n+2}} = P_3 + P_4$$

Proposed by Angel Plaza-Spain

Solution by Naren Bhandari-Bajura-Nepal

To solve the problem we shall deduce the generating function for n −th pell number.

So we define:

$$P(x) = \sum_{n=0}^{\infty} P_n x^n \Rightarrow 2xP(x) = 2\sum_{n=0}^{\infty} P_n x^{n+1} \Rightarrow x^2P(x) = \sum_{n=0}^{\infty} P_n x^{n+2}$$

And then $(1 - 2x - x^2)P(x) = P_0 + P_1x - 2P_0x$ as we have $P_{n+1} = 2P_n + P_{n-1}$.

Also, we have $P_0 = 0$ and $P_1 = 1$ we get:

$$P(x) = \sum_{n=0}^{\infty} P_n x^n = \frac{x}{1 - 2x - x^2}; \quad (1)$$

Now, we head to the main problem. Since

$$\sum_{n=0}^{\infty} 4T_n \cdot \frac{P_n}{3^{n+2}} = \frac{4}{9}\sum_{n=0}^{\infty}\binom{n+1}{2}\frac{P_n}{3^n} = \frac{2}{9}\sum_{n=0}^{\infty}\frac{n(n+1)}{3^n}\cdot P_n\,; \quad (2)$$

To evaluate (2) we use the (1). First we different the $P(x)$ with respect to x and the multiply by x^2 giving us

$$x^2P'(x) = \sum_{n=0}^{\infty} nP_n x^{n+1} = x^2\left\{\frac{D[x](1-2x-x^2) - D[1-2x-x^2]x}{(1-2x-x^2)^2}\right\}$$

$$= \frac{x^4+x^2}{(1-2x-x^2)^2}; \quad (3)$$

We again differentiate (3) with respect to x with respect to x which gives us

$$\left(xP'(x)\right) = \sum_{n=0}^{\infty} n(n+1)P_n x^n = \frac{2x+6x^3-4x^4}{(1-2x-x^2)^3}$$

Set $x = \frac{1}{3}$ we get:

$$\sum_{n=0}^{\infty}\frac{n(n+1)P_n}{3^n} = \frac{2}{9}\cdot\frac{\frac{2}{3}-\frac{4}{81}+\frac{6}{27}}{\left(1-\frac{1}{9}-\frac{2}{3}\right)^2} = \frac{2}{9}\cdot\frac{253}{2} = 17$$

Since $17 = 5 + 12$ and $P_3 = 5, P_4 = 12$ we are done.

Note by editor:

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