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For positive integers $1 \le j \le n$, prove or disprove that:

$$\sum_{k=1}^{N} \sum_{j=1}^{k} \binom{2N}{2k} \binom{k}{j} = \frac{1}{\left(1+\sqrt{2}\right)^{2n}} + \frac{2P_{2n}}{\sqrt{2}} - \frac{4^n}{2}$$

Where P_n is n −th Pell number.

Proposed by Naren Bhandari-Bajura-Nepal

Solution 1 by Surjeet Singhania-India, Solution 2 by Kamel Benaicha-Algiers-Algerie

Solution 1 by Surjeet Singhania-India

$$\sum_{r=1}^{k} \binom{k}{r} = 2^k - 1$$

$$\sum_{k=1}^{n} \sum_{r=1}^{k} \binom{2n}{2k} \binom{k}{r} = \sum_{k=1}^{n} \binom{2n}{2k} \left(2^k - 1\right)$$

$$\sum_{k=0}^{2n} \binom{2n}{k} = 4^n; \quad \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} = 0$$

$$\sum_{k=1}^{n} \binom{2n}{2k} = -1 + 2^{2n-1}$$

$$(1+x)^{2n}=\sum_{k=0}^{2n}\binom{2n}{k}x^k$$

Put: $x=\pm\sqrt{2}$ **and adding, it follows that:**

$$\sum_{k=1}^{n}\binom{2n}{2k}2^k=-1+\frac{\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}}{2}=-1+\sqrt{2}P_{2n}+\frac{1}{\left(1+\sqrt{2}\right)^{2n}}-\frac{4^n}{2}$$

$$\sum_{k=1}^{n}\sum_{r=1}^{k}\binom{2n}{2k}\binom{k}{r}=\sum_{k=1}^{n}\binom{2n}{2k}(2^k-1)$$

Combining all result, we get:

$$\sum_{k=1}^{n}\sum_{r=1}^{k}\binom{2n}{2k}\binom{k}{r}=\frac{1}{\left(1+\sqrt{2}\right)^{2n}}+\frac{2P_{2n}}{\sqrt{2}}-\frac{4^n}{2}$$

Solution 2 by Kamel Benaicha-Algiers-Algerie

$$\Omega(n)=\sum_{k=1}^{n}\sum_{j=1}^{k}\binom{2n}{2k}\binom{k}{j},\ n\in\mathbb{N}^*$$

$$=\sum_{k=1}^{n}\left[\binom{2n}{2k}\sum_{j=0}^{k}\binom{k}{j}-\binom{2n}{2k}\right]=\sum_{k=1}^{n}(2^k-1)\binom{2n}{2k}$$

Put: $\Phi(n)=\sum_{k=1}^{n}\left[\left(\sqrt{2}\right)^{2k-1}-1\right]\binom{2n}{2k-1}$

$$\Omega(n)+\Phi(n)=\sum_{k=0}^{2n}\left[\left(\sqrt{2}\right)^k-1\right]\binom{2n}{k}=\left(1+\sqrt{2}\right)^{2n}-2^{2n}=\left(1+\sqrt{2}\right)^{2n}-4^n;\ \ (A)$$

$$\sum_{k=0}^{2n}\left[\left(-\sqrt{2}\right)^k-(-1)^k\right]\binom{2n}{k}=\left(1-\sqrt{2}\right)^{2n}=$$

$$=\sum_{k=0}^{n}\left[\left(\sqrt{2}\right)^{2k}-1\right]\binom{2n}{2k}-\sum_{k=1}^{n}\left[\left(\sqrt{2}\right)^{2k-1}-1\right]\binom{2n}{2k-1}=\Omega(n)-\Phi(n);\ \ (B)$$

$$\Omega(n)-\Phi(n)=\left(1-\sqrt{2}\right)^{2n};\ \ (B)$$

$$A+B\Leftrightarrow 2\Omega(n)=\left(1+\sqrt{2}\right)^{2n}+\left(1-\sqrt{2}\right)^{2n}-4^n$$

$$\Omega(n)=\frac{\left(\sqrt{2}+1\right)^{2n}-\left(\sqrt{2}-1\right)^{2n}}{2}+\left(\sqrt{2}-1\right)^{2n}-\frac{4^n}{2}=$$

$$=\frac{\left(\sqrt{2}+1\right)^{2n}-\left(\sqrt{2}-1\right)^{2n}}{2\sqrt{2}}\cdot\sqrt{2}+\frac{1}{\left(\sqrt{2}+1\right)^{2n}}-\frac{4^n}{2}=$$

$$=\frac{1}{\left(\sqrt{2}+1\right)^{2n}}+\frac{2P_{2n}}{\sqrt{2}}-\frac{4^n}{2}$$

Therefore,

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\binom{2n}{2k}\binom{k}{j}=\frac{1}{\left(1+\sqrt{2}\right)^{2n}}+\frac{2P_{2n}}{\sqrt{2}}-\frac{4^n}{2}$$

Where $P_n=\frac{\left(1+\sqrt{2}\right)^n-\left(1-\sqrt{2}\right)^n}{2\sqrt{2}}$ is the n –th Pell number.

Note by editor:

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