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Given $(1+x)^{2023}(1-x+x^2)^{2022}=a_0+a_1x+a_2x^2+\dots+a_{6067}x^{6067}$. Also, if the equations $a_r\equiv 1\ (mod\ 5)$ and $a_r\equiv 2\ (mod\ 5)$, $r\in\{0,1,2,3,\dots,6067\}$ has m and n solutions respectively, then prove that:

$$\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\cdots}}}}=\sqrt{1+mn}$$

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Solution by Kamel Gandouli Rezgui-Tunisia

$$(1+x)^{2023}(1-x+x^2)^{2022}=(1+x)(x^3+1)^{2022}=(1+x)\sum_{k=0}^{2022}\binom{2022}{k}x^{3k}=$$

$$=\sum_{k=0}^{2022}\binom{2022}{k}x^{3k+1}+\sum_{k=0}^{2022}x^{3k}\Rightarrow a_{3k+2}=0,a_{3k}=\binom{2022}{k},a_{3k+1}=\binom{2022}{k}$$

$$\binom{2022}{k}\equiv 1\ (mod\ 5)\Rightarrow\frac{2022!}{k!\,(2022-k)!}\equiv 1\ (mod\ 5)$$

$$a_r\equiv 1\ (mod\ 5)\Rightarrow\binom{2022}{k}\equiv 1\ (mod\ 5),\binom{2022}{0}\equiv 1\ (mod\ 5)$$

$$\binom{2022}{1}=2022\equiv 2\ (mod\ 5)$$

$$\binom{2022}{2}=\frac{2022\cdot 2021}{2}=1011\cdot 2021\equiv 1\ (mod\ 5)$$

$$\binom{2022}{3},\dots,\binom{2022}{2019}\equiv 0\ (mod\ 5)$$

$$\binom{2022}{2021}=2022\equiv 2\ (mod\ 5),\binom{2022}{2022}=1\equiv 1\ (mod\ 5)$$

So, $m=4$ and $n=2$, then $mn+1=8\Rightarrow\sqrt{mn+1}=3$

$$3=\sqrt{1+8}=\sqrt{1+2\cdot 4}=\sqrt{1+2\sqrt{1+15}}=\sqrt{1+2\sqrt{1+3\sqrt{25}}}=$$

$$=\sqrt{1+2\sqrt{1+3\sqrt{1+24}}}=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{36}}}}\dots\ (Ramanujan)$$