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In ΔABC the following relationship holds:

$$4(R+r)^3 \ge \frac{1}{16}\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right) \ge 27R^2 r$$

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Solution by Soumava Chakraborty-Kolkata-India

$$\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}} = \frac{a^2(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} + \frac{2bc\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$= \frac{a^2(s-a)+2\left(b\sqrt{s-b}\right)\left(c\sqrt{s-c}\right)}{\sqrt{s(s-a)(s-b)(s-c)}}$$

$$\overset{A-G}{\le} \frac{a^2(s-a)+b^2(s-b)+c^2(s-c)}{rs} = \frac{s\left(\sum a^2\right)-\sum a^3}{rs}$$

$$= \frac{2s(s^2-4Rr-r^2)-2s(s^2-6Rr-3r^2)}{rs} = \frac{2s(2Rr+2r^2)}{rs} = 4(R+r)$$

$$\therefore \frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}} \le 4(R+r) \text{ and analogs } \overset{\text{multiplying together}}{\Rightarrow} \left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right) \le 64(R+r)^3$$

$$\Rightarrow 4(R+r)^3 \overset{(1)}{\ge} \frac{1}{16}\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right)$$

Also, $\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}=\frac{a^2}{r_a}+\frac{bc}{\sqrt{r_b r_c}}+\frac{bc}{\sqrt{r_b r_c}}\overset{A-G}{\gtrapprox} 3\sqrt[3]{\left(\frac{a^2}{r_a}\right)\left(\frac{bc}{\sqrt{r_b r_c}}\right)\left(\frac{bc}{\sqrt{r_b r_c}}\right)}=3\sqrt[3]{\frac{16R^2r^2s^2}{rs^2}}$

$$\therefore \frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\ge 3\sqrt[3]{16R^2r} \text{ and analogs}$$

$$\overset{\text{multiplying together}}{\Rightarrow}\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right)\ge 27.16R^2r$$

$$\Rightarrow \frac{1}{16}\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right)\overset{(2)}{\gtrapprox} 27R^2r$$

$(1),(2)\Rightarrow 4(R+r)^3\ge \frac{1}{16}\left(\frac{a^2}{r_a}+\frac{2bc}{\sqrt{r_b r_c}}\right)\left(\frac{b^2}{r_b}+\frac{2ca}{\sqrt{r_c r_a}}\right)\left(\frac{c^2}{r_c}+\frac{2ab}{\sqrt{r_a r_b}}\right)\ge 27R^2r$ (Proved)