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In acute ΔABC the following relationship holds:

$$\cos(A-B)\cos(B-C)\cos(C-A)\le\frac{8abc}{(a+b)(b+c)(c+a)}$$

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$$\cos(A-B)\cos(B-C)\cos(C-A)$$

$$=\left(2\cos^2\frac{A-B}{2}-1\right)\left(2\cos^2\frac{B-C}{2}-1\right)\left(2\cos^2\frac{C-A}{2}-1\right)$$

$$\overset{(a)}{=}8\prod\cos^2\frac{B-C}{2}-4\left(\prod\cos^2\frac{B-C}{2}\right)\sum\sec^2\frac{B-C}{2}+2\sum\cos^2\frac{B-C}{2}-1$$

$$\text{Now, }\sum\cos^2\frac{B-C}{2}=\sum\frac{(b+c)^2\sin^2\frac{A}{2}}{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}=\frac{1}{16R^2s}\sum\frac{bc(b+c)^2}{s-a}$$

$$=\frac{1}{16R^2s}\sum\frac{bc(s+s-a)^2}{s-a}$$

$$=\frac{1}{16R^2s}\sum\left\{\frac{bcs^2}{s-a}+2sbc+bc(s-a)\right\}=\frac{1}{16R^2s}\left\{s^3\sum\sec^2\frac{A}{2}+3s\sum ab-3abc\right\}$$

$$=\frac{1}{16R^2s}\left[s^3\left\{\frac{s^2+(4R+r)^2}{s^2}\right\}+3s(s^2+4Rr+r^2)-12Rrs\right]=\frac{4s^2+(4R+r)^2+3r^2}{16R^2}$$

$$\Rightarrow\sum\cos^2\frac{B-C}{2}\overset{(1)}{=}\frac{4s^2+(4R+r)^2+3r^2}{16R^2}$$

Again, $\sum \sec^2\frac{B-C}{2} = \sum \frac{16R^2\sin^2\frac{A}{2}\cos^2\frac{A}{2}}{(b+c)^2\sin^2\frac{A}{2}} = \sum \frac{16R^2s(s-a)a}{4Rrs(b+c)^2}$

$$= \frac{2R}{r}\sum \frac{a(b+c-a)}{(b+c)^2} \overset{(2)}{\cong} \frac{2R}{r}\left\{\sum \frac{a}{b+c} - \sum \frac{a^2}{(b+c)^2}\right\}$$

Now, $\sum \frac{a}{b+c} = \frac{\sum a(c+a)(a+b)}{\prod(b+c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2+2Rr+r^2)}$

$$= \frac{2s(s^2+4Rr+r^2) + 2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)} \overset{(3)}{\cong} \frac{2s^2-2Rr-2r^2}{s^2+2Rr+r^2}$$

and, $\sum \frac{a^2}{(b+c)^2} = \sum \frac{\left(2s-(b+c)\right)^2}{(b+c)^2}$

$$= \sum \frac{4s^2-4s(b+c)+(b+c)^2}{(b+c)^2} \overset{(i)}{\cong} 4s^2\left[\frac{\sum\{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2}\right]$$

$$-4s\left[\frac{\sum(c+a)(a+b)}{\prod(b+c)}\right] + 3$$

$$\sum\{(c+a)^2(a+b)^2\} = \sum\left(\sum ab + a^2\right)^2 = \sum\left\{\left(\sum ab\right)^2 + 2a^2\sum ab + a^4\right\}$$

$$= 3\left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 - 2\sum a^2b^2$$

$$= \left(\sum ab\right)^2 + 2\left(\sum ab\right)(\sum a^2) + (\sum a^2)^2 + 2\sum a^2b^2 + 4abc(2s) - 2\sum a^2b^2$$

$$= \left(\sum ab + \sum a^2\right)^2 + 32Rrs^2$$

$$= (3s^2-4Rr-r^2)^2 + 32Rrs^2$$

$$\therefore \sum\{(c+a)^2(a+b)^2\} \overset{(ii)}{\cong} (3s^2-4Rr-r^2)^2 + 32Rrs^2$$

Again, $\sum(c+a)(a+b) = \sum\left(\sum ab + a^2\right) = 3\sum ab + \sum a^2$

$$= \sum a^2 + 2\sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2$$

$$\therefore \sum(c+a)(a+b) \overset{(iii)}{\cong} 5s^2 + 4Rr + r^2$$

$\because \prod(b+c) = s^2+2Rr+r^2 \therefore$ (i), (ii), (iii) $\Rightarrow \sum \frac{a^2}{(b+c)^2}$

$$= \frac{4s^2\{(3s^2-4Rr-r^2)^2 + 32Rrs^2\}}{4s^2(s^2+2Rr+r^2)^2} - \frac{4s(5s^2+4Rr+r^2)}{2s(s^2+2Rr+r^2)} + 3$$

$$= \frac{(3s^2-4Rr-r^2)^2 + 32Rrs^2 - 2(5s^2+4Rr+r^2)(s^2+2Rr+r^2) + 3(s^2+2Rr+r^2)^2}{(s^2+2Rr+r^2)^2}$$

$$= \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}$$

$$\Rightarrow \sum \frac{a^2}{(b+c)^2} \overset{(4)}{=} \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}$$

$$(2),(3),(4) \Rightarrow \sum \sec^2\frac{B-C}{2} = \frac{2R}{r}\left\{\frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} - \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}\right\}$$

$$\overset{(5)}{=} \frac{2R}{r}\left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2}\right]$$

Also, $8\prod \cos^2\frac{B-C}{2} = 8\prod \frac{(b+c)^2\sin^2\frac{A}{2}}{a^2}$

$$= 8\left\{\frac{4s^2(s^2 + 2Rr + r^2)^2}{16R^2r^2s^2}\right\}\left(\frac{r^2}{16R^2}\right) \overset{(6)}{=} \frac{(s^2 + 2Rr + r^2)^2}{8R^4}$$

$$(a),(1),(5),(8) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) = \frac{(s^2 + 2Rr + r^2)^2}{8R^4} - \left\{\frac{(s^2 + 2Rr + r^2)^2}{16R^4}\right\}\frac{2R}{r}\left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2}\right] + \frac{4s^2 + (4R + r)^2 + 3r^2}{8R^2} - 1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) \overset{(m)}{=} \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r}$$

(where $\sigma = (2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}$)

Now, $\frac{8abc}{(a+b)(b+c)(c+a)} \geq \frac{2r}{R} \Leftrightarrow \frac{32Rrs}{2s(s^2 + 2Rr + r^2)} \geq \frac{2r}{R} \Leftrightarrow 8R^2 \overset{(iv)}{\gtreqless} s^2 + 2Rr + r^2$

Now, RHS of (iv) $\overset{\text{Gerretsen}}{\leq} 4R^2 + 6Rr + 4r^2 \overset{?}{\leq} 8R^2 \Leftrightarrow (R - 2r)(2R + r) \overset{?}{\geq} 0 \to$ true

$$\Rightarrow \text{(iv) is true} \therefore \frac{8abc}{(a+b)(b+c)(c+a)} \overset{(n)}{\geq} \frac{2r}{R}$$

$\therefore$ (m), (n) $\Rightarrow$ it suffices to prove

$$: \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} - \frac{2r}{R} \leq 0$$

$$\Leftrightarrow \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r - 16R^3r^2}{8R^4r} \leq 0$$

$$\Leftrightarrow s^4+8R^4-s^2(6R^2+8Rr-2r^2)+8R^3r+22R^2r^2+8Rr^3+r^4 \overset{(x)}{\leq} 0$$

$\because \Delta$ **ABC is acute − angled, Walker and Gerretsen**

$$\Rightarrow (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2) \leq 0$$

$\Rightarrow$ **in order to prove (x),**

it suffices to prove : $s^4+8R^4-s^2(6R^2+8Rr-2r^2)+8R^3r+22R^2r^2+8Rr^3+r^4$
$\leq (s^2-2R^2-8Rr-3r^2)(s^2-4R^2-4Rr-3r^2)$

$$\Leftrightarrow (R+2r)s^2 \overset{(y)}{\leq} 8R^3+7R^2r+7Rr^2+2r^3$$

Now, $(R+2r)s^2 \overset{\text{Gerretsen}}{\leq} (R+2r)(4R^2+4Rr+3r^2) \overset{?}{\leq} 8R^3+7R^2r+7Rr^2+2r^3$

$$\Leftrightarrow 4t^3-5t^2-4t-4 \overset{?}{\geq} 0 \left(\text{where } t=\frac{R}{r}\right)$$

$\Leftrightarrow (t-2)(4t^2+3t+2) \overset{?}{\geq} 0 \rightarrow$ **true** $\because t \overset{\text{Euler}}{\geq} 2 \Rightarrow (x) \Rightarrow (y)$ **is true**

$$\therefore \cos(A-B)\cos(B-C)\cos(C-A) \leq \frac{8abc}{(a+b)(b+c)(c+a)} \text{ (Proved)}$$