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In ΔABC then following relationship holds:

$$\frac{2}{3}\left(\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}\right)\le\frac{R}{2r}$$

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Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum xy\le\sum x^2\Rightarrow\frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2}\le\frac{m_a^2m_b^2+m_b^2m_c^2+m_c^2m_a^2}{9S^2}$$

$$=\frac{\left(\frac{9}{16}\right)\sum a^2b^2}{9S^2}\overset{\text{Goldstone}}{\le}\frac{\left(\frac{9}{16}\right)4R^2s^2}{9r^2s^2}=\left(\frac{R}{2r}\right)^2$$

$$\therefore\left(\frac{R}{2r}\right)^2\overset{(1)}{\ge}\frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2}$$

$$\text{Now, }\frac{R}{2r}\ge\frac{1}{4}+\frac{\sum a^3}{4abc}=\frac{4Rrs+2s(s^2-6Rr-3r^2)}{16Rrs}=\frac{s^2-4Rr-3r^2}{8Rr}$$

$$\Leftrightarrow s^2-4Rr-3r^2\le4R^2\Leftrightarrow s^2\le4R^2+4Rr+3r^2\to\text{true}$$

$$(\text{Gerretsen})\ \therefore\frac{abc(a+b+c)}{16S^2}\ge\frac{1}{4}+\frac{\sum a^3}{4abc}\text{ applying which on a triangle with sides}$$

$$\frac{2m_a}{3},\frac{2m_b}{3},\frac{2m_c}{3}\text{ whose area of course }=\frac{S}{3}$$

we get : $\frac{\frac{8}{27}m_am_bm_c\left\{\frac{2}{3}(m_a+m_b+m_c)\right\}}{16\left(\frac{S^2}{9}\right)} \geq \frac{1}{4}+\frac{\frac{8}{27}(m_a^3+m_b^3+m_c^3)}{4\left(\frac{8}{27}\right)m_am_bm_c}$

$$\Rightarrow \frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2} \geq \frac{1}{4}+\frac{m_a^3+m_b^3+m_c^3}{4m_am_bm_c}$$

$\therefore \frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2} \overset{(2)}{\geq} \frac{1}{4}+\frac{m_a^3+m_b^3+m_c^3}{4m_am_bm_c}$

Again, $\frac{1}{4}+\frac{\sum a^3}{4abc} \geq \left(\frac{\sum a^2}{\sum ab}\right)^2 \Leftrightarrow \frac{1}{4}+\frac{3abc+(\sum a)(\sum a^2-\sum ab)}{4abc} \geq \left(\frac{\sum a^2}{\sum ab}\right)^2$

$$\Leftrightarrow \frac{(\sum a)(\sum a^2-\sum ab)}{4abc} \geq \left(\frac{\sum a^2}{\sum ab}\right)^2-1$$

$\Leftrightarrow \frac{(\sum a)(\sum a^2-\sum ab)}{4abc} \geq \frac{(\sum a^2-\sum ab)(\sum a^2+\sum ab)}{(\sum ab)^2}$

$\Leftrightarrow \left(\sum a^2-\sum ab\right)\left\{\frac{\sum a}{4abc}-\frac{\sum a^2+\sum ab}{(\sum ab)^2}\right\} \geq 0$ $\therefore$ in order to prove :

$\frac{1}{4}+\frac{\sum a^3}{4abc} \geq \left(\frac{\sum a^2}{\sum ab}\right)^2$, it suffices to prove : $\frac{\sum a}{4abc} > \frac{\sum a^2+\sum ab}{(\sum ab)^2} \Leftrightarrow (s^2+4Rr+r^2)^2$
$> 8Rr(3s^2-4Rr-r^2)$

$\Leftrightarrow s^4-s^2(16Rr-2r^2)+r^2(48R^2+16Rr+r^2) \overset{(i)}{>} 0$

Now, LHS of (i) $\overset{\text{Gerretsen}}{\geq}$ $s^2(16Rr-5r^2)-s^2(16Rr-2r^2)+r^2(48R^2+16Rr+r^2)$
$= r^2(48R^2+16Rr+r^2-3s^2)$

$\overset{\text{Gerretsen}}{\geq}$ $r^2(48R^2+16Rr+r^2-12R^2-12Rr-9r^2)$

$= 4r^2\left(9R^2+r(R-2r)\right) \overset{\text{Euler}}{\geq} 36R^2r^2 > 0 \Rightarrow$

(i) *is true* $\therefore \frac{1}{4}+\frac{\sum a^3}{4abc} \geq \left(\frac{\sum a^2}{\sum ab}\right)^2$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we get

$$: \frac{1}{4}+\frac{\frac{8}{27}(m_a^3+m_b^3+m_c^3)}{4\left(\frac{8}{27}\right)m_am_bm_c} \geq \left(\frac{\frac{4}{9}(m_a^2+m_b^2+m_c^2)}{\frac{4}{9}(m_am_b+m_bm_c+m_cm_a)}\right)^2$$

$\Rightarrow \frac{1}{4}+\frac{m_a^3+m_b^3+m_c^3}{4m_am_bm_c} \overset{(3)}{\geq} \left(\frac{m_a^2+m_b^2+m_c^2}{m_am_b+m_bm_c+m_cm_a}\right)^2 \therefore (1),(2),(3) \Rightarrow$

$$\left(\frac{R}{2r}\right)^2 \geq \frac{m_a m_b m_c(m_a+m_b+m_c)}{9S^2} \geq \frac{1}{4}+\frac{m_a^3+m_b^3+m_c^3}{4m_a m_b m_c}$$

$$\geq \left(\frac{m_a^2+m_b^2+m_c^2}{m_a m_b+m_b m_c+m_c m_a}\right)^2 \Rightarrow \frac{R}{2r} \overset{(4)}{\geq} \frac{m_a^2+m_b^2+m_c^2}{m_a m_b+m_b m_c+m_c m_a}$$

Now, $\frac{2}{3}\sum\frac{a}{b+c} \leq \frac{\sum a^2}{\sum ab} \Leftrightarrow \left(\frac{2}{3}\right)\frac{\sum a(c+a)(a+b)}{2s(s^2+2Rr+r^2)} \leq \frac{2(s^2-4Rr-r^2)}{s^2+4Rr+r^2}$

$$\Leftrightarrow \left(\frac{1}{3}\right)\frac{\sum\{a(ab+a^2)\}}{2s(s^2+2Rr+r^2)} \leq \frac{s^2-4Rr-r^2}{s^2+4Rr+r^2}$$

$$\Leftrightarrow \left(\frac{1}{3}\right)\frac{2s(s^2+4Rr+r^2)+2s(s^2-6Rr-3r^2)}{2s(s^2+2Rr+r^2)} \leq \frac{s^2-4Rr-r^2}{s^2+4Rr+r^2}$$

$$\Leftrightarrow 3(s^2-4Rr-r^2)(s^2+2Rr+r^2) \geq (s^2+4Rr+r^2)(2s^2-2Rr-2r^2)$$

$$\Leftrightarrow s^4-12Rrs^2-r^2(4R+r)^2 \overset{(ii)}{\geq} 0$$

Now, LHS of (ii) $\overset{\text{Gerretsen}}{\geq} s^2(4Rr-5r^2)$

$$-r^2(4R+r)^2 \overset{\text{Gerretsen}}{\geq} r^2\{(16R-5r)(4R-5r)-(4R+r)^2\}$$

$$=12r^2(R-2r)(4R-r) \overset{\text{Euler}}{\geq} 0$$

$\Rightarrow$ (ii) is true $\therefore \frac{2}{3}\sum\frac{a}{b+c}$

$\leq \frac{\sum a^2}{\sum ab}$ applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we get :

$$\frac{2}{3}\sum\left\{\frac{\frac{2}{3}m_a}{\frac{2}{3}(m_b+m_c)}\right\} \leq \frac{\left(\frac{4}{9}\right)\sum m_a^2}{\left(\frac{4}{9}\right)\sum m_a m_b} \Rightarrow \frac{2}{3}\left(\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}\right)$$

$$\leq \frac{m_a^2+m_b^2+m_c^2}{m_a m_b+m_b m_c+m_c m_a} \overset{\text{by }(4)}{\leq} \frac{R}{2r} \text{ (Proved)}$$

Solution 2 by Bogdan Fuştei-Romania

In any ΔABC and $\Delta A_1B_1C_1$ we have:

$$\frac{2}{3}\left(\frac{a}{a_1}+\frac{b}{b_1}+\frac{c}{c_1}\right) \leq \frac{R}{r_1} (Bogdan\ Fustei-result)$$

Proof: $\frac{2}{3}\left(\frac{a}{a_1}+\frac{b}{b_1}+\frac{c}{c_1}\right) \overset{CBS}{\leq} \frac{2}{3}\sqrt{a^2+b^2+c^2}\cdot\sqrt{\frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2}}$

$$\begin{cases} a^2+b^2+c^2 \leq 9R^2 (Leibniz) \\ \frac{1}{a_1^2}+\frac{1}{b_1^2}+\frac{1}{c_1^2} \leq \frac{1}{4r_1^2} \end{cases} \Rightarrow \frac{2}{3}\left(\frac{a}{a_1}+\frac{b}{b_1}+\frac{c}{c_1}\right) \leq \frac{2}{3}\cdot 3R\cdot\frac{1}{2r_1} \Rightarrow$$

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$$\frac{2}{3}\left(\frac{a}{a_1}+\frac{b}{b_1}+\frac{c}{c_1}\right)\le\frac{R}{r_1}$$

Let be $a_1=b+c, b_1=c+a, c_1=a+b, s_1=\frac{a_1+b_1+c_1}{2}=2s$

$$2sr_1=\sqrt{2s(2s-b-c)(2s-c-a)(2s-a-b)}=\sqrt{2sabc}\overset{abc=4RS=4Rrs}{\Longleftrightarrow}$$

$$2sr_1=\sqrt{2s\cdot 4Rrs}=2s\sqrt{2Rr}\Rightarrow r_1=\sqrt{2Rr}$$

So, we have the following relationship holds:

$$\frac{2}{3}\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right)\le\sqrt{\frac{R^2}{2Rr}}=\sqrt{\frac{R}{2r}}$$

m_a, m_b, m_c can be the sides of the triangle, then we get:

$$\frac{2}{3}\left(\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}\right)\le\sqrt{\frac{R_m}{2r_m}};(1)$$

$$S_m=\frac{3}{4}S; abc=4RS\Rightarrow m_am_bm_c=4R_mS_m=4R_m\cdot\frac{3}{4}S=3SR_m$$

$$R_m=\frac{m_am_bm_c}{3S}; S_m=s_mr_m; s_m=\frac{m_a+m_b+m_c}{2}$$

$$\frac{3}{4}S=r_m\cdot\frac{m_a+m_b+m_c}{2}\Rightarrow r_m=\frac{3S}{2(m_a+m_b+m_c)}$$

$$\Rightarrow\frac{R_m}{2r_m}=\frac{m_am_bm_c}{3S}\cdot\frac{m_a+m_b+m_c}{3S}=\frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2}$$

Using $\frac{1}{2}s^2R\ge m_am_bm_c$

$$m_a+m_b+m_c\le 4R+r; 4R+r\le\frac{9R}{2}\Leftrightarrow 8R+2r\le 9R\Leftrightarrow R\ge 2r(Euler)$$

$$\Rightarrow m_a+m_b+m_c\le\frac{9R}{2}$$

$$\frac{1}{2}Rs^2\cdot\frac{9R}{2}\ge\frac{m_am_bm_c(m_a+m_b+m_c)}{s^2}\overset{\cdot\frac{1}{r^2}}{\Rightarrow}$$

$$\frac{R^2}{4r^2}\ge\frac{m_am_bm_c(m_a+m_b+m_c)}{9S^2}=\frac{R_m}{2r_m}\Rightarrow\frac{R}{2r}\ge\sqrt{\frac{R_m}{2r_m}};(2)$$

From (1),(2) we get:

$$\frac{2}{3}\left(\frac{m_a}{m_b+m_c}+\frac{m_b}{m_c+m_a}+\frac{m_c}{m_a+m_b}\right)\le\frac{R}{2r}$$

Note by editor:

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