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ROMANIAN MATHEMATICAL MAGAZINE

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Let $\Delta A'B'C'$ be the circumcevian triangle of I –incenter in acute ΔABC.

If $\varphi_1, \varphi_2, \varphi_3$ are inradii of $\Delta BA'C, \Delta CB'A, \Delta AC'B$ then:

$$h_a\varphi_a + h_b\varphi_b + h_c\varphi_c = s(AI + BI + CI - s)$$

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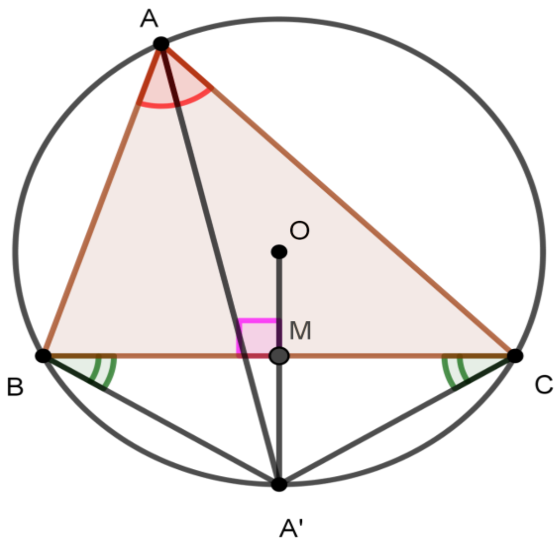
Solution by Tran Hong-Dong Thap-Vietnam



We have: $\angle CBA' = \angle BCA' = \frac{1}{2}\angle BAC \Rightarrow A'B = A'C$

$$OA' \perp BC \Rightarrow MB = MC = \frac{a}{2} \Rightarrow$$

$$OM = \sqrt{OB^2 - BM^2} = \sqrt{R^2 - \frac{a^2}{4}} = \sqrt{R^2 - R^2\sin^2 A} = R\cos A;$$

$$(\Delta ABC \text{ acute} \Rightarrow \cos A > 0)$$

$$MA' = R - R\cos A = 2R\sin^2\frac{A}{2}$$

$$[A'BC] = \frac{1}{2}BC \cdot MA' = \frac{1}{2} \cdot a \cdot 2R\sin^2\frac{A}{2} = 2R^2\sin A \cdot \sin^2\frac{A}{2}$$

$$[A'BC] = \frac{1}{2} \cdot A'B \cdot A'C \cdot \sin(BA'C) = \frac{1}{2} \cdot A'B^2 \cdot \sin A = 2R^2\sin A \cdot \sin^2\frac{A}{2}$$

$$\Rightarrow A'B = A'C = 2R\sin\frac{A}{2}$$

$$\Rightarrow p_A=\frac{A'B+A'C+BC}{2}=\frac{4R\sin\frac{A}{2}+2R\sin A}{2}=\frac{4R\sin\frac{A}{2}+4R\sin\frac{A}{2}\cos\frac{A}{2}}{2}=$$

$$=2R\sin\frac{A}{2}\left(1+\cos\frac{A}{2}\right)$$

$$\Rightarrow \varphi_A=\frac{[A'BC]}{p_A}=\frac{R\cdot\sin A\sin\frac{A}{2}}{1+\cos\frac{A}{2}}=\frac{a}{2}\cdot\frac{\sin\frac{A}{2}}{1+\cos\frac{A}{2}}=\frac{a}{2}\cdot\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}$$

$$\Rightarrow h_a\varphi_a=\frac{2S}{a}\cdot\frac{a}{2}\cdot\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}=s\left(\frac{r}{\sin\frac{A}{2}}-r\cdot\cot\frac{A}{2}\right)=s\left(AI-r\cdot\cot\frac{A}{2}\right)$$

Therefore,

$$h_a\varphi_a+h_b\varphi_b+h_c\varphi_c=s\left(\sum_{cyc}AI-r\sum_{cyc}\cot\frac{A}{2}\right)\overset{\sum\cot\frac{A}{2}=\frac{s}{r}}{=}$$

$$=s\left(\sum_{cyc}AI-r\cdot\frac{s}{r}\right)=s(AI+BI+CI-s)$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.