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HÖLDER's AND BERGSTRÖM's INEQUALITIES-NEW APPLICATIONS

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BERGSTROM's INEQUALITY (B)

$$\frac{a^2}{x}+\frac{b^2}{y}\geq\frac{(a+b)^2}{x+y};a,b\in\mathbb{R};x,y\in(0,\infty)$$

$$\frac{a^2}{x}+\frac{b^2}{y}+\frac{c^2}{z}\geq\frac{(a+b+c)^2}{x+y+z};a,b,c\in\mathbb{R};x,y,z\in(0,\infty)$$

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\geq\frac{(a+b+c)^2}{ax+by+cz};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a_1^2}{x_1}+\frac{a_2^2}{x_2}+\cdots+\frac{a_n^2}{x_n}\geq\frac{(a_1+a_2+\cdots+a_n)^2}{x_1+x_2+\cdots+x_n};a_i\in\mathbb{R};x_i>0;i\in\overline{1,n}$$

HOLDER's INEQUALITY (H)

$$\frac{a^3}{x}+\frac{b^3}{y}+\frac{c^3}{z}\geq\frac{(a+b+c)^3}{3(x+y+z)};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^4}{x}+\frac{b^4}{y}+\frac{c^4}{z}\geq\frac{(a+b+c)^4}{9(x+y+z)};a,b,c,x,y,z\in(0,\infty)$$

$$\frac{a^n}{x}+\frac{b^n}{y}+\frac{c^n}{z}\geq\frac{(a+b+c)^n}{3^{n-2}(x+y+z)};a,b,c,x,y,z\in(0,\infty);n\in\mathbb{N};n\geq 2$$

$$\frac{a^n}{x}+\frac{b^n}{y}\geq\frac{(a+b)^n}{2^{n-2}(x+y)};a,b,x,y\in(0,\infty);\ n\geq 2;n\in\mathbb{N}$$

$$\frac{x^3}{a^2}+\frac{y^3}{b^2}+\frac{z^3}{c^2}\geq\frac{(x+y+z)^3}{(a+b+c)^2};x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^4}{a^3}+\frac{y^4}{b^3}+\frac{z^4}{c^3}\geq\frac{(x+y+z)^4}{(a+b+c)^3};x,y,z,a,b,c\in(0,\infty)$$

$$\frac{x^{n+1}}{a^n}+\frac{y^{n+1}}{b^n}+\frac{z^{n+1}}{c^n}\geq\frac{(x+y+z)^{n+1}}{(a+b+c)^n};x,y,z,a,b,c\in(0,\infty);\ n\in\mathbb{N}$$

1) If $x, y, z > 0$ then prove:

$$\sum_{cyc}\frac{y}{x^3(1+y)}\geq\frac{3}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

Proof.

$$\sum_{cyc}\frac{y}{x^3(1+y)}=\sum_{cyc}\frac{\frac{1}{x^3}}{\frac{1+y}{y}}=\sum_{cyc}\frac{\frac{1}{x^3}}{1+\frac{1}{y}}\overset{(H)}{\geq}\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3\right)}=$$

$$=\frac{1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3}\overset{(1)}{\geq}\frac{1}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{1}{24}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3}{8}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

From $t=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$; $(1)\Leftrightarrow\frac{t^3}{t+3}\geq\frac{5t^2-9}{8}\Leftrightarrow(t-3)^2(t+1)\geq 0,\forall t>0$

$$(2)\Leftrightarrow\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2\geq 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)\Leftrightarrow\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\geq\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}$$

$$(3)\Leftrightarrow x^2+y^2+z^2\geq xy+yz+zx$$

$$(H)\Leftrightarrow\frac{x^3}{a}+\frac{y^3}{b}+\frac{z^3}{c}\geq\frac{(x+y+z)^3}{3(a+b+c)};(Holder)$$

$$(B)\Leftrightarrow\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\geq\frac{(x+y+z)^2}{a+b+c};(Bergstrom)$$

2) If $x, y, z > 0$ then prove:

$$\sum_{cyc}\frac{x(y^3+z^3)}{(1+x)y^3z^3}\geq\frac{3}{4}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

Proof.

$$\sum_{cyc}\frac{x(y^3+z^3)}{(1+x)y^3z^3}=\sum_{cyc}\frac{\frac{y^3+z^3}{y^3z^3}}{\frac{1+x}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}+\frac{1}{z^3}}{1+\frac{1}{x}}\overset{(H)}{\geq}\frac{2\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{2}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{1}{12}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{1}{12}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3}{4}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

3) If $x,y,z>0, n\geq 0$ then prove:

$$\sum_{cyc}\frac{x(ny^3+z^3)}{(1+x)y^3z^3}\geq\frac{3(n+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

Proof.

$$\sum_{cyc}\frac{x(ny^3+z^3)}{(1+x)y^3z^3}=\sum_{cyc}\frac{\frac{ny^3+z^3}{y^3z^3}}{\frac{1+x}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}+\frac{n}{z^3}}{1+\frac{1}{x}}=\sum_{cyc}\frac{\frac{1}{y^3}}{1+\frac{1}{x}}+\sum_{cyc}\frac{\frac{n}{z^3}}{1+\frac{1}{x}}\overset{(H)}{\geq}$$

$$\overset{(H)}{\geq}\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}+\frac{n\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)}=\frac{n+1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}\overset{(1)}{\geq}$$

$$\overset{(1)}{\geq}\frac{n+1}{3}\cdot\frac{5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9}{8}=\frac{n+1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{n+1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9\right]\overset{(B)}{\geq}$$

$$\overset{(B)}{\geq}\frac{n+1}{24}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9\right)\overset{(3)}{\geq}\frac{3(n+1)}{8}\left(\frac{15}{x^2+y^2+z^2}-1\right)$$

4) If $x, y, z > 0, n > 0$ then prove:

$$\sum_{cyc} \frac{y}{x^3(1+ny)} \ge \frac{3}{8} \cdot \left(\frac{15}{x^2+y^2+z^2} - n^2\right)$$

Proof.

$$\sum_{cyc} \frac{y}{x^3(1+ny)} = \sum_{cyc} \frac{\frac{1}{x^3}}{\frac{1}{y}+n} \overset{(H)}{\ge} \frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{3\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n\right)} =$$

$$= \frac{n^3\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{3n\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3\right)} \overset{(1)}{\ge} \frac{n^2}{3} \cdot \frac{5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2 - 9}{8} =$$

$$= \frac{n^2}{24} \cdot \left[5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2 - 9\right] \overset{(2)}{\ge} \frac{1}{24} \cdot \left[5 \cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right) - 9n^2\right] \overset{(B)}{\ge}$$

$$\overset{(B)}{\ge} \frac{1}{24} \cdot \left(5 \cdot \frac{27}{xy+yz+zx} - 9n^2\right) \overset{(3)}{\ge} \frac{3}{8} \cdot \left(\frac{15}{x^2+y^2+z^2} - n^2\right)$$

5) If $x, y, z > 0, n > 0$ then prove:

$$\sum_{cyc} \frac{(x^3+y^3)z}{x^3y^3(1+nz)} \ge \frac{3}{4} \cdot \left(\frac{15}{x^2+y^2+z^2} - n^2\right)$$

Proof.

$$\sum_{cyc} \frac{(x^3+y^3)z}{x^3y^3(1+nz)} = \sum_{cyc} \frac{\frac{1}{x^3}+\frac{1}{y^3}}{\frac{1}{z}+n} \overset{(H)}{\ge} \frac{2}{3} \cdot \frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n} =$$

$$= \frac{2}{3} \cdot \frac{n^3\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{n\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3\right)} = \frac{2n^2}{3} \cdot \frac{\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3} \overset{(1)}{\ge}$$

$$\overset{(1)}{\ge} \frac{2n^2}{3} \cdot \frac{5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2 - 9}{8} = \frac{n^2}{12} \cdot \left[5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2 - 9\right] \overset{(2)}{\ge}$$

$$\overset{(2)}{\geq}\frac{1}{12}\cdot\left(5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9n^2\right)\overset{(B)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{xy+yz+zx}-9n^2\right)\overset{(3)}{\geq}$$

$$\overset{(3)}{\geq}\frac{1}{12}\cdot\left(5\cdot\frac{27}{x^2+y^2+z^2}-9n^2\right)=\frac{3}{4}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

6) If $x,y,z>0,n,k>0$ then prove:

$$\sum_{cyc}\frac{(kx^3+y^3)z}{x^3y^3(1+nz)}\geq\frac{3(k+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

Proof.

$$\sum_{cyc}\frac{(kx^3+y^3)z}{x^3y^3(1+nz)}=\sum_{cyc}\frac{\frac{1}{x^3}+\frac{k}{y^3}}{\frac{1}{z}+n}=\sum_{cyc}\frac{\frac{1}{x^3}}{\frac{1}{z}+n}+\sum_{cyc}\frac{\frac{k}{y^3}}{\frac{1}{z}+n}\overset{(H)}{\geq}$$

$$\overset{(H)}{\geq}\frac{1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}+\frac{k}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}=\frac{k+1}{3}\cdot\frac{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3}{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}+3n}=$$

$$=\frac{k+1}{3}\cdot\frac{n^3\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^3}{n\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}+3\right)}\overset{(1)}{\geq}\frac{n^2(k+1)}{3}\cdot\frac{5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9}{8}=$$

$$=\frac{n^2(k+1)}{24}\cdot\left[5\left(\frac{1}{nx}+\frac{1}{ny}+\frac{1}{nz}\right)^2-9\right]=\frac{k+1}{24}\cdot\left[5\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^2-9n^2\right]\overset{(2)}{\geq}$$

$$\overset{(2)}{\geq}\frac{k+1}{24}\cdot\left[5\cdot 3\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)-9n^2\right]\overset{(B)}{\geq}\frac{k+1}{24}\cdot\left(5\cdot\frac{27}{x^2+y^2+z^2}-9n^2\right)=$$

$$=\frac{3(k+1)}{8}\cdot\left(\frac{15}{x^2+y^2+z^2}-n^2\right)$$

References:

[1]. Marin Chirciu -INEGALITĂŢI ALGEBRICE-de la iniţiere la performanţă, Editura „Paralela 45”-2014

[2]. Daniel Sitaru-ALGEBRAIC PHENOMENON-Publishing House”Paralela 45”-2017

[3]. ROMANIAN MATHEMATICAL MAGAZINE-www.ssmrmh.ro