

### A. GENERALIZATION OF KOUTRAS' THEOREM

### B. CHARACTERISTIC LINE ( $g$ ) OF TRIANGLE

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### A. GENERALIZATION of KOUTRAS' THEOREM

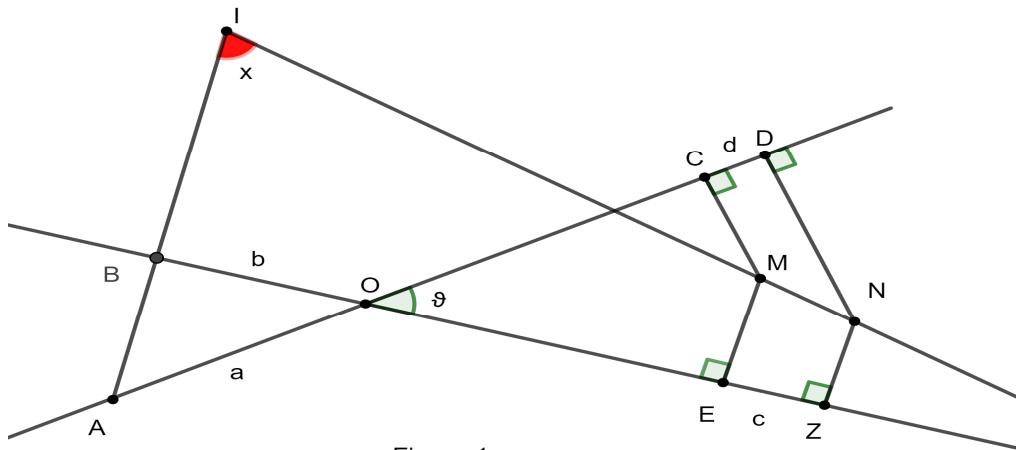


Figure-1

Let:  $OA = a$ ,  $OB = b$ ,  $EZ = c$ .  $CD = d$ ,  $\frac{OA}{OB} = \frac{a}{b} = m$ ,  $\frac{EZ}{CD} = k \cdot \frac{OA}{OB} \Rightarrow \frac{c}{d} = k \cdot m$ ,  $k \neq 0$

Then holds:

$$\tan x = \frac{k \cdot m^2 - (k + 1)m \cdot \cos \vartheta + 1}{(k - 1)m \cdot \sin \vartheta}$$

- If  $k = 1$ , then  $\tan x = \infty \Rightarrow x = 90^\circ$

*Stathis Koutras' theorem*

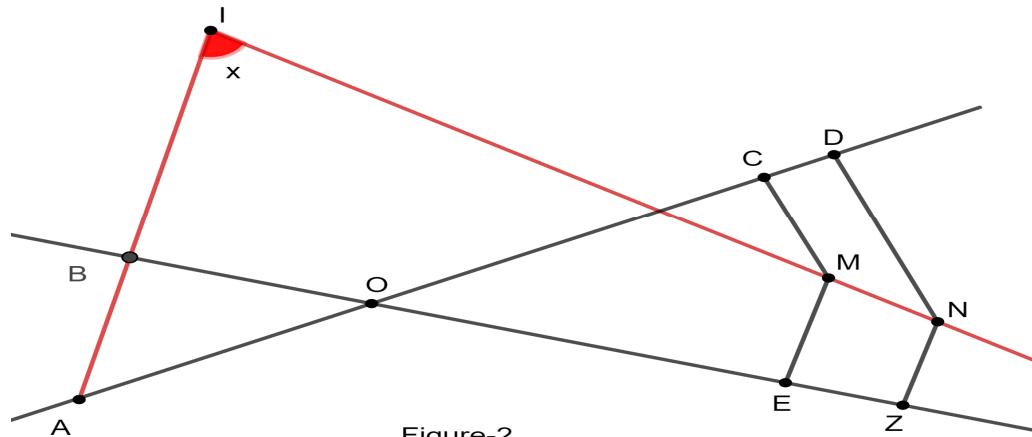


Figure-2



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$$\frac{EZ}{CD} = \frac{OA}{OB} \Leftrightarrow MN \perp AB$$

Proof: (on figure 1)

PLAGIAGONAL SYSTEM:  $OE \equiv Ox, OC \equiv Oy$

Let:  $OA = a, OB = b, OE = e, OZ = z, OC = c, OD = d$

$$AB: y = -\frac{a}{b}x + a, \lambda_{AB} = \lambda_1 = -\frac{a}{b}; \quad (E_1)$$

$$EM: y = -\frac{1}{\cos\vartheta}x + \frac{e}{\cos\vartheta}; \quad (1)$$

$$CM: y = -\cos\vartheta \cdot x + c; \quad (2)$$

$$ZN: y = -\frac{1}{\cos\vartheta}x + \frac{z}{\cos\vartheta}; \quad (3)$$

$$DN: y = -\cos\vartheta \cdot x + d; \quad (4)$$

$$(1), (2): M\left(\frac{e - c \cdot \cos\vartheta}{\sin^2\vartheta}, \frac{c - e \cdot \cos\vartheta}{\sin^2\vartheta}\right), M(m_1, m_2)$$

$$(3), (4): N\left(\frac{z - d \cdot \cos\vartheta}{\sin^2\vartheta}, \frac{d - z \cdot \cos\vartheta}{\sin^2\vartheta}\right), N(n_1, n_2)$$

$$\lambda_{MN} = \lambda_2 = \frac{m_2 - n_2}{m_1 - n_1} \Rightarrow \lambda_2 = \frac{d - c \cdot \cos\vartheta}{c - d \cdot \cos\vartheta}; \quad (E_2)$$

$$\tan x = \frac{(\lambda_2 - \lambda_1) \cdot \sin\vartheta}{(\lambda_2 + \lambda_1) \cdot \cos\vartheta + \lambda_2 \lambda_1 + 1} \stackrel{E_1/E_2}{\iff}$$

$$\tan x = \frac{\left(1 + \frac{a}{b} \cdot \frac{c}{d}\right) - \left(\frac{c}{d} + \frac{a}{b}\right) \cdot \cos\vartheta}{\left(\frac{c}{d} - \frac{a}{b}\right) \cdot \sin\vartheta} = \frac{1 + k \cdot \frac{a^2}{b^2} - \frac{a}{b}(k+1) \cdot \cos\vartheta}{\frac{a}{b}(k-1) \cdot \sin\vartheta}$$

$$\tan x = \frac{k \cdot m^2 - (k+1)m \cdot \cos\vartheta + 1}{(k-1)m \cdot \sin\vartheta}$$

So,

$$\frac{EZ}{CD} = k \cdot \frac{OA}{OB}, k \in \mathbb{R} - \{0, 1\} \Leftrightarrow \tan x = \frac{k \cdot m^2 - (k+1)m \cdot \cos\vartheta + 1}{(k-1)m \cdot \sin\vartheta}$$

### B. CHARACTERISTIC LINE ( $g$ ) of TRIANGLE

Given triangle  $ABC$  and circle  $(\omega)$  with center  $K$  passes through the vertices  $B, C$  and intersects the sides  $AB, AC$  at points  $D, E$  respectively. Let point  $G$  is the perpendicular projection of  $D$  on the side  $BC$  and line  $(g)$  is perpendicular bisector to the segment  $DG$ . Let circles  $(\omega_1), (\omega_2)$  with centers random points  $K_1, K_2$  belonging to the line  $(g)$

and radius  $GK_1, GK_2$  respectively, intersect  $AB$  at points  $D_1, D_2$  and  $ED$  at points  $C_1, C_2$  respectively.

If  $\angle ACB = \vartheta, \frac{AD}{DE} = m$ , then the ratio  $\frac{c_1 c_2}{D_1 D_2} = \frac{c}{d}$  depends only on the parameters  $\vartheta$  and  $m$ .

Holds that  $\frac{c}{d} = k \cdot m$ , where  $k = \frac{m \cdot \cos 2\vartheta - \cos \vartheta}{m(m \cdot \cos \vartheta - 1)}$ ;  $k \neq 0, \cos \vartheta \neq \frac{1}{m}$

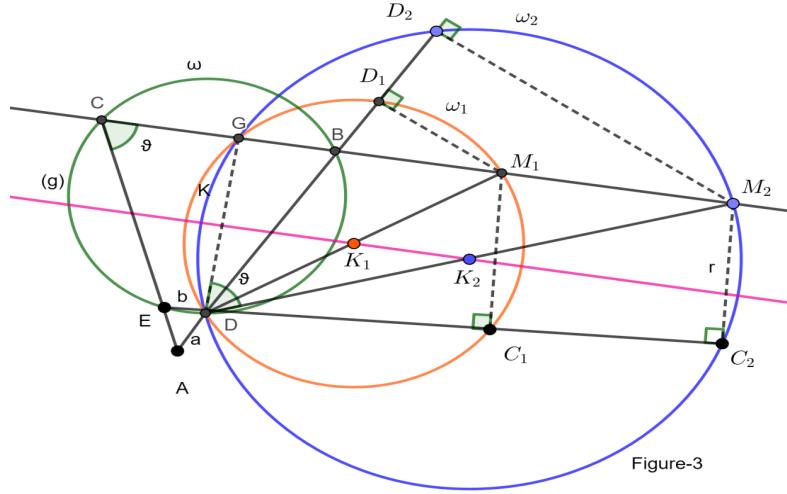


Figure-3

Let  $AD = a, ED = b, \frac{a}{b} = m, \angle ACB = x, \angle BDC_1 = \vartheta$

$$DK_1 \cap (\omega_1) = M_1, DK_2 \cap (\omega_2) = M_2, \frac{c}{d} = k \cdot m, k \neq 1$$

Is:  $BCED$  — cyclic  $\Rightarrow x = \vartheta$

$DM_1$  — diameter of  $(\omega_1) \Rightarrow \angle DC_1 M_1 = \angle DD_1 M_1 = 90^\circ$

$DM_2$  — diameter of  $(\omega_2) \Rightarrow \angle DC_2 M_2 = \angle DD_2 M_2 = 90^\circ$

$$\text{Is: } \tan x = \frac{k \cdot m^2 - (k+1)m \cdot \cos \vartheta + 1}{(k-1)m \cdot \sin \vartheta} \Rightarrow \frac{\sin \vartheta}{\cos \vartheta} = \frac{k \cdot m^2 - (k+1)m \cdot \cos \vartheta + 1}{(k-1)m \cdot \sin \vartheta}$$

$$km \cdot \sin^2 \vartheta - m \cdot \sin^2 \vartheta = km^2 \cdot \cos \vartheta - km \cdot \cos^2 \vartheta + \cos \vartheta$$

$$km \cdot \cos^2 \vartheta + km \cdot \sin^2 \vartheta - km^2 \cdot \cos \vartheta = m \cdot \sin^2 \vartheta - m \cdot \cos^2 \vartheta + \cos \vartheta$$

$$km - km^2 \cdot \cos \vartheta = m(\sin^2 \vartheta - \cos^2 \vartheta) + \cos \vartheta$$

$$km(m \cdot \cos \vartheta - 1) = m \cdot \cos 2\vartheta - \cos \vartheta$$

$$k = \frac{m \cdot \cos 2\vartheta - \cos \vartheta}{m(m \cdot \cos \vartheta - 1)}$$

$$m \cdot \cos \vartheta - 1 \neq 0 \Rightarrow \cos \vartheta \neq \frac{1}{m} \Rightarrow \cos \vartheta \neq \frac{a}{b}$$

$$\text{If } \vartheta = 90^\circ \Leftrightarrow k = \frac{m \cdot (-1) - 0}{m(m \cdot 0 - 1)} \Leftrightarrow k = 1 \text{ and } \frac{c}{d} = \frac{a}{b}$$

**Koutras' theorem**

Note: If more circles are written with centers  $K_i, i = 1, 2, 3 \dots, K_i \in (g)$  and points

$C_i, D_i$  respectively, then holds that:

$$\frac{C_i C_{i+j}}{D_i D_{i+j}} = k \cdot \frac{a}{b}, j = 1, 2, 3, \dots; i \neq j$$

This is the characteristic property of line  $(g)$

### Application 1.

In the figure 1 it is given that:

$\vartheta = 60^\circ, \frac{OA}{OB} = 2, \frac{EZ}{CD} = 3 \cdot \frac{OA}{OB}$ . Find angle  $x$ .

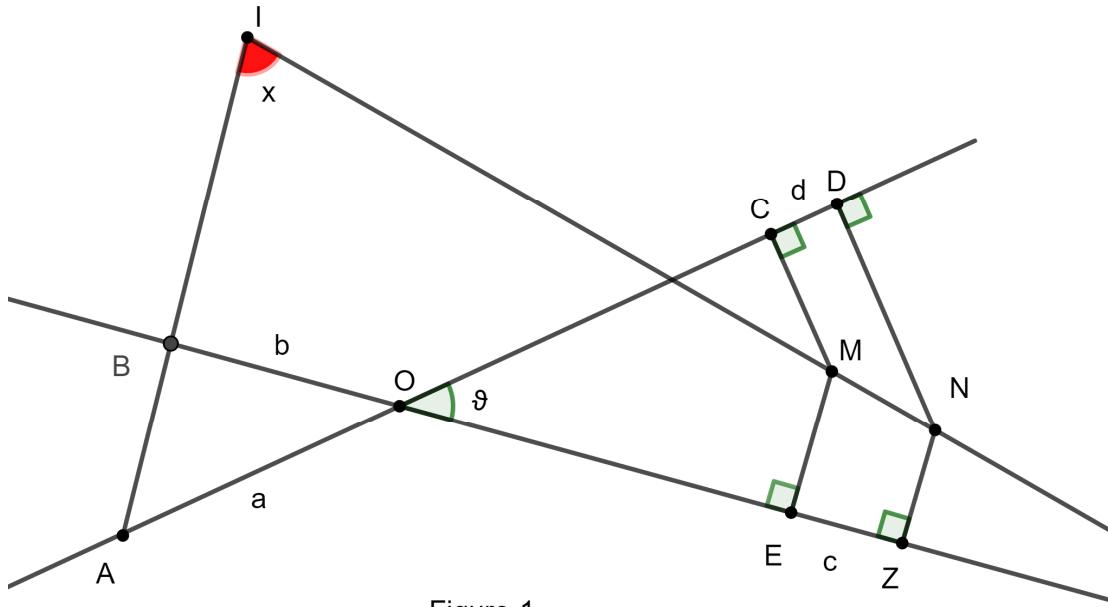


Figure-1

**Solution.**

Let  $OA = a, OB = b, EZ = c, CD = d$

$$\text{Is: } \frac{a}{b} = m = 2, \frac{c}{d} = k \cdot \frac{a}{b} \Rightarrow k = 3$$

$$\tan x = \frac{k \cdot m^2 - (k + 1)m \cdot \cos \vartheta + 1}{(k - 1)m \cdot \sin \vartheta} = \frac{3 \cdot 2^2 - (3 + 1) \cdot 2 \cdot \cos 60^\circ + 1}{(3 - 1) \cdot 2 \cdot \sin 60^\circ} = \frac{3\sqrt{3}}{2}$$

$$x = \tan^{-1} \left( \frac{3\sqrt{3}}{2} \right) \approx 68,9 \cdot 83^\circ$$

**Application 2:**

In the figure 3 it is given that:

$\vartheta = 30^\circ, \frac{OA}{OB} = 2, \frac{EZ}{CD} = k \cdot \frac{OA}{OB}, k = \frac{3}{4}(1 + \sqrt{3})$ . Find angle x.

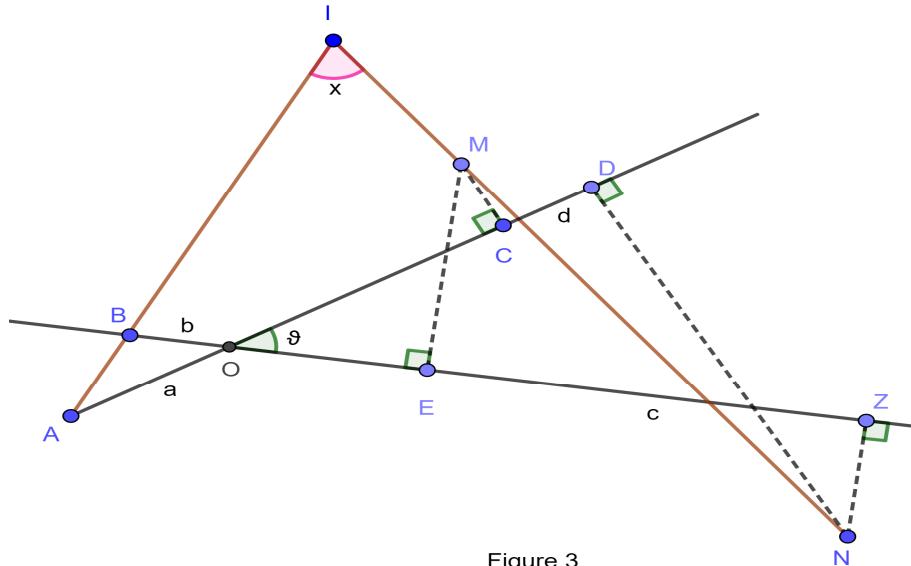


Figure 3

**Solution.**

Let  $OA = a, OB = b, EZ = c, CD = d$

$$\text{Is } \frac{a}{b} = 2 = m, \frac{c}{d} = k \cdot \frac{a}{b} = k \cdot m$$

$$\tan x = \frac{k \cdot m^2 - (k+1)m \cdot \cos \vartheta + 1}{(k-1)m \cdot \sin \vartheta} =$$

$$= \frac{\frac{3}{4}(1 + \sqrt{3}) \cdot 2^2 - [\frac{3}{4}(1 + \sqrt{3}) + 1] \cdot 2 \cos 30^\circ + 1}{[\frac{3}{4}(1 + \sqrt{3}) - 1] \cdot 2 \sin 30^\circ} = 2 + \sqrt{3} \Rightarrow$$

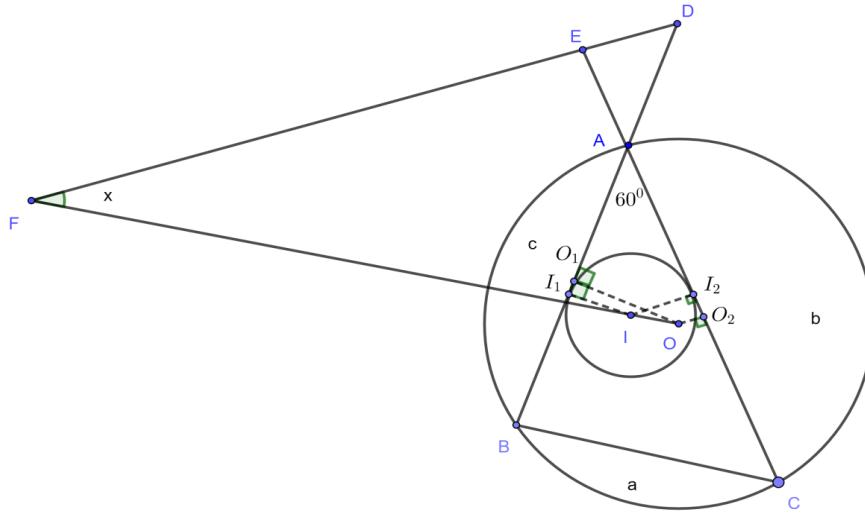
$$\tan \vartheta = 2 + \sqrt{3} \Rightarrow \vartheta = 75^\circ$$

**Application 3.**

Given triangle  $ABC$  with lengths of sides  $a, b, c$  and  $b > a > c, 2b + c = 3a$ ,  $\angle BAC = 60^\circ$ . Let points  $D, E$  on the extensions of the sides  $BA$  to point  $A$  and  $CA$  to point  $A$  respectively, such  $2BD = 3a - c, CE = 3b - 2a$ .

Denote  $I$  the incenter and  $O$  the circumcenter of  $\triangle ABC$ .  $DE$  and  $OI$  intersect at point

F. Prove that:  $\angle IFE = 30^\circ$



Solution.

$$\begin{aligned}
 2b + c &= 3a \Rightarrow \frac{b - a}{a - c} = \frac{1}{2} \\
 AI_1 &= \frac{-a + b + c}{2}, OA_1 = \frac{c}{2} \Rightarrow \overline{O_1 I_1} = \frac{b - a}{2} \\
 AI_2 &= \frac{-a + b + c}{2}, OA_2 = \frac{b}{2} \Rightarrow \overline{O_2 I_2} = -\frac{a - c}{2} \\
 \Rightarrow \frac{\overline{O_1 I_1}}{\overline{O_2 I_2}} &= -\frac{b - a}{a - c} = -\frac{1}{2} \\
 \overline{DA} &= \overline{DB} - \overline{AB} = \frac{3a - c}{2} - c = \frac{3}{2}(a - c) \\
 \overline{EA} &= \overline{EC} - \overline{AC} = 3b - 2a - b = 2(b - a) \\
 \Rightarrow \frac{\overline{AE}}{\overline{AD}} &= \frac{4}{3} \cdot \frac{b - a}{a - c} = \frac{4}{3} \cdot \frac{1}{2} = \frac{2}{3} = m \\
 \frac{\overline{O_1 I_1}}{\overline{O_2 I_2}} &= k \cdot m \Rightarrow -\frac{1}{2} = k \cdot \frac{2}{3} \Rightarrow k = -\frac{3}{4} \\
 \tan(180^\circ - x) &= -\tan x = \frac{km^2 - (k+1)m \cdot \cos 60^\circ}{(k-1)m \cdot \sin 60^\circ}
 \end{aligned}$$

$$\Rightarrow -\tan x = \frac{-\frac{3}{4}\left(\frac{2}{3}\right)^2 - \left(1 - \frac{3}{4}\right) \cdot \frac{2}{3} \cdot \frac{1}{2} + 1}{\left(-\frac{3}{4} - 1\right) \cdot \frac{2}{3} \cdot \frac{\sqrt{3}}{2}} = -\frac{\sqrt{3}}{3} \Rightarrow x = 30^\circ$$

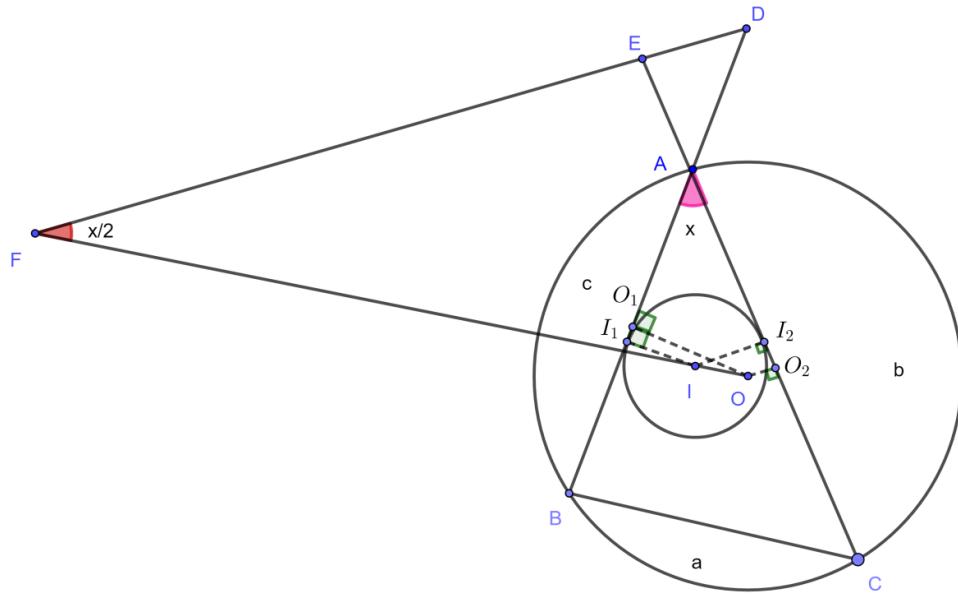
### Application 4.

Given triangle  $ABC$  with lengths sides  $a, b, c$  and  $b > a > c$ ,  $4b + 3c = 7a$ . Let points

$D, E$  on the extensions of the sides  $BA$  to point  $A$  and  $CA$  to point  $A$ , such  $\frac{AE}{AD} = \frac{16(b-a)}{21(a-c)}$ .

Denote  $I$  – the incenter and  $O$  – the circumcenter of  $\triangle ABC$ .  $DE$  and  $OI$  intersect at

point  $F$ . If  $\angle BAC = 2 \cdot \angle IFC = x$ , find the value of  $x$ .



Solution.

$$4b + 3c = 7a \Rightarrow \frac{b-a}{a-c} = \frac{3}{4}; \quad (1)$$

$$AI_1 = \frac{-a+b+c}{2}, AO_1 = \frac{c}{2} \Rightarrow \overline{O_1 I_1} = \frac{b-a}{2}$$

$$AI_2 = \frac{-a+b+c}{2}, OA_2 = \frac{b}{2} \Rightarrow \overline{O_2 I_2} = -\frac{a-c}{2}$$

$$\Rightarrow \frac{\overline{O_1 I_1}}{\overline{O_2 I_2}} = -\frac{b-a}{a-c} = -\frac{3}{4}; \quad (2)$$

$$\Rightarrow \frac{\overline{AE}}{\overline{AD}} = \frac{16}{21} \cdot \frac{b-a}{a-c} \stackrel{(2)}{=} \frac{16}{21} \cdot \frac{3}{4} = \frac{4}{7} = m; \quad (3)$$

$$\frac{\overline{O_1 I_1}}{\overline{O_2 I_2}} = k \cdot m \xrightarrow{(2)/(3)} -\frac{3}{4} = k \cdot \frac{4}{7} \Rightarrow k = -\frac{21}{16}; \quad (4)$$

$$\tan\left(180^\circ - \frac{x}{2}\right) = -\tan\frac{x}{2} = \frac{km^2 - (k+1)m \cdot \cos x}{(k-1)m \cdot \sin x}$$

$$\Rightarrow -\tan x = \frac{-\frac{21}{16}\left(\frac{4}{7}\right)^2 - \left(1 - \frac{21}{16}\right) \cdot \frac{4}{7} \cdot \cos x + 1}{\left(-\frac{21}{16} - 1\right) \cdot \frac{4}{7} \cdot \sin x} = -\frac{\sqrt{3}}{3} \Rightarrow x = 30^\circ$$

$$\Rightarrow \frac{3}{2}\left(\cos^2 \frac{x}{2} - \sin^2 \frac{x}{2}\right) = \frac{3}{4} \Rightarrow \cos x = \frac{1}{2} \Rightarrow x = 60^\circ$$

### Application 5.

Cyclic quadrilateral  $CBED$  is given. The extension of side  $CE$  to point  $E$  and the

extension of side  $BD$  to point  $D$  intersect at point  $A$  is  $\frac{DA}{DE} = \frac{3}{2}$ .

Let  $G$  be the vertical projection of the point  $D$  on the  $CB$  and line  $(g)$  is the perpendicular bisector of the segment  $DG$ . Random distinct points  $K_1, K_2$  belonging to the line  $(g)$  are the centers of circles  $(\omega_1), (\omega_2)$  with radius  $GK_1, GK_2$  respectively. Circles  $(\omega_1), (\omega_2)$  intersect the line  $BD$  at points  $D_1, D_2$  and the line  $ED$  at points

$C_1, C_2$  respectively. If  $\frac{C_1 C_2}{D_1 D_2} = S$ , find the angle  $BCE$ .

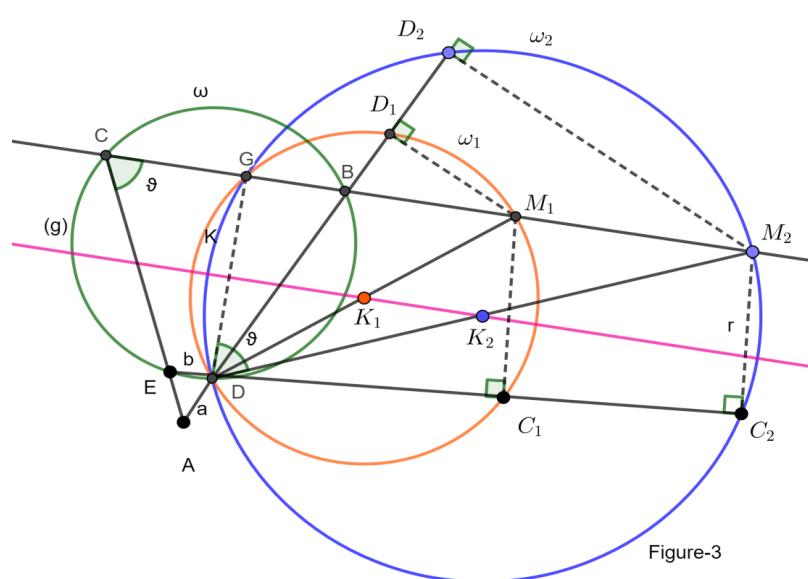


Figure-3



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Solution.

Let  $\angle BCE = \angle BDC_1$ , let  $BC \cap (\omega_1) = M_1$ ,  $\angle G = 90^\circ \Rightarrow M_1 \in OK_1$

Let  $BC \cap (\omega_2) = M_2$ ,  $\angle G = 90^\circ \Rightarrow M_2 \in OK_2$

Is  $M_1D_1 \perp AD$ ,  $M_2D_2 \perp AD$ ,  $M_1C_1 \perp ED$ ,  $M_2C_2 \perp ED$

Let  $\frac{DA}{DE} = m = \frac{3}{2}$ ,  $\frac{C_1C_2}{D_1D_2} = S$  and let  $k: \frac{c}{d} = k \cdot m \Rightarrow k = \frac{10}{3}$

Is  $k = \frac{m \cdot \cos 2\vartheta - \cos \vartheta}{m(m \cdot \cos \vartheta - 1)} \xrightarrow{k=\frac{10}{3}, m=\frac{3}{2}} 6 \cdot \cos^2 \vartheta - 17 \cos \vartheta + 7 = 0 \xrightarrow{\cos \vartheta < 1}$

$$\cos \vartheta = \frac{1}{2} \Rightarrow \angle BCE = 60^\circ$$

Reference:

Romanian Mathematical Magazine-[www.ssmrmh.ro](http://www.ssmrmh.ro)