Elliptic Integrals

1 Introduction

It is known that the complete elliptic integral of the first kind is given by,

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2 sin^2(\theta)}} = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}$$

and the complete elliptic integral of the second kind is given by,

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2 sin^2(\theta)}d\theta = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx$$

where $0 < k < 1$.
In this paper some elementary properties and generalisations of $K(k)$ and $E(k)$ are proved.

2 Prerequisite

Theorem 2.1. If $m, n \in \mathbb{Z}^+$, then

$$\int_0^1 x^{m-1} ln^{n-1}(x)dx = \frac{(-1)^{n-1}(n-1)!}{m^n}$$

Theorem 2.2. If $m, n, p \in \mathbb{R}^+$, then

$$\int_0^1 x^{m-1}(1-x^p)^{n-1}dx = \frac{B(\frac{m}{p}, n)}{p}$$

where $B(x, y)$ is the beta function.

Theorem 2.3. If $K(k)$ is the complete elliptic integral of the first kind, then

$$\int_0^1 K(k)dk = 2G$$

where G is Catalan's constant.

Theorem 2.4. If $f(x)$ is a function satisfying Dirichlet's conditions in the interval $(0,1)$, then its Fourier series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(2\pi nx) + \sum_{n=1}^{\infty} b_n sin(2\pi nx)$$

where,

$$a_0 = 2\int_0^1 f(x)dx$$

$$a_n = 2\int_0^1 f(x)cos(2\pi nx)dx$$

$$b_n = 2\int_0^1 f(x)sin(2\pi nx)dx$$

Theorem 2.5. If $a, b, c \in \mathbb{R}$ such that $c > b > 0$ and $c > a + b$ then,

$${}_2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

where ${}_2F_1(a, b; c; k)$ is the hypergeometric function.

3 Main proof

Theorem 3.1. Let $m \in \mathbb{N}$, then

$$\int_0^1 ln^{m-1}(k)K(k)dk = \frac{(-1)^{m-1}\pi\Gamma(m)}{2}\sum_{n=0}^{\infty}\frac{(1/2)^{(n)}(1/2)^{(n)}}{n!^2(2n+1)^m}$$

Proof: From the definition of $K(k)$,

$$\int_0^1 ln^{m-1}(k)K(k)dk = \int_0^1 ln^{m-1}(k)\int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}dk$$

Inverting the order of integration and expanding $(1-k^2x^2)^{-1/2}$ using binomial theorem, we have,

$$\int_0^1 ln^{m-1}(k)K(k)dk = \int_0^1 \frac{1}{\sqrt{1-x^2}}\int_0^1 ln^{m-1}(k)\sum_{n=0}^{\infty}\frac{(-1)^n(-1/2)_n(kx)^{2n}}{n!}dkdx$$

Inverting the order of integration and summation we have,

$$\int_0^1 ln^{m-1}(k)K(k)dk = \int_0^1 \frac{1}{\sqrt{1-x^2}} \sum_{n=0}^{\infty} \frac{(-1)^n(-1/2)_n x^{2n}}{n!} \left(\int_0^1 k^{2n} ln^{m-1}(k)dk \right) dx$$

Using (2.1) and simplifying it further, we have,

$$\int_0^1 ln^{m-1}(k)K(k)dk = (-1)^{m-1}(m-1)! \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!(2n+1)^m} \left(\int_0^1 \frac{x^{2n}}{\sqrt{1-x^2}} dx \right)$$

employing (2.2), we have

$$\int_0^1 ln^{m-1}(k)K(k)dk = (-1)^{m-1}(m-1)! \sum_{n=0}^{\infty} \frac{(1/2)^{(n)}}{n!(2n+1)^m} \frac{B(\frac{2n+1}{2}, \frac{1}{2})}{2}$$

simplifying it again, we obtain the desired result.

Corollary 3.1. Substituting $m = 1$ and using (2.3) we have,

$$\sum_{n=0}^{\infty} \frac{(1/2)^{(n)}(1/2)^{(n)}}{n!^2(2n+1)} = \frac{4G}{\pi}$$

Theorem 3.2. Let $\alpha := \alpha(n, \theta)$ and $\beta := \beta(n, \theta)$ be functions of n and θ, such that,

$$\alpha + i\beta = \int_0^1 \frac{e^{2\pi i n x}}{\sqrt{1 - x^2 sin^2(\theta)}} dx$$

then,

$$K(k) = 2G + 2\sum_{n=1}^{\infty} \int_0^{\pi/2} \alpha d\theta cos(2\pi nk) + 2\sum_{n=1}^{\infty} \int_0^{\pi/2} \beta d\theta sin(2\pi nk)$$

Proof: Substituting $f(x) = K(x)$ in (2.4), we obtain,

$$a_0 = 2\int_0^1 K(k)dk = 2G$$

$$a_n = 2\int_0^1 K(k)cos(2\pi nk)dk = 2\int_0^1 cos(2\pi nk) \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 sin^2(\theta)}} dk$$

$$= 2\int_0^{\pi/2}\Big(\int_0^1 \frac{cos(2\pi nk)}{\sqrt{1-k^2sin^2(\theta)}}dk\Big)d\theta = 2\int_0^{\pi/2}\alpha d\theta$$

Similarly,

$$b_n = 2\int_0^1 K(k)sin(2\pi nk)dk = 2\int_0^{\pi/2}\Big(\int_0^1 \frac{sin(2\pi nk)}{\sqrt{1-k^2sin^2(\theta)}}dk\Big)d\theta = 2\int_0^{\pi/2}\beta d\theta$$

this completes the proof.

Corollary 3.2. Substituting $k = 1/2$, we have,

$$K\Big(\frac{1}{2}\Big) = 2\Big(G - \int_0^{\pi/2}\alpha(1,\theta)d\theta + \int_0^{\pi/2}\alpha(2,\theta)d\theta - \int_0^{\pi/2}\alpha(3,\theta)d\theta..\Big)$$

4 Generalisation

Let us define a function by,

$$I(a,b,c;k) = \int_0^1 \frac{x^a}{(\sqrt{1-x^2})^b(\sqrt{1-k^2x^2})^c}dx$$

hence $I(a,b,c;k)$ is the generalised version of elliptic integrals.

It can be easily shown that,

1. $$K(k) = I(0,1,1;k)$$

2. $$E(k) = I(0,1,-1;k)$$

3. $$I(a,b,c;k) = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{2-b}{2})}{2\Gamma(\frac{a-b+3}{2})}{}_2F_1\Big(\frac{a+1}{2},\frac{c}{2},\frac{a-b+3}{2};k^2\Big)$$

4. $$I(a,b,c;1) = \frac{\Gamma(\frac{a+1}{2})\Gamma(\frac{2-b-c}{2})}{2\Gamma(\frac{a-b-c+3}{2})}$$

5.
$$(\sqrt{1-k^2})^c I(a,b,c;k) = I\Big(1-b, 1-a, c; \sqrt{\frac{k^2}{k^2-1}}\Big)$$

6.
$$I(a,b,c,k) = \sum_{n=0}^{\infty} \frac{I(a+2n,0,c;k)(b/2)^{(n)}}{n!}$$

7.
$$I(a,b,c;k) - k^2 I(a+2,b,c;k) = I(a,b,c-2;k)$$

8.
$$\frac{dI}{dk} = kcI(a+2,b,c+2;k)$$

5 Proof of properties of $I(a,b;c;k)$

Proof of 4.1: Using the definition of $K(k)$.

Proof of 4.2: Using the definition of $E(k)$.

Proof of 4.3: Expanding $(1-k^2x^2)^{-c/2}$ using binomial theorem and then using the definition of beta function and hypergeometric function.

Proof of 4.4: Substituting $k=1$ in (4.3) and then using (2.5).

Proof of 4.5: Substituting $x^2 = 1-y^2$ in the definition of $I(a,b;c;k)$ and simplifying it further to form a transformation formula.

Proof of 4.6: Expanding $(1-x^2)^{-b/2}$ using binomial theorem and then using the definition of $I(a,b;c;k)$.

Proof of 4.7: Simply manipulating the terms and then using the definition of $I(a,b;c;k)$ repeatedly.

Proof of 4.8: Differentiating $I(a,b;c;k)$ w.r.t. k and adjusting some terms to get a partial differential equation.

6 References

[1] Some Summation Theorems for Generalized Hypergeometric Functions, Mohammad Masjed-Jamei and Wolfram Koepf.

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