RMT and $\int_0^\infty \frac{sin(x)}{x}dx$

1 Introduction

It is known that,

$$\int_0^\infty \frac{sin(x)}{x}dx = \frac{\pi}{2}$$

The above result can be easily derived using contour integration, Laplace transform, Leibniz rule, Riemann sum, infinite series, double integration or by using any other technique. Here, I prove this result using Ramanujan's master theorem.

2 Main proof

Ramanujan's master theorem states that, if the Taylor seires of a function is given by,

$$f(x) = \sum_{k=0}^{\infty} \frac{\phi(k)(-x)^k}{k!}$$

then, its Mellin transform is given by,

$$\int_0^\infty x^{n-1} f(x)dx = \Gamma(n)\phi(-n)$$

Now let, $f(x) = sin(x)$, then $\phi(k) = -sin(\frac{\pi k}{2})$, thus Mellin transform of $sin(x)$ is given by,

$$\int_0^\infty x^{n-1} sin(x)dx = -\Gamma(n)sin\Big(\frac{\pi(-n)}{2}\Big) = \Gamma(n)sin\Big(\frac{\pi n}{2}\Big) \tag{1}$$

using Euler's reflection formula for gamma function, we have

$$\int_0^\infty x^{n-1} sin(x)dx = \frac{\pi}{\Gamma(1-n)sin(\pi n)} sin\Big(\frac{\pi n}{2}\Big)$$

taking limit of both the sides as $n \to 0$, we get,

$$\int_0^\infty \frac{sin(x)}{x}dx = \lim_{n\to 0} \frac{\pi}{\Gamma(1-n)sin(\pi n)} sin\Big(\frac{\pi n}{2}\Big) = \pi \lim_{n\to 0} \frac{sin(\frac{\pi n}{2})}{sin(\pi n)}$$

thus,

$$\int_0^\infty \frac{sin(x)}{x}dx = \frac{\pi}{2}$$

hence, proved.
(1) can be further generalised by substituting $x = t^m$ and then replacing t by x and n by $\frac{n}{m}$,

$$\int_0^\infty x^{n-1}sin(x^m)dx = \frac{1}{m}\Gamma\Big(\frac{n}{m}\Big)sin\Big(\frac{\pi n}{2m}\Big) \tag{2}$$

Thus, substituting $n = 1$ and $m = 2$, we obtain,

$$\int_0^\infty sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

the famous Fresenel integral.
Differentiating (2) w.r.t. n, we obtain,

$$\int_0^\infty x^{n-1}sin(x^m)ln(x)dx = \frac{1}{m^2}\Gamma\Big(\frac{n}{m}\Big)\Big(\frac{\pi}{2}cos\Big(\frac{\pi n}{2m}\Big) + \psi\Big(\frac{n}{m}\Big)sin\Big(\frac{\pi n}{2m}\Big)\Big)$$

Thus, we can obtain the following equalities,

$$\int_0^\infty \frac{sin(x)ln(x)}{\sqrt{x}}dx = \sqrt{\frac{\pi}{2}}\Big(\frac{\pi}{2} - \gamma - ln(4)\Big)$$

$$\int_0^\infty \frac{sin(x)ln(x)}{x}dx = -\frac{\pi\gamma}{2}$$

3 References

[1] B.C. Berndt, Ramanujans Notebooks: Part I. New York: SpringerVerlag, p. 298, 1985.

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