Ekpo, Samuel O.
ekpoduduabasi@gmail.com

# AN AMAZING IDENTITY INVOLVING THE MODIFIED BESSEL FUNCTION OF THE FIRST KIND

Here, i present a proof for an amazing identity involving the Modified Bessel function of the first kind proposed by Narendra Bhandari in the Romanian mathematical magazine.

$$\boldsymbol{Let\ the\ sequence\ (Q_k)_{k\geq 1} = n_k, \forall n \geq 1\ with\ P(Q_k) = \prod_{l=1}^{k} Q_l\ , S(Q_k) = \sum_{l=1}^{k} Q_l}$$

$$\boldsymbol{and \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k} \frac{1}{P(Q_k)\,S(Q_k)} = F(k)}$$

$$\boldsymbol{then\ prove\ or\ disprove\ that}$$

$$\boldsymbol{\sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)\sum_{i=1}^{k}\sum_{q=0}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\frac{(F(k))^{-1}}{(k-q+1)^{k+1}} = 2\sqrt[4]{e}+\sqrt{e}I_0(2^{-1})-3}$$

$$\boldsymbol{where\ I_0\ is\ the\ modified\ Bessel\ function\ of\ the\ first\ kind}$$

$$for\ the\ given\ sequence;\ (Q_k)_{k\geq 1} = n_k, \forall n \geq 1$$

$$P(Q_k) = \prod_{l=1}^{k} Q_l = (n_1 n_2 n_3 \cdots\cdots n_k)\ and\ S(Q_k) = \sum_{l=1}^{k} Q_l = (n_1+n_2+n_3+\cdots\cdots+n_k)$$

$$\therefore F(k) = \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k}\frac{1}{P(Q_k)\,S(Q_k)} = \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k}\frac{1}{(n_1n_2n_3\cdots\cdots n_k)(n_1+n_2+n_3+\cdots\cdots+n_k)}$$

$$= \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k}\frac{1}{(n_1n_2n_3\cdots\cdots n_k)}\int_0^1 a^{n_1+n_2+n_3+\cdots\cdots+n_k-1}da = \int_0^1 \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k}\frac{1}{(n_1n_2n_3\cdots\cdots n_k)}a^{n_1+n_2+n_3+\cdots\cdots+n_k}\frac{da}{a}$$

$$= \int_0^1\left[\left(\sum_{n_1}\frac{a^{n_1}}{n_1}\right)\left(\sum_{n_2}\frac{a^{n_2}}{n_2}\right)\left(\sum_{n_3}\frac{a^{n_3}}{n_3}\right)\cdots\cdots\cdots\left(\sum_{n_k}\frac{a^{n_k}}{n_k}\right)\right]\frac{da}{a} = \int_0^1[(-\ln(1-a))(-\ln(1-a))(-\ln(1-a))\cdots\cdots(-\ln(1-a))]\frac{da}{a}$$

$$= \int_0^1(-\ln(1-a))^k\frac{da}{a} = (-1)^k\int_0^1\frac{\ln^k(1-a)}{a}da = (-1)^k\int_0^1\frac{\ln^k(a)}{1-a}da = (-1)^k\sum_{n=0}^{\infty}\int_0^1 a^n\ln^k(a)da$$

$$= (-1)^k\sum_{n=0}^{\infty}\frac{(-1)^k k!}{(n+1)^{k+1}} = k!\,\zeta(k+1)$$

$$\boxed{\boldsymbol{\therefore F(k) = \sum_{1\leq n_1\leq n_2\leq\cdots\cdots\leq n_k}\frac{1}{P(Q_k)\,S(Q_k)} = k!\,\zeta(k+1)}}$$

$$let\ J = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)\sum_{i=1}^{k}\sum_{q=0}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\frac{(F(k))^{-1}}{(k-q+1)^{k+1}} = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)\sum_{i=1}^{k}\sum_{q=0}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\frac{1}{(k-q+1)^{k+1}k!\,\zeta(k+1)}$$

$$Let\ J_1=\sum_{i=1}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\ and\ J_2=\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)$$

$$\therefore J=\sum_{k=1}^{\infty}\sum_{q=1}^{k}\frac{J_1 J_2}{(k-q+1)^{k+1}k!\,\zeta(k+1)}=\sum_{k=1}^{\infty}\sum_{q=0}^{\infty}\frac{J_1 J_2}{(q+1)^{k+1}k!\,\zeta(k+1)}=\sum_{k=1}^{\infty}\frac{J_1 J_2}{k!\,\zeta(k+1)}\sum_{q=0}^{\infty}\frac{1}{(q+1)^{k+1}}=\sum_{k=1}^{\infty}\frac{J_1 J_2\zeta(k+1)}{k!\,\zeta(k+1)}$$

$$\therefore \boldsymbol{J}=\sum_{\boldsymbol{k=1}}^{\infty}\frac{\boldsymbol{J_1 J_2}}{\boldsymbol{k}!}$$

$$considering\ J_1=\sum_{i=1}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}=\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}H_i=\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\int_0^1\frac{1-x^i}{1-x}dx=\int_0^1\frac{1}{1-x}\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}\left(1-x^i\right)dx$$

$$=\int_0^1\frac{1}{1-x}\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}-\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}x^i dx$$

$$we\ know\ that\ \sum_{i=1}^{k}(-1)^{i}\binom{k}{i}x^i=(1-x)^k-1\ \therefore\sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}x^i=1-(1-x)^k\ and\ \sum_{i=1}^{k}(-1)^{i-1}\binom{k}{i}=1$$

$$\therefore J_1=\int_0^1\frac{1-\left(1-(1-x)^k\right)}{1-x}dx=\int_0^1\frac{(1-x)^k}{1-x}dx=\int_0^1(1-x)^{k-1}dx=-\frac{(1-x)^k}{k}\bigg|_0^1=\frac{1}{k}$$

$$\therefore\ \boldsymbol{J_1}=\sum_{\boldsymbol{i=1}}^{\boldsymbol{k}}\frac{(\boldsymbol{-1})^{\boldsymbol{i-1}}}{(\boldsymbol{H_i})^{\boldsymbol{-1}}}\binom{\boldsymbol{k}}{\boldsymbol{i}}=\frac{\boldsymbol{1}}{\boldsymbol{k}}$$

$$considering\ \ J_2=\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)=1+\cos^{2k}\left(\frac{\pi}{k}\right)+\cos^{2k}\left(\frac{2\pi}{k}\right)+\cos^{2k}\left(\frac{3\pi}{k}\right)+\cdots\cdots\cdots+\cos^{2k}\left(\frac{(k-1)\pi}{k}\right)$$

$$if\ z=cos\left(\frac{2\pi}{k}\right)+i\ sin\left(\frac{2\pi}{k}\right)\ then\ z^j=\left(cos\left(\frac{2\pi}{k}\right)+i\ sin\left(\frac{2\pi}{k}\right)\right)^j=cos\left(\frac{2\pi j}{k}\right)+i\ sin\left(\frac{2\pi j}{k}\right)$$

$$z^j=2cos^2\left(\frac{\pi j}{k}\right)-1+2isin\left(\frac{\pi j}{k}\right)cos\left(\frac{\pi j}{k}\right)$$

$$\therefore\left(1+z^j\right)=2cos^2\left(\frac{\pi j}{k}\right)+2isin\left(\frac{\pi j}{k}\right)cos\left(\frac{\pi j}{k}\right)=2cos\left(\frac{\pi j}{k}\right)\left[cos\left(\frac{\pi j}{k}\right)+i\ sin\left(\frac{\pi j}{k}\right)\right]$$

$$hence\ \left(1+z^j\right)^{2k}=\left(2cos\left(\frac{\pi j}{k}\right)\left[cos\left(\frac{\pi j}{k}\right)+i\ sin\left(\frac{\pi j}{k}\right)\right]\right)^{2k}=2^{2k}cos^{2k}\left(\frac{\pi j}{k}\right)[cos(2\pi j)+i\ sin(2\pi j)]$$

$$\therefore\left(1+z^j\right)^{2k}=2^{2k}cos^{2k}\left(\frac{\pi j}{k}\right)\ and\ cos^{2k}\left(\frac{\pi j}{k}\right)=4^{-k}\left(1+z^j\right)^{2k}$$

$$\therefore\ J_2=\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)=4^{-k}\sum_{j=0}^{k-1}\left(1+z^j\right)^{2k}=4^{-k}\sum_{j=0}^{k-1}\sum_{a=0}^{2k}\binom{2k}{a}z^{ja}=4^{-k}\sum_{j=0}^{k-1}\sum_{a=0}^{2k}\binom{2k}{a}z^{ja}$$

$$=4^{-k}\sum_{j=0}^{k-1}\sum_{a=0}^{2k}\binom{2k}{a}z^{ja}=4^{-k}\underbrace{\sum_{j=0}^{k-1}\left(\binom{2k}{0}+\binom{2k}{1}z^j+\binom{2k}{2}z^{2j}+\binom{2k}{3}z^{3j}+\cdots\cdots\cdots+\binom{2k}{k}z^{kj}+\binom{2k}{2k-1}z^{(2k-1)j}+\binom{2k}{2k}\right)}_{S_k}$$

$$by\ roots\ of\ unity\ filter;\ \frac{S_k}{k}=\binom{2k}{0}+\binom{2k}{k}+\binom{2k}{2k}=\left(2+\binom{2k}{k}\right)\ \therefore S_k=\ k\left(2+\binom{2k}{k}\right)$$

$$hence\ J_2 = 4^{-k}\left(2k + k\binom{2k}{k}\right)$$

$$\boldsymbol{J_2 = \sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right) = 4^{-k}\left(2k + k\binom{2k}{k}\right) = \frac{k}{4^k}\left(2 + \binom{2k}{k}\right)}$$

$$recall\ J = \sum_{k=1}^{\infty}\frac{J_1 J_2}{k!} = \sum_{k=1}^{\infty}\frac{\frac{k}{4^k}\left(2+\binom{2k}{k}\right)\frac{1}{k}}{k!} = \sum_{k=1}^{\infty}\frac{\left(2+\binom{2k}{k}\right)}{4^k k!} = \sum_{k=1}^{\infty}\frac{2}{4^k k!} + \sum_{k=1}^{\infty}\frac{\binom{2k}{k}}{4^k k!} = 2\left(\sum_{k=0}^{\infty}\frac{\left(\frac{1}{4}\right)^k}{k!} - 1\right) + \left(\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!} - 1\right)$$

$$= 2\left(\sqrt[4]{e} - 1\right) + \left(\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!} - 1\right) = 2\sqrt[4]{e} - 3 + \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!}$$

$$\sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!} = \sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k+1)k!} = \sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\sqrt{\pi}\,\Gamma(k+1)\Gamma\left(\frac{1}{2}\right)k!} = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(k+1)k!} = \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{\beta\left(k+\frac{1}{2},\frac{1}{2}\right)}{k!}$$

$$= \frac{1}{\pi}\sum_{k=0}^{\infty}\frac{2}{k!}\int_0^{\frac{\pi}{2}}\cos^{2n}(x)dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\sum_{k=0}^{\infty}\frac{\cos^{2n}(x)}{k!}dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}e^{\cos^2(x)}dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}e^{\frac{\cos(2x)}{2}+\frac{1}{2}}dx = e^{\frac{1}{2}}\frac{2}{\pi}\int_0^{\frac{\pi}{2}}e^{\frac{\cos(2x)}{2}}dx = e^{\frac{1}{2}}\frac{1}{\pi}\int_0^{\pi}e^{\frac{\cos(x)}{2}}dx$$

$$recall\ I_v(z) = \sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k+v}}{\Gamma(k+v+1)k!}\ and\ when\ v=0;\ I_0(z) = \sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{(k!)^2} = \frac{1}{\pi}\int_0^{\pi}e^{z\cos(x)}dx$$

$$\therefore \frac{1}{\pi}\int_0^{\pi}e^{\frac{\cos(x)}{2}}dx = I_0\left(\frac{1}{2}\right)$$

$$hence\ \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!} = e^{\frac{1}{2}}I_0\left(\frac{1}{2}\right) = \sqrt{e}\,I_0(2^{-1})$$

$$\therefore\ J = \sum_{k=1}^{\infty}\frac{J_1 J_2}{k!} = 2\sqrt[4]{e} - 3 + \sum_{k=0}^{\infty}\frac{\binom{2k}{k}}{4^k k!} = 2\sqrt[4]{e} - 3 + \sqrt{e}\,I_0(2^{-1}) = 2\sqrt[4]{e} + \sqrt{e}\,I_0(2^{-1}) - 3$$

$$\boxed{\boldsymbol{J = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1}\cos^{2k}\left(\frac{j\pi}{k}\right)\sum_{i=1}^{k}\sum_{q=1}^{k}\frac{(-1)^{i-1}}{(H_i)^{-1}}\binom{k}{i}\frac{(F(k))^{-1}}{(k-q+1)^{k+1}} = 2\sqrt[4]{e} + \sqrt{e}\,I_0\left(2^{-1}\right) - 3}}$$

***proved***!!