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Let $x^{2020}+a_{2019}x^{2019}+a_{2018}x^{2018}+\cdots+a_0\in\mathbb{Z}[x]$ and all roots of this polynomials are positive real numbers.

Find the smallest possible value of coefficient a_{1010}.

Proposed by Gantumur Choijilsuren-Mongolia

Solution by Abdul Hannan-Tezpur-India

Let $x^{4n}+a_{4n-1}x^{4n-1}+\cdots+a_1x+a_0\in\mathbb{Z}[x]$ and all roots of this polynomial are positive real numbers. Find the smallest possible value of coefficient a_{2n}.

Here is a small observation that we will need later:

$$\frac{\binom{4n-1}{2n-1}}{\binom{4n}{2n}}=\frac{(4n-1)!}{(2n)!\,(2n-1)!}\cdot\frac{(2n)!\,(2n)!}{(4n)!}=\frac{2n}{4n}=\frac{1}{2}$$

Let $\beta_1,\beta_2,\beta_3,\ldots,\beta_{4n-1},\beta_{4n}$ be the (positive real) roots. Then $a_0=\beta_1\beta_2\cdot\ldots\cdot\beta_{4n}>0$.

Being an integer, we must have, $a_0\geq 1$. Also, we have

$$a_{2n}=\sum_{1\leq i_1\leq i_2\leq\cdots\leq i_{2n}\leq 4n}\beta_{i_1}\beta_{i_2}\ldots\beta_{i_{2n}}\overset{AGM}{\geq}$$

$$\overset{AGM}{\geq}\binom{4n}{2n}\left(\prod_{1\le i_1\le i_2\le\cdots\le i_{2n}\le 4n}\beta_{i_1}\beta_{i_2}\dots\beta_{i_{2n}}\right)^{\frac{1}{\binom{4n}{2n}}}=$$

$$=\binom{4n}{2n}\left(\beta_{i_1}\beta_{i_2}\dots\beta_{i_{2n}}\right)^{\frac{\binom{4n-1}{2n-1}}{\binom{4n}{2n}}}=\binom{4n}{2n}\sqrt{\beta_{i_1}\beta_{i_2}\dots\beta_{i_{2n}}}=\binom{4n}{2n}\sqrt{a_0}\geq\binom{4n}{2n};\ a_0\geq 1$$

If we put $\beta_1=\beta_2=\cdots=\beta_{4n}=1$, then we see that

$$a_{2n}=\sum_{1\le i_1\le i_2\le\cdots\le i_{2n}\le 4n}1=\binom{4n}{2n}$$

Since $\binom{4n}{2n}$ is achieved, it is indeed the minimum value of a_{2n}.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.