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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-V

By Marin Chirciu – Romania

1) In ΔABC the following inequality holds:

$$\sum \frac{r_a - h_a}{r_a + h_a} + \frac{2r}{R} \le 1$$

Proposed by Rahim Shahbazov–Azerbaijan

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \frac{r_a - h_a}{r_a + h_a} = \frac{s^2 - 7r^2 - 10Rr}{s^2 + r^2 + 2Rr}$$

Proof. *Using the formulas* $r_a = \frac{S}{s-a}$ *and* $h_a = \frac{2S}{a}$ *we obtain:*

$$\sum \frac{r_a - h_a}{r_a + h_a} = \sum \frac{\frac{S}{s-a} - \frac{2S}{a}}{\frac{S}{s-a} + \frac{2S}{a}} = \sum \frac{3a - 2s}{2s - a} = \sum \frac{2a - b - c}{b + c} = \sum \left(\frac{2a}{b+c} - 1\right) =$$

$$= 2\sum \frac{a}{b+c} - 3 = 2 \cdot \frac{2(s^2 - r^2 - Rr)}{s^2 + r^2 + 2Rr} - 3 = \frac{s^2 - 7r^2 - 10Rr}{s^2 + r^2 + 2Rr}$$

Let's get back to the main problem: Using the Lemma we write the inequality:

$\frac{s^2-7r^2-10Rr}{s^2+r^2+2Rr} + \frac{2r}{R} \le 1 \Leftrightarrow s^2 \le 6R^2 + 2Rr - r^2$, *which follows from Gerretsen's inequality*

$s^2 \le 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$4R^2 + 4Rr + 3r^2 \le 6R^2 + 2Rr - r^2 \Leftrightarrow R^2 - Rr - 2r^2 \ge 0 \Leftrightarrow (R - 2r)(R + r) \ge 0,$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

3) In ΔABC the following relationship holds:

$$\sum \frac{r_a - h_a}{r_a + h_a} + \frac{r}{R - r} \le 1$$

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Solution *Using the Lemma we write the inequality:*

$\frac{s^2-7r^2-10Rr}{s^2+r^2+2Rr}+\frac{r}{R-r}\leq 1 \Leftrightarrow s^2\leq 12R^2-6Rr-9r^2$, *which follows from Gerretsen's inequality* $s^2\leq 4R^2+4Rr+3r^2$.*It remains to prove that:*

$$4R^2+4Rr+3r^2\leq 12R^2-6Rr-9r^2 \Leftrightarrow 4R^2-5Rr-6r^2\geq 0 \Leftrightarrow$$
$$\Leftrightarrow (R-2r)(4R+3r)\geq 0, \text{ obviously from Euler's inequality.}$$

Equality holds if and only if the triangle is equilateral.

Remark.*Inequality 1) can be strengthened.*

4) In ΔABC the following relationship holds:

$$\sum\frac{r_a-h_a}{r_a+h_a}+\frac{3r}{R+r}\leq 1$$

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Solution: *Using the Lemma the inequality can be written:*

$\frac{s^2-7r^2-10Rr}{s^2+r^2+2Rr}+\frac{3r}{R+r}\leq 1 \Leftrightarrow s^2\leq 12R^2+14Rr+5r^2$, *which follows from Gerretsen's inequality* $s^2\leq 4R^2+4Rr+3r^2$.*It remains to prove that:*

$$3(4R^2+4Rr+3r^2)\leq 12R^2+14Rr+5r^2 \Leftrightarrow R\geq 2r \text{ (Euler's inequality)}$$

Equality holds if and only if the triangle is equilateral.

Remark.*Inequality 4) is stronger than inequality 1).*

5) In ΔABC the following relationship holds:

$$\sum\frac{r_a-h_a}{r_a+h_a}+\frac{2r}{R}\leq\sum\frac{r_a-h_a}{r_a+h_a}+\frac{3r}{R+r}\leq 1$$

Solution:*See inequality 4) and* $\frac{2r}{R}\leq\frac{3r}{R+r}\Leftrightarrow R\geq 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark. *We can write the inequalities*

6) In ΔABC the following inequality holds:

$$\sum\frac{r_a-h_a}{r_a+h_a}+\frac{r}{R-r}\leq\sum\frac{r_a-h_a}{r_a+h_a}+\frac{2r}{R}\leq\sum\frac{r_a-h_a}{r_a+h_a}+\frac{3r}{R+r}\leq 1$$

Solution:*See inequalities 3) and 5).Equality holds if and only if the triangle is equilateral.*

Reference:

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