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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-IV

By Marin Chirciu – Romania

1) Let be $x, y, z, t > 0$ such that $xyzt = 1$. Prove that:

$$\frac{x^2+1}{x^5+3}+\frac{y^2+1}{y^5+3}+\frac{z^2+1}{z^5+3}+\frac{t^2+1}{t^5+3}\le 2$$

Proposed by Rahim Shabazov – Azerbaijan

Solution *We prove the following lemma:*

Lemma.

If $x > 0$ then:

$$\frac{x^2+1}{x^5+3}\le\frac{2}{x+3}$$

Proof.

Inequality $\frac{x^2+1}{x^5+3}\le\frac{2}{x+3}$ *is equivalent with:*

$2x^5-x^3-3x^2-x+3\ge 0 \Leftrightarrow (x-1)^2(2x^3+4x^2+5x+3)\ge 0$, *obviously with equality for* $x=1$.

Let's get back to the main problem.

Using the Lemma we obtain:

$M_x=\sum\frac{x^2+1}{x^5+3}\le\sum\frac{2}{x+3}\overset{(1)}{\le}2=M_d$, *where (1)* $\Leftrightarrow \sum\frac{1}{x+3}\Leftrightarrow\sum\frac{x}{x+3}\ge 1$, *which follows from Bergström's inequality:*

$$\sum\frac{x}{x+3}=\sum\frac{\left(\sqrt{x}\right)^2}{x+3}\ge\frac{\left(\sum\sqrt{x}\right)^2}{\sum(x+3)}\overset{(2)}{\ge}1,\ \text{where (2)}\Leftrightarrow$$

$$\Leftrightarrow\frac{x+y+z+t+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{xt}+2\sqrt{yz}+2\sqrt{yt}+2\sqrt{zt}}{x+y+z+t+12}\ge 1\Leftrightarrow$$

$\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+2\sqrt{yz}+\sqrt{yt}+\sqrt{zt}\ge 6$, *which follows from means inequality and the condition from hypothesis* $xyzt=1$.

We deduce that the equality from enunciation holds, with equality if and only if $x=y=z=t=1$.

Remark. *The inequality can be developed*

2) Let be $x, y, z, t > 0$ such that $xyzt = 1$ and $3 \le n \le 4$. Prove that:

$$\frac{x^2+1}{x^5+n}+\frac{y^2+1}{y^5+n}+\frac{z^2+1}{z^5+n}+\frac{t^2+1}{t^5+n}\le\frac{8}{n+1}$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

If $x > 0$ and $\frac{3}{2} \le n \le 4$ then:

$$\frac{x^2+1}{x^5+n}\le\frac{2}{x(4-n)+2n-3}$$

Proof. *Inequality* $\frac{x^2+1}{x^5+n}\le\frac{2}{x(4-n)+2n-3}$ *is equivalent with:*

$$2x^5+(n-4)x^3+(3-2n)x^2+(n-4)x+3\ge 0 \Leftrightarrow$$

$$\Leftrightarrow (x-1)^2(2x^2+4x^2+(n+2)x+3)\ge 0 \text{ obviously with equality for } x=1.$$

Let's get back to the main problem.

Using the Lemma we obtain:

$$M_x=\sum\frac{x^2+1}{x^5+n}\le\sum\frac{2}{x(4-n)+2n-3}\overset{(1)}{\le}\frac{8}{n+1}=M_d,\text{ where (1)}$$

$$\Leftrightarrow\sum\frac{1}{x(4-n)+2n-3}\le\frac{4}{n+1}\Leftrightarrow(4-n)\sum\frac{x}{x(4-n)+2n-3}\ge(4-n)\frac{4}{n+1},\text{ which follows from}$$

$$(4-n)\ge 0 \text{ and } \sum\frac{1}{x(4-n)+2n-3}\ge\frac{4}{n+1},\text{ true from Bergström's inequality:}$$

$$\sum\frac{x}{x(4-n)+2n-3}=\sum\frac{(\sqrt{x})^2}{x(4-n)+2n-3}\ge\frac{(\sum\sqrt{x})^2}{\sum(x(4-n)+2n-3)}\overset{(2)}{\ge}\frac{4}{n+1},\text{ where (2)}\Leftrightarrow$$

$$\Leftrightarrow\frac{x+y+z+t+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{xt}+2\sqrt{yz}+2\sqrt{yt}+2\sqrt{zt}}{(4-n)(x+y+z+t)+4(2n-3)}\ge\frac{4}{n+1}\Leftrightarrow$$

$$(n+1)\left(x+y+z+t+2\sqrt{xy}+2\sqrt{xz}+2\sqrt{xt}+2\sqrt{yz}+2\sqrt{yt}+2\sqrt{zt}\right)\ge$$

$$\ge 4(4-n)(x+y+z+t)+16(2n-3)\Leftrightarrow$$

$$\Leftrightarrow(5n-15)(x+y+z+t)+(2n+2)\left(\sqrt{xy}+\sqrt{xz}+\sqrt{xt}+\sqrt{yz}+\sqrt{yt}+\sqrt{zt}\right)\ge 16(2n-3),$$

which follows from means inequality and the conditions $n \ge 3, xyzt = 1$, assured by the *hypothesis. We deduce that the inequality from enunciation holds, with equality if and only if* $x = y = t = 1$.

Note.

1) For $n = 3$ *we obtain the proposed problem by Rahim Shahbazov în RMM 11/2019.*

2) For $n = 4$ *we obtain the inequality:*

3) Let $x, y, z, t > 0$ such that $xyzt = 1$. Prove that:

$$\frac{x^2+1}{x^5+4}+\frac{y^2+1}{y^5+4}+\frac{z^2+1}{z^5+4}+\frac{t^2+1}{t^5+4}\le\frac{8}{5}$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

If $x > 0$ then:

$$\frac{x^2+1}{x^5+4}\le\frac{2}{5}$$

Proof. *Inequality* $\frac{x^2+1}{x^5+4}\le\frac{2}{5}$ *is equivalent with:*

$2x^5 - x^2 + 3 \ge 0 \Leftrightarrow (x-1)^2(2x^3+4x^2+6x+3) \ge 0$, *obviously with equality for* $x = 1$. *Let's get back to the main problem. Using the Lemma we obtain:*

$$M_x = \sum\frac{x^2+1}{x^5+4}\le\sum\frac{2}{5}=4\cdot\frac{2}{5}=\frac{8}{5}=M_d$$

We deduce that the inequality from enunciation holds, with equality if and only if $x = y = z = t = 1$.

Remark. *Inequality can be reduced to three variables:*

4) Let $x, y, z > 0$ such that $xyz = 1$ and $\frac{11}{4}\le n\le 4$. Prove that:

$$\frac{x^2+1}{x^5+n}+\frac{y^2+1}{y^5+n}+\frac{z^2+1}{z^5+n}\le\frac{6}{n+1}$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma $\frac{x^2+1}{x^5+n}\le\frac{2}{x(4-n)+2n-3}, x>0$ *and* $\frac{3}{2}\le n\le 4$ *we obtain:*

$$M_x=\sum\frac{x^2+1}{x^5+n}\le\sum\frac{2}{x(4-n)+2n-3}\overset{(1)}{\le}\frac{6}{n+1}=M_d,\ \text{where (1)}$$

$$\Leftrightarrow\sum\frac{1}{x(4-n)+2n-3}\le\frac{3}{n+1}\Leftrightarrow(4-n)\sum\frac{x}{x(4-n)+2n-3}\ge(4-n)\frac{3}{n+1},\ \text{which follows from}$$

$$(4-n)\ge 0\ \text{and}\sum\frac{1}{x(4-n)+2n-3}\ge\frac{3}{n+1},\ \text{true from Bergström's inequality:}$$

$$\sum\frac{x}{x(4-n)+2n-3}=\sum\frac{(\sqrt{x})^2}{x(4n-n)+2n-3}\ge\frac{(\sum\sqrt{x})^2}{\sum(x(4-n)+2n-3)}\overset{(2)}{\ge}\frac{3}{n+1},\ \text{where (2)}\Leftrightarrow$$

$$\Leftrightarrow\frac{x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}}{(4-n)(x+y+z)+3(2n-3)}\ge\frac{3}{n+1}\Leftrightarrow$$

$$(n+1)\left(x+y+z+2\sqrt{xy}+2\sqrt{yz}+2\sqrt{zx}\right)\ge 3(4-n)(x+y+z)+12(2n-3)\Leftrightarrow$$

$$\Leftrightarrow(4n-11)(x+y+z)+(2n+2)\left(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}\right)\ge 12(2n-3)$$

which follows from means inequality and the conditions $n\ge\frac{11}{4}, xyz=1$*, assured by the enunciation. We deduce that the inequality from enunciation holds, with equality if and only if* $x=y=z=1$.

Remark.*Inequality can be strengthened to five variables:*

5) Let $x_1,x_2,x_3,x_4,x_5>0$ such that $x_1\cdot x_2\cdot x_3\cdot x_4\cdot x_5=1$ and $\frac{19}{6}\le n\le 4$. Prove that:

$$\frac{x_1^2+1}{x_1^5+n}+\frac{x_2^2+1}{x_2^5+n}+\frac{x_3^2+1}{x_3^5+n}+\frac{x_5^2+1}{x_4^5+n}+\frac{x_5^2+1}{x_5^5+n}\le\frac{10}{n+1}$$

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Solution *Using the Lemma* $\frac{x^2+1}{x^5+n}\le\frac{2}{x(4-n)+2n-3}, x>0$ *and* $\frac{3}{2}\le n\le 4$ *we obtain:*

$$M_x=\sum\frac{x_1^2+1}{x_1^5+n}\le\sum\frac{2}{x_1(4-n)+2n-3}\overset{(1)}{\le}\frac{10}{n+1}=M_d,\ \text{where (1)}$$

$$\Leftrightarrow\sum\frac{1}{x_1(4-n)+2n-3}\le\frac{5}{n+1}\Leftrightarrow(4-n)\sum\frac{x}{x_1(4-n)+2n-3}\ge(4-n)\frac{5}{n+1},\ \text{which follows from}$$

$$(4-n)\ge 0\ \text{and}\sum\frac{1}{x_1(4-n)+2n-3}\ge\frac{5}{n+1}\ \text{true from Bergström's inequality:}$$

$$\sum\frac{x_1}{x_1(4-n)+2n-3}=\sum\frac{(\sqrt{x_1})^2}{x_1(4-n)+2n-3}\ge\frac{(\sum\sqrt{x_1})^2}{\sum(x_1(4-n)+2n-3)}\overset{(2)}{\ge}\frac{5}{n+1},\ \text{where (2)}\Leftrightarrow$$

$$\Leftrightarrow\frac{\sum x_1+2\sum\sqrt{x_1x_2}}{(4-n)\sum x_1+5(2n-3)}\ge\frac{5}{n+1}\Leftrightarrow$$

$$\Leftrightarrow(n+1)\left(\sum x_1+2\sum\sqrt{x_1x_2}\right)\ge 5(4-n)\sum x_1+25(2n-3)\Leftrightarrow$$

$$(6n-19)\sum x_1+(2n+2)\sum x_1x_2\ge 25(2n-3)$$

which follows from means inequality and the condition $n\ge\frac{19}{6}, x_1\cdot x_2\cdot x_3\cdot x_4\cdot x_5=1$, *assured by the enunciation.*

We obtain $(6n-19)\sum x_1+(2n+2)\sum x_1x_2\ge(6n-19)\cdot 5+(2n+2)\cdot 10=25(2n-3)$

We deduce that the inequality from enunciation holds, with equality if and only if

$$x_1=x_2=x_3=x_4=x_5=1$$

Remark. *We can generalize the inequality:*

6) Let $x_1,x_2,\dots,x_k>0$ ***such that*** $x_1\cdot x_2\cdot\ldots\cdot x_k=1$ ***and*** $\frac{4k-1}{k+1}\le n\le 4$***.***

Prove that:

$$\frac{x_1^2+1}{x_1^5+n}+\frac{x_2^2+1}{x_2^5+n}+\dots+\frac{x_k^2+1}{x_k^5+n}\le\frac{2k}{n+1}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemma* $\frac{x^2+1}{x^5+n}\le\frac{2}{x(4-n)+2n-3}, x>0$ *and* $\frac{3}{2}\le n\le 4$ *we obtain:*

$$M_x=\sum\frac{x_1^2+1}{x_1^5+n}\le\sum\frac{2}{x_1(4-n)+2n-3}\overset{(1)}{\le}\frac{2k}{n+1}=M_d,\ \text{where (1)}$$

$$\Leftrightarrow\sum\frac{1}{x_1(4-n)+2n-3}\le\frac{k}{n+1}\Leftrightarrow(4-n)\sum\frac{x}{x_1(4-n)+2n-3}\ge(4-n)\frac{k}{n+1},\ \text{which follows from}$$

$$(4-n)\ge 0\ \text{and}\sum\frac{1}{x_1(4-n)+2n-3}\ge\frac{k}{n+1},\ \text{true from Bergström's inequality:}$$

$$\sum\frac{x_1}{x_1(4-n)+2n-3}=\sum\frac{(\sqrt{x_1})^2}{x_1(4-n)+2n-3}\ge\frac{(\sum\sqrt{x_1})^2}{\sum(x_1(4-n)+2n-3)}\overset{(2)}{\ge}\frac{k}{n+1},\ \text{where (2)}\Leftrightarrow$$

$$\Leftrightarrow\frac{\sum x_1+2\sum\sqrt{x_1x_2}}{(4-n)\sum x_1+k(2n-3)}\ge\frac{k}{n+1}\Leftrightarrow$$

$$\Leftrightarrow(n+1)\left(\sum x_1+2\sum\sqrt{x_1x_2}\right)\ge k(4-n)\sum x_1+k^2(2n-3)\Leftrightarrow$$

$$\Leftrightarrow(n(k+1)+1-4k)\sum x_1+(2n+2)\sum x_1x_2\ge k^2(2n-3)$$

which follows from means inequality and the conditions $n\ge\frac{4k-1}{k+1}, x_1\cdot x_2\cdot\ldots\cdot x_k=1$, *assured by the enunciation.*

We deduce that the inequality from enunciation holds, with equality if and only if

$$x_1=x_2=\dots=x_k=1.$$

Reference:

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