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ABOUT AN INEQUALITY BY RAHIM SHAHBAZOV-III

By Marin Chirciu – Romania

1) Let $a, b, c > 0$. Prove that:

$$\sqrt[2019]{\frac{a^3}{a^3 - ab^2 + b^3}} + \sqrt[2019]{\frac{b^3}{b^3 - bc^2 + c^3}} + \sqrt[2019]{\frac{c^3}{c^3 - ca^2 + a^3}} \le 3$$

Proposed by Rahim Shahbazov-Azerbaijan

Solution.

Denote $\frac{a^3}{a^3-ab^2+b^3} = m, \frac{b^3}{b^3-bc^2+c^3} = n$ *and* $\frac{c^3}{c^3-ca^2+a^3} = s$,

we prove that $m + n + p \le 3$, *where from means inequality it follows*

$$\sqrt[2019]{m} + \sqrt[2019]{n} + \sqrt[2019]{p} \le 3.$$

2) Let $a, b, c > 0$. Prove that:

$$\frac{a^3}{a^3 - ab^2 + b^3} + \frac{b^3}{b^3 - bc^2 + c^3} + \frac{c^3}{c^3 - ca^2 + a^3} \le 3$$

Solution

Denoting $\frac{b}{a} = x, \frac{c}{b} = y, \frac{a}{c} = z$, *we have* $x, y, z > 0, xyz = 1$ *and the above inequality can be written:*

$$\frac{1}{x^3 - x^2 + 1} + \frac{1}{y^3 - y^2 + 1} + \frac{1}{z^3 - z^2 + 1} \le 3$$

As $x^3 - x^2 + 1 \ge x^2 - x + 1 \Leftrightarrow x(x-1)^2 \ge 0$, *obviously with equality for* $x = 1$, *it suffices to prove that:* $\frac{1}{x^2-x+1} + \frac{1}{y^2-y+1} + \frac{1}{z^2-z+1} \le 3$

"Flipping" the last inequality, we obtain: $\frac{(2x-1)^2}{x^2-x+1} + \frac{(2y-1)^2}{y^2-y+1} + \frac{(2z-1)^2}{z^2-z+1} \ge 3$, *which follows from Bergström's inequality. We obtain* $\frac{(2x-1)^2}{x^2-x+1} + \frac{(2y-1)^2}{y^2-y+1} + \frac{(2z-1)^2}{z^2-z+1} \ge \frac{(2x+2y+2z-3)^2}{x^2+y^2+z^2-x-y-x+3} \overset{(1)}{\ge} 3$,

where (1) $\Leftrightarrow (x+y+z)^2 + 6(xy+yz+zx) \ge 9(x+y+z)$ *(2), true from:*

$$(xy+yz+zx)^2 \ge 3xyz(x+y+z) = 3(x+y+z) = 3t = u^2$$

We have $xy+yz+zx\ge u$ *and* $x+y+z=\frac{u^2}{3}$*. It suffices to prove equality (2) and to prove that:*

$$\left(\frac{u^2}{3}\right)^2+6u\ge 9\cdot\frac{u^2}{3}\Leftrightarrow u^4-27u^2+54u\ge 0\Leftrightarrow u(u-3)^2(u+6)\ge 0$$

Equality holds if and only if $a=b=c$.

3) Let $a,b,c>0$, real numbers and $n\ge 1,k\ge 2$, natural numbers.

Prove that:

$$\sqrt[k]{\frac{a^{n+1}}{a^{n+1}-ab^n+b^{n+1}}}+\sqrt[k]{\frac{b^{n+1}}{b^{n+1}-bc^n+c^{n+1}}}+\sqrt[k]{\frac{c^{n+1}}{c^{n+1}-ca^n+a^{n+1}}}\le 3$$

Proposed by Marin Chirciu – Romania

Solution

Denoting $\frac{a^{n+1}}{a^{n+1}-ab^n+b^{n+1}}=m,\frac{b^{n+1}}{b^{n+1}-bc^n+c^{n+1}}=n$ *and* $\frac{c^{n+1}}{c^{n+1}-ca^n+a^{n+1}}=p$*, we prove that* $m+n+p\le 3$*, wherefrom means inequality it follows that* $\sqrt[k]{m}+\sqrt[k]{n}+\sqrt[k]{p}\le 3$.

4) Let $a,b,c>0$ and $n\ge 1$, natural number. Prove that:

$$\frac{a^{n+1}}{a^{n+1}-ab^n+b^{n+1}}+\frac{b^{n+1}}{b^{n+1}-bc^n+c^{n+1}}+\frac{c^{n+1}}{c^{n+1}-ca^n+a^{n+1}}\le 3$$

Proposed by Marin Chirciu – Romania

Solution

Denoting $\frac{b}{a}=x,\frac{c}{b}=y,\frac{a}{c}=z$*, we have* $x,y,z>0,xyz=1$ *and we write the inequality that we have to prove:* $\frac{1}{x^{n+1}-x^n+1}+\frac{1}{y^{n+1}-y^n+1}+\frac{1}{z^{n+1}-z^n+1}\le 3$.

As $x^{n+1}-x^n+1\ge x^2-x+1\Leftrightarrow x(x-1)^2(x^{n-2}+x^{n-3}+\dots+x+1)\ge 0$*, obviously with equality for* $x=1$*, it suffices to prove that:*

$\frac{1}{x^2-x+1}+\frac{1}{y^2-y+1}+\frac{1}{z^2-z+1}\le 3$*. "Flipping" the last inequality, we obtain*

$\frac{(2x-1)^2}{x^2-x+1}+\frac{(2y-1)^2}{y^2-y+1}+\frac{(2z-1)^2}{z^2-z+1}\ge 3$*, which follows from Bergström's inequality.*

We obtain $\frac{(2x-1)^2}{x^2-x+1}+\frac{(2y-1)^2}{y^2-y+1}+\frac{(2z-1)^2}{z^2-z+1}\ge\frac{(2x+2y+2z-3)^2}{x^2+y^2+z^2-x-y-x+3}\overset{(1)}{\ge} 3$*, where (1)*

$$\Leftrightarrow (x+y+z)^2+6(xy+yz+zx)\ge 9(x+y+z)\ (2)\text{, true from:}$$

$$(xy+yz+zx)^2\ge 3xyz(x+y+z)=3(x+y+z)=3t=u^2$$

We have $xy+yz+zx\ge u$ *and* $x+y+z=\frac{u^2}{3}$

It suffices for proving inequality (2) and to prove that;

$$\left(\frac{u^2}{3}\right)^2+6u\ge 9\cdot\frac{u^2}{3}\Leftrightarrow u^4-27u^2+54u\ge 0\Leftrightarrow u(u-3)^2(u+6)\ge 0$$

Equality holds if and only if $a=b=c$.

Note.

For $n=2$ *and* $k=2019$ *we obtain the problem from RMM 2019, proposed by Rahim Shahbazov.*

Reference:

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