# ABOUT AN INEQUALITY BY MEHMET SAHIN-I

***By Marin Chirciu – Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$\sum \frac{b+c}{a}\cdot\frac{r_a}{h_a}\ge 6$$

***Proposed by Mehmet Şahin – Ankara – Turkey***

***Solution*** *We prove the following lemma:*

***Lemma.***

***2) In $\Delta ABC$ the following relationship holds:***

$$\sum \frac{b+c}{a}\cdot\frac{r_a}{h_a}=2\left(1+\frac{R}{r}\right)$$

***Proof.*** *Using* $r_a=\frac{S}{s-a}$ *and* $h_a=\frac{2S}{a}$ *we obtain:*

$\frac{r_a}{h_a}=\frac{\frac{S}{s-a}}{\frac{2S}{a}}=\frac{a}{2(s-a)}$, *wherefrom* $\sum\frac{b+c}{a}\cdot\frac{r_a}{h_a}=\sum\frac{b+c}{a}\cdot\frac{a}{2(s-a)}=\frac{1}{2}\sum\frac{b+c}{s-a}=2\left(1+\frac{R}{r}\right)$, *which follows from the known identity in triangle* $\sum\frac{b+c}{s-a}=4\left(1+\frac{R}{r}\right)$. *Let's get back to the main problem. Using the Lemma the inequality can be rewritten:*

$$2\left(1+\frac{R}{r}\right)\ge 6\Leftrightarrow R\ge 2r \text{ (Euler's inequality)}$$

*Equality holds if and only if the triangle is equilateral.*

***Solution*** *Let's find an inequality having an opposite sense:*

***3) In $\Delta ABC$ the following relationship holds:***

$$\sum \frac{b+c}{a}\cdot\frac{r_a}{h_a}\le\frac{3R}{r}$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*Using Lemma we write the inequality:*

$$2\left(1+\frac{R}{r}\right)\le\frac{3R}{r}\Leftrightarrow R\ge 2r \text{ (Euler's inequality)}$$

*Equality holds if and only if the inequality is equilateral.*

***Remark.***

*We can write the double inequality:*

**4) In ΔABC the following relationship holds:**

$$6 \le \sum \frac{b+c}{a} \cdot \frac{r_a}{h_a} \le \frac{3R}{r}$$

***Solution***

*See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

***Remark.***

*If we interchange $r_a$ with $h_a$ we propose:*

**5) In ΔABC the following relationship holds:**

$$6 \le \sum \frac{b+c}{a} \cdot \frac{h_a}{r_a} \le \left(\frac{2R}{r}\right)^2 - 10$$

***Proposed by Marin Chirciu – Romania***

***Solution***

*We prove the following lemma:*

***Lemma.***

**6) In ΔABC the following relationship holds:**

$$\sum \frac{b+c}{a} \cdot \frac{h_a}{r_a} = \frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)}{4R^2s^2}$$

***Proof.***

*Using* $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:* $\frac{h_a}{r_a} = \frac{\frac{2S}{a}}{\frac{S}{s-a}} = \frac{2(s-a)}{a}$, *wherefrom*

$$\sum \frac{b+c}{a} \cdot \frac{h_a}{r_a} = \sum \frac{b+c}{a} \cdot \frac{2(s-a)}{a} = 2\sum \frac{(b+c)(s-a)}{a^2}$$

$$= \frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)}{4R^2r^2}$$

*which follows from the known identity in triangle*

$$\sum\frac{(b+c)(s-a)}{a^2}=\frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr-r^2)}{8R^2r^2}\ and$$

$$\sum b^2c^2\,(b+c)(s-a)=2s^2[s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)]$$

*Let's get back to the main problem.*

*LHS inequality. Using the lemma the inequality can be written:*

$$\frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)}{4R^2r^2}\geq 6\Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\geq 24R^2r^2$, *which follows from Gerretsen's inequality* $s^2\geq 16Rr-5r^2$. *It remains to prove that:*

$$(16Rr-5r^2)(16Rr-5r^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\geq 24R^2r^2\Leftrightarrow$$

$\Leftrightarrow 3R^2-7Rr+2r^2\geq 0\Leftrightarrow (R-2r)(3R-r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$. *Equality holds if and only if the triangle is equilateral.*

*Using Lemma the inequality can be rewritten:*

$$\frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)}{4R^2r^2}\leq\left(\frac{2R}{r}\right)^2-10\Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\leq 4R^2r^2(4R^2-10r^2)$, *which follows from Gerretsen's inequality* $s^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$(4R^2+4Rr+3r^2)(4R^2+4Rr+3r^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\leq 4R^2(4R^2-10r^2)$$

$\Leftrightarrow 3R^3-6R^2r+Rr^2-2r^3\geq 0\Leftrightarrow (R-2r)(3R^2+r^2)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$. *Equality holds if and only if the triangle is equilateral.*

***Remark.***

*Between* $\sum\frac{b+c}{a}\cdot\frac{r_a}{h_a}$ *and* $\sum\frac{b+c}{a}\cdot\frac{h_a}{r_a}$ *the following relationship holds:*

## *7) In ΔABC the following relationship holds:*

$$\sum\frac{\boldsymbol{b+c}}{\boldsymbol{a}}\cdot\frac{\boldsymbol{h_a}}{\boldsymbol{r}}\geq\frac{\boldsymbol{2r}}{\boldsymbol{R}}\sum\frac{\boldsymbol{b+c}}{\boldsymbol{a}}\cdot\frac{\boldsymbol{r_a}}{\boldsymbol{h_a}}$$

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***Solution*** *Using the lemmas from above, we write the inequality:*

$$\frac{s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)}{4R^2r^2}\geq\frac{2r}{R}\cdot 2\left(1+\frac{R}{r}\right)\Leftrightarrow$$

$\Leftrightarrow s^2(s^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\geq 16Rr^2(R+r)$, *which follows from*

*Gerretsen's inequality* $s^2\geq 16Rr-5r^2$. *It remains to prove that:*

$(16Rr-5r^2)(16Rr-5r^2+2r^2-14Rr)+r^2(16R^2+2Rr+r^2)\geq 16Rr^2(R+r)\Leftrightarrow$

$\Leftrightarrow 4R^2-9Rr+2r^2\geq 0\Leftrightarrow (R-2r)(4R-r)\geq 0$, *obviously from Euler's inequality*

$R\geq 2r$. *Equality holds if and only if the triangle is equilateral.*

***Remark.***

*We write the double sequence of inequalities:*

***8) In $\Delta ABC$ the following inequality holds:***

$$\frac{3R}{r}\geq\sum\frac{b+c}{a}\cdot\frac{h_a}{r_a}\geq\frac{2r}{R}\sum\frac{b+c}{a}\cdot\frac{r_a}{h_a}\geq\frac{12r}{R}$$

*See inequalities 1), 4) and 8). Equality holds if and only if the triangle is equilateral.*

**Reference:**

**Romanian Mathematical Magazine-www.ssmrmh.ro**