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ABOUT AN INEQUALITY OF MARIAN URSĂRESCU-I

By Marin Chirciu – Romania

1) Let be $\Delta ABC, AD, BE$ and CF the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 18r^2$$

Proposed by Marian Ursărescu – Romania

Solution

We prove the following lemma:

Lemma.

2) Let be $\Delta ABC, AD, BE$ and CF the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB=2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}$$

Solution

Using the bisector's theorem in ΔABC the following relationship holds:

$BD=\frac{ac}{b+c}, CE=\frac{ab}{a+c}, AF=\frac{bc}{a+b}$. *It follows that:*

$$AF\cdot BC+BD\cdot AC+CE\cdot AB=\frac{bc}{a+b}\cdot a+\frac{ab}{a+c}\cdot c+\frac{ab}{a+c}\cdot c=abc\sum\frac{1}{b+c}=$$

$=4Rrs\cdot\frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}=2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}$, *which follows from* $\sum\frac{1}{b+c}=\frac{5s^2+r^2+4Rr}{2s(s^2+r^2+2Rr)}$.

Let's get back to the main problem.

Using Lemma the inequality from enunciation can be written:

$2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}\ge 18r^2\Leftrightarrow s^2(5R-9r)\ge r(9r^2+17Rr-4R^2)$, *which follows from*

Gerretsen's inequality $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$(16Rr-5r^2)(5R-9r)\ge r(9r^2+17Rr-4R^2)\Leftrightarrow 14R^2-31Rr+6r^2\ge 0\Leftrightarrow$

$(R-2r)(14R-3r)\ge 0$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark.

The inequality can be strengthened:

3) Let be $\Delta ABC, AD, BE$ and CF be the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 9Rr$$

Proposed by Marin Chirciu – Romania

Solution

Using the Lemma the inequality can be written:

$2Rr\cdot\frac{5s^2+r^2+4Rr}{s^2+r^2+2Rr}\ge 9Rr\Leftrightarrow s^2\ge 10Rr+7r^2$ *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:* $16Rr-5r^2\ge 10Rr+7r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality). Equality holds if and only if the triangle is equilateral.*

Remark.

Inequality 3) is stronger than inequality 1):

4) Let be $\Delta ABC, AD, BE$ and CF be the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\ge 9Rr\ge 18r^2$$

Solution

See inequalities 1), 3) and $9Rr\ge 18r^2\Leftrightarrow R\ge 2r$ *(Euler's inequality)*

Equality holds if and only if the triangle is equilateral.

Remark.

Let's find an inequality having an opposite sense:

5) Let be $\Delta ABC, AD, BE$ and CF the internal bisectors. Prove that:

$$AF\cdot BC+BD\cdot AC+CE\cdot AB\le\frac{9R^2}{4}$$

Proposed by Marin Chirciu – Romania

Solution

Using Lemma the inequality can be written:

$$2Rr \cdot \frac{5s^2 + r^2 + 4Rr}{s^2 + r^2 + 2Rr} \leq \frac{9R^2}{2} \Leftrightarrow s^2(9R - 20r) + r(18R^2 - 7Rr - 4r^2) \geq 0$$

We distinguish the following cases:

Case 1). If $(9R - 20r) \geq 0$, *the inequality is obvious.*

Case 2). If $(9R - 20r) < 0$, *the inequality can be rewritten:*

$r(18R^2 - 7Rr - 4r^2) \geq s^2(20r - 9R)$, *which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r(18R^2 - 7Rr - 4r^2) \geq (4R^2 + 4Rr + 3r^2)(20r - 9R) \Leftrightarrow$$

$$\Leftrightarrow 18R^3 - 13R^2r - 30Rr^2 - 32r^3 \geq 0 \Leftrightarrow (R - 2r)(18R^2 + 23Rr + 16r^2) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$. *Equality holds if and only if the triangle is equilateral.*

Remark.

We can write the double inequality:

6) Let be $\Delta ABC, AD, BE$ ***and*** CF ***the internal bisectors. Prove that:***

$$9Rr \leq AF \cdot BC + BD \cdot AC + CE \cdot AB \leq \frac{9R^2}{2}$$

Proposed by Marin Chirciu – Romania

Solution:

See inequalities 3) and 5). Equality holds if and only if the triangle is equilateral.

Reference:

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