R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-II

By Marin Chirciu-Romania

1) In any ΔABC the following relationship:

$$\sum_{cyc} \frac{1}{1-\cos A} \geq 2\sum_{cyc} \frac{m_a}{r_a}$$

Proposed by Bogdan Fuştei-Romania

Solution:

2) Lemma 1.
In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{1-\cos A} = \frac{s^2+r^2-8Rr}{2r^2}$$

Using $\cos x = 1-2\sin^2\frac{x}{2}$ we get

$$\sum_{cyc} \frac{1}{1-\cos A} = \sum_{cyc} \frac{1}{2\sin^2\frac{A}{2}} = \frac{1}{2}\sum_{cyc} \frac{1}{\sin^2\frac{A}{2}} = \frac{1}{2}\cdot\frac{s^2+r^2-8Rr}{r^2} = \frac{s^2+r^2-8Rr}{2r^2}$$

which result from known relation $\sum_{cyc} \frac{1}{\sin^2\frac{A}{2}} = \frac{s^2+r^2-8Rr}{r^2}$ in any triangle .

3) Lemma 2.
In any ΔABC the following relationship:

$$\sum_{cyc} \frac{m_a}{r_a} \leq \frac{s^2+r^2-8Rr}{4r^2}$$

Proof.

Using Panaitopol' Inequality $\frac{m_a}{h_a} \leq \frac{R}{2r}$ $(GM.11-1982)$ we get

$\frac{m_a}{r_a} \leq \frac{h_a\cdot\frac{R}{2r}}{r_a} = \frac{R}{2r}\cdot\frac{h_a}{r_a} \Rightarrow \sum_{cyc}\frac{m_a}{r_a} \leq \sum_{cyc}\frac{R}{2r}\cdot\frac{h_a}{r_a} = \frac{R}{2r}\sum_{cyc}\frac{h_a}{r_a} = \frac{R}{2r}\cdot\frac{s^2+r^2-8Rr}{2Rr} = \frac{s^2+r^2-8Rr}{4r^2}$ which result from known identity $\sum_{cyc}\frac{h_a}{r_a} = \frac{s^2+r^2-8Rr}{2Rr}$

Let's solve the proposed problem.

$$Lhs=\sum_{cyc}\frac{1}{1-cosA}\overset{Lemma1}{=}\frac{s^2+r^2-8Rr}{2r^2}\overset{Lemma2}{\geq}2\sum_{cyc}\frac{m_a}{r_a}=Rhs.$$

Equality holds if and only if the triangle is equilateral.

Remark. In same class of problems:

4) In any ΔABC the following relationship:

$$\sum_{cyc}\frac{1}{1-cosA}\geq 2\sum_{cyc}\frac{m_a}{h_a}$$

Proposed by Marin Chirciu-Romania

5) In any ΔABC the following relationship:

$$\sum_{cyc}\frac{1}{1-cosA}=\frac{s^2+r^2-8Rr}{2r^2}$$

Proposed by Marin Chirciu-Romania

Proof.

Using $cos\,x=1-2sin^2\frac{x}{2}$ we get

$$\sum_{cyc}\frac{1}{1-cosA}=\sum_{cyc}\frac{1}{2sin^2\frac{A}{2}}=\frac{1}{2}\sum_{cyc}\frac{1}{sin^2\frac{A}{2}}=\frac{1}{2}\cdot\frac{s^2+r^2-8Rr}{r^2}=\frac{s^2+r^2-8Rr}{2r^2}$$

which result from known relation $\sum_{cyc}\frac{1}{sin^2\frac{A}{2}}=\frac{s^2+r^2-8Rr}{r^2}$ in any triangle .

Lemma 2. 6) In any ΔABC the following relationship: $\sum_{cyc}\frac{m_a}{h_a}\leq\frac{s^2+r^2-8Rr}{4r^2}$

Proof. Using Panaitopol' Inequality $\frac{m_a}{h_a}\leq\frac{R}{2r}$ $(GM.11-1982)$ we get

$$\sum_{cyc}\frac{m_a}{h_a}\leq\sum_{cyc}\frac{R}{2r}=\frac{3R}{2r}$$

Let's solve the proposed problem.

$$Lhs=\sum_{cyc}\frac{1}{1-cosA}\overset{Lemma1}{=}\frac{s^2+r^2-8Rr}{2r^2}\overset{Lemma2}{\geq}2\sum_{cyc}\frac{m_a}{h_a}=Rhs\Leftrightarrow$$

$\frac{s^2+r^2-8Rr}{2r^2}\geq\frac{3R}{r}\Leftrightarrow s^2\geq 14Rr-r^2$ which result from $s^2\geq 16Rr-5r^2(Gerretsen)$

Remains to prove that: $16Rr-5r^2\geq 14Rr-r^2\Leftrightarrow R\geq 2r(Euler)$.

Equality holds if and only if the triangle is equilateral.

Reference:

Romanian Mathematical Magazine-www.ssmrmh.ro