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ABOUT AN INEQUALITY BY BOGDAN FUȘTEI-I

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{r_b + r_c} \le 3$$

Proposed by Bogdan Fuștei – Romania

Solution *We prove the following lemma:*

Lemma.

2) In ΔABC the following relationship holds:

$$\sum \frac{h_b + h_c}{r_b + r_c} = 2 + \frac{2r}{R}$$

Proof. *Using the formulas* $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{h_b + h_c}{r_b + r_c} = \sum \frac{\frac{2S}{b} + \frac{2S}{c}}{\frac{S}{s-b} + \frac{S}{s-c}} = \frac{2}{abc} \sum (b+c)(s-b)(s-c) = \frac{2}{4Rrs} \cdot 4rs(R+r) =$$

$$= \frac{2(R+r)}{R} = 2 + \frac{2r}{R}$$

Let's get back to the main problem.

Using the Lemma it follows:

$M_s = \sum \frac{h_b+h_c}{r_b+r_c} = 2 + \frac{2r}{R} \le 3 = M_d$, *which follows from Euler's inequality* $R \ge 2r$.

Remark. *Let's find an inequality having an opposite sense:*

3) In ΔABC the following inequality holds:

$$\sum \frac{h_b + h_c}{r_b + r_c} \ge \frac{6r}{R}$$

Proposed by Marin Chirciu – Romania

Solution *Using Lemma it follows:*

$M_s = \sum \frac{h_b+h_c}{r_b+r_c} = 2 + \frac{2r}{R} \ge \frac{6r}{R} = M_d$, *which follows from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

4) In ΔABC the following inequality holds:

$$\frac{6r}{R} \le \sum \frac{h_b + h_c}{r_b + r_c} \le 3$$

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Solution *See inequalities 1) and 3). Equality holds if and only if the triangle is equilateral.*

5) In ΔABC the following relationship holds:

$$3 \le \sum \frac{r_b + r_c}{h_b + h_c} \le \frac{3R}{2r}$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

6) In ΔABC the following relationship holds:

$$\sum \frac{r_b + r_c}{h_b + h_c} = \frac{R}{r} \cdot \frac{s^2 + 5r^2 + 8Rr}{s^2 + r^2 + 2Rr}$$

Proof. *Using the following lemmas* $h_a = \frac{2S}{a}$ *and* $r_a = \frac{S}{s-a}$ *we obtain:*

$$\sum \frac{r_b + r_c}{h_b + h_c} = \sum \frac{\frac{S}{s-b} + \frac{S}{s-c}}{\frac{2S}{b} + \frac{2S}{c}} = \frac{abc}{2} \sum \frac{1}{(b+c)(s-b)(s-c)} =$$

$$= \frac{4Rrs}{2} \cdot \frac{s^2+5r^2+8Rr}{2r^2 s(s^2+r^2+2Rr)} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr},$$ *which follows from the identities in triangle:*

$$\sum \frac{1}{(b+c)(s-b)(s-c)} = \frac{s^2+5r^2+8Rr}{r^2 s(s^2+r^2+2Rr)}$$ *and*

$$\sum (a+b)(a+c)(s-a) = s(s^2 + 5r^2 + 8Rr)$$

Let's get back to the main problem.RHS inequality: Using the Lemma it follows:

$$M_s = \sum \frac{r_b+r_c}{h_b+h_c} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \overset{(1)}{\le} \frac{R}{r} \cdot \frac{3}{2} = \frac{3R}{2r} = M_a \text{ where } (1) \Leftrightarrow \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \le \frac{3}{2} \Leftrightarrow$$

$\Leftrightarrow s^2 \ge 10Rr + 7r^2$, *which follows Gerretsen's inequality* $s^2 \ge 16Rr - 5r^2$.

It remains to prove that $16Rr - 5r^2 \geq 10Rr + 7r^2 \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

Equality holds if and only if the triangle is equilateral.

LHS inequality: Using the Lemma it follows:

$$M_s = \sum \frac{r_b+r_c}{h_b+h_c} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \overset{(2)}{\geq} 3 = M_d, \text{ where (1)} \Leftrightarrow \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr} \geq 3 \Leftrightarrow$$

$$\Leftrightarrow s^2(R-3r) + r(8R^2 - Rr - 3r^2) \geq 0$$

We distinguish the following cases:

Case 1) If $(R-3r) \geq 0$ *the inequality is obvious.*

Case 2) If $(R-3r) < 0$ *the inequality rewrites itself:* $r(8R^2 - Rr - 3r^2) \geq s^2(3r - R)$,

which follows from Gerretsen's inequality $s^2 \leq 4R^2 + 4Rr + 3r^2$.

It remains to prove that:

$$r(8R^2 - Rr - 3r^2) \geq (4R^2 + 4Rr + 3r^2)(3r - R) \Leftrightarrow 2R^3 - 9Rr^2 - 6r^3 \geq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(2R^2 + 4Rr + 3r^2) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

Remark. *Between the sums* $\sum \frac{r_b+r_c}{h_b+h_c}$ *and* $\sum \frac{h_b+h_c}{r_b+r_c}$ *the following relationship holds:*

7) In ΔABC the following relationship holds:

$$\sum \frac{r_b + r_c}{h_b + h_c} \geq \sum \frac{h_b + h_c}{r_b + r_c}$$

Proposed by Marin Chirciu – Romania

Solution *Using the Lemmas* $\sum \frac{r_b+r_c}{h_b+h_c} = \frac{R}{r} \cdot \frac{s^2+5r^2+8Rr}{s^2+r^2+2Rr}$ *and* $\sum \frac{h_b+h_c}{r_b+r_c} = \frac{2(R+r)}{r}$ *the inequality can be written:*

$$\frac{R(s^2+5r^2+8Rr)}{r(s^2+r^2+2Rr)} \geq \frac{2(R+r)}{R} \Leftrightarrow s^2(R^2 - 2Rr - 2r^2) + r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq 0.$$

We distinguish the following cases:

Case 1) If $(R^2 - 2Rr - 2r^2) \geq 0$ *the inequality is obvious.*

Case 2) If $(R^2 - 2Rr - 2r^2) < 0$ *the inequality rewrites itself:*

$r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq s^2(2r^2 + 2Rr - R^2)$, *which follows from Gerretsen's inequality* $s^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$r(8R^3 + R^2r - 6Rr^2 - 2r^3) \geq (4R^2 + 4Rr + 3r^2)(2r^2 + 2Rr - R^2) \Leftrightarrow$$

$\Leftrightarrow R^4 + R^3r - 3R^2r^2 - 5Rr^3 - 2r^4 \geq 0 \Leftrightarrow (R-2r)(R+r)^3 \geq 0$ *obviously from Euler's inequality* $R \geq 2r$*. Equality holds if and only if the triangle is equilateral.*

Remark. *We can write the sequence of inequalities:*

8) In ΔABC ***the following relationship holds:***

$$\frac{3R}{2r} \geq \sum \frac{r_b + r_c}{h_b + h_c} \geq \sum \frac{h_b + h_c}{r_b + r_c} \geq \frac{6r}{R}$$

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Solution *See inequalities 4), 5) and 7). Equality holds if and only if the triangle is equilateral.*

Reference:

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