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ROMANIAN MATHEMATICAL MAGAZINE

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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-VI

By Marin Chirciu-Romania

1) In any ΔABC the following relationship holds:

$$\frac{\sum h_a^{-1}}{\sum m_a^{-1}} \le \left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution

$$\sum h_a^{-1} = \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}; \sum m_a^{-1} = \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \ge \frac{9}{\sum m_a} \ge \frac{9}{4R+r} \Rightarrow$$

$$Lhs = \frac{\sum h_a^{-1}}{\sum m_a^{-1}} \le \frac{\frac{1}{r}}{\frac{9}{4R+r}} = \frac{4R+r}{9r} \overset{(1)}{\le} \left(\frac{R}{2r}\right)^2 = Rhs.$$

$$(1) \Leftrightarrow \frac{4R+r}{9r} \le \frac{R^2}{4r^2} \Leftrightarrow 9R^2 - 16Rr - 4r^2 \ge 0 \Leftrightarrow (R-2r)(9R+2r) \ge 0$$

true from $R \ge 2r(Euler)$. Equality holds if and only if triangle is equilateral.

2) In any ΔABC the following relationship holds:

$$\frac{\sum r_a^{-1}}{\sum m_a^{-1}} \le \left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution

$$\sum r_a^{-1} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}; \sum m_a^{-1} = \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \ge \frac{9}{\sum m_a} \ge \frac{9}{4R+r} \Rightarrow$$

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Proposed by Marin Chirciu-Romania

Solution

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true from $R \geq 2r (Euler)$ .Equality holds if and only if triangle is equilateral.

4) In any ΔABC the following relationship holds:

$$\frac{\sum h_a^{-1}}{\sum w_a^{-1}} \leq \left(\frac{R}{2r}\right)^2$$

Proposed by Marin Chirciu-Romania

Solution

$$\sum h_a^{-1} = \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r};\ \sum w_a^{-1} \geq \sum m_a^{-1} = \frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{9}{\sum m_a} \geq \frac{9}{4R+r} \Rightarrow$$

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Reference:

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