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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-V

By Marin Chirciu – Romania

1) In ΔABC the following inequality holds:

$$(r_a+r_b+r_c)\left(\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}\right)\ge\frac{9(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Adil Abdullayev –Azerbaijan

Solution: *Using the known identities in triangle:*

$\sum r_a=4R+r,\sum\frac{1}{r_a}=\frac{1}{r},\prod(b+c)=2s(s^2+r^2+2Rr)$ *and* $abc=4Rrs$, *the inequality rewrites itself:*

$$(4R+r)\cdot\frac{1}{r}\ge\frac{9\cdot 2s(s^2+r^2+2Rr)}{8\cdot 4Rrs}\Leftrightarrow 16R(4R+r)\ge 9(s^2+r^2+2Rr)\Leftrightarrow$$

$\Leftrightarrow 9s^2\le 64R^2-2Rr-9r^2$, *which follows from Gerretsen's inequality* $s^2\le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$9(4R^2+4Rr+3r^2)\le 64R^2-2Rr-9r^2\Leftrightarrow 14R^2-19Rr-18r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow (R-2r)(14R+9r)\ge 2$, *obviously from Euler's inequality* $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

2) In ΔABC the following relationship holds:

$$(h_a+h_b+h_c)\left(\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c}\right)\le\frac{9(a+b)(b+c)(c+a)}{8abc}$$

Proposed by Marin Chirciu – Romania

Solution: *Using the known identities in triangle:*

$\sum h_a=\frac{s^2+r(4R+r)}{2R},\sum\frac{1}{h_a}=\frac{1}{r},\prod(b+c)=2s(s^2+r^2+2Rr)$ *and* $abc=4Rrs$, *the inequality can be written:*

$\frac{s^2+r(4R+r)}{2R}\cdot\frac{1}{r}\le\frac{9\cdot 2s(s^2+r^2+2Rr)}{8\cdot 4Rrs}\Leftrightarrow s^2\ge 14Rr-r^2$, *which follows from Gerretsen's inequality* $s^2\ge 16Rr-5r^2$. *It remains to prove that:*

$16Rr-5r^2\ge 14Rr-r^2\Leftrightarrow R\ge 2r$, *obviously from Euler's inequality.*

Equality holds if and only if the triangle is equilateral.

Remark. *We can write the double inequality:*

3) In ΔABC the following inequality holds:

$$(h_a + h_b + h_c)\left(\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c}\right) \le \frac{9(a+b)(b+c)(c+a)}{8abc}$$

$$\le (r_a + r_b + r_c)\left(\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c}\right)$$

Solution

See inequalities 1) and 2).

Equality holds if and only if the triangle is equilateral.

Reference:

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