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# ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-IV

***By Marin Chirciu – Romania***

**1) In $\Delta ABC$ the following relationship holds:**

$$\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}\geq 3\frac{h_ah_bh_c}{m_am_bm_c}$$

***Proposed by Adil Abdullayev– Azerbaijan***

***Solution*** *We prove the following lemmas:*

***Lemma 1.***

$$2)\ \sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}\geq\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{4R^2s^2}$$

***Proof.***

*Using GM-HM we obtain:*

$$\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}=\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}\cdot 1}\geq\sum\frac{2}{\frac{r_b^2+r_c^2}{2r_br_c}+1}=\sum\frac{4r_br_c}{(r_b+r_c)^2}=$$

$$=4\cdot\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}=\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{4R^2s^2}$$, *which follows from:*

$$\sum\frac{r_br_c}{(r_b+r_c)^2}=\sum\frac{\frac{S}{s-b}\cdot\frac{S}{s-c}}{\left(\frac{S}{s-b}+\frac{S}{s-c}\right)^2}=\sum\frac{(s-b)(s-c)}{a^2}=\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{16R^2s^2}.$$

***Lemma 2.***

***3) In $\Delta ABC$ the following inequality holds:***

$$\frac{h_ah_bh_c}{m_am_bm_c}\leq\frac{2r}{R}$$

***Proof.*** *From* $m_a\geq\sqrt{s(s-a)}$ *it follows* $m_am_bm_c\geq sS=rs^2$ *(1)*

*From* $h_a=\frac{2S}{a}$ *it follows* $h_ah_bh_c=\frac{2r^2s^2}{R}$ *(2)*

*From (1) and (2) we obtain* $\frac{h_ah_bh_c}{m_am_bm_c}\leq\frac{\frac{2r^2s^2}{R}}{rs^2}=\frac{2r}{R}$

*Let's get back to the main problem.*

*Using the above Lemmas, we obtain:*

$$M_s=\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}\overset{Lemma\ 1}{\geq}\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{4R^2s^2}\overset{(1)}{\geq}\frac{6r}{R}\overset{Lemma\ 2}{\geq}3\frac{h_ah_bh_c}{m_am_bm_c}=M_d,$$

*where (1)* $\Leftrightarrow\frac{s^4+s^2(2r^2-4Rr)+r(4R+r)^3}{4R^2s^2}\geq\frac{6r}{R}\Leftrightarrow$

$$\Leftrightarrow s^4+s^2(2r^2-28Rr)+r(4R+r)^3\geq 0\Leftrightarrow s^2(s^2+2r^2-28Rr)+r(4R+r)^3\geq 0$$

*We distinguish the following cases:*

*Case 1). If* $(s^2+2r^2-28Rr)\geq 0$ *the inequality is obvioius.*

*Case 2). If* $(s^2+2r^2-28Rr)<0$ *the inequality can be written:*

$r(4R+r)^3\geq s^2(28Rr-2r^2-s^2)$, *which follows from Blundon-Gerretsen's inequality*

$$16Rr-5r^2\leq s^2\leq\frac{R(4R+r)^2}{2(2R-r)}$$

*It remains to prove that:*

$$r(4R+r)^3\geq\frac{R(4R+r)^2}{2(2R-r)}(28Rr-2r^2-16Rr+5r^2)\Leftrightarrow 4R^2-7Rr-2r^2\geq 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(R+r)\geq 0$, *obviously from Euler's inequality* $R\geq 2r$.

*Equality holds if and only if the triangle is equilateral.*

***Remark.***

*If we replace* $\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}$ *with* $\sum\sqrt{\frac{2h_bh_c}{h_b^2+h_c^2}}$ *we can propose:*

***4) In $\Delta ABC$ the following inequality holds:***

$$\sum\sqrt{\frac{2h_bh_c}{h_b^2+h_c^2}}\geq 3\frac{h_ah_bh_c}{m_am_bm_c}$$

***Proposed by Marin Chirciu – Romania***

***Solution*** *We prove the following lemmas:*

***Lemma 1.***

$$5)\ \sum\sqrt{\frac{2h_bh_c}{h_b^2+h_c^2}}\geq\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{s^2(s^2+r^2+2Rr)^2}$$

***Proof.***

*Using the means inequality GM-HM we obtain:*

$$\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}=\sum\sqrt{\frac{2h_bh_c}{h_b^2+h_c^2}\cdot 1}\ge\sum\frac{2}{\frac{h_b^2+h_c^2}{h_bh_c}+1}=\sum\frac{4h_bh_c}{(h_b+h_c)^2}=$$

$$=4\cdot\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{4s^2(s^2+r^2+2Rr)^2}=$$

$=\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{s^2(s^2+r^2+2Rr)^2}$, *which follows from:*

$$\sum\frac{h_bh_c}{(h_b+h_c)^2}=\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{4s^2(s^2+r^2+2Rr)^2}$$

$$\sum\frac{bc}{(b+c)^2}=\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{4s^2(s^2+r^2+2Rr)^2}$$

*true from:*

$$\sum\frac{h_bh_c}{(h_b+h_c)^2}=\sum\frac{\frac{2S}{b}\cdot\frac{2S}{c}}{\left(\frac{2S}{b}+\frac{2S}{c}\right)^2}=\sum\frac{bc}{(b+c)^2}=\frac{\sum bc(a+b)^2(a+c)^2}{\prod(b+c)^2}=$$

$\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{4s^2(s^2+r^2+2Rr)^2}$, *which follows from*

$$\sum bc(a+b)^2(a+c)^2=s^6+s^4(36Rr+3r^2)+$$

$+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^2$ *and* $\prod(b+c)^2=4s^2(s^2+r^2+2Rr)^2$.

***Lemma 2.***

***6) In $\Delta ABC$ the following inequality holds:***

$$\frac{h_ah_bh_c}{m_am_bm_c}\le\frac{2r}{R}$$

***Proof.*** *From* $m_a\ge\sqrt{s(s-a)}$ *it follows* $m_am_bm_c\ge sS=rs^2$ *(1)*

*From* $h_a=\frac{2S}{a}$ *it follows* $h_ah_bh_c=\frac{2r^2s^2}{R}$ *(2)*

*From (1) and (2) we obtain* $\frac{h_ah_bh_c}{m_am_bm_c}\le\frac{\frac{2r^2s^2}{R}}{rs^2}=\frac{2r}{R}$

*Let's get back to the main problem. Using the above Lemmas we obtain:*

$$M_s=\sum\sqrt{\frac{2r_br_c}{r_b^2+r_c^2}}\overset{Lemma\ 1}{\ge}\frac{s^6+s^4(36Rr+3r^2)+s^2r^2(64R^2+16Rr+3r^2)+r^3(4R+r)^3}{s^2(s^2+r^2+2Rr)^2}\overset{(1)}{\ge}$$

$$\overset{(1)}{\geq} \frac{6r}{R} \overset{Lemma\ 2}{\geq} 3\frac{h_a h_b h_c}{m_a m_b m_c} = M_d, \text{ where (1)} \Leftrightarrow$$

$$\Leftrightarrow \frac{s^6 + s^4(36Rr + 3r^2) + s^2r^2(64R^2 + 16Rr + 3r^2) + r^3(4R + r)^3}{s^2(s^2 + r^2 + 2Rr)^2} \geq \frac{6r}{R} \Leftrightarrow$$

$$\Leftrightarrow s^6(R - 6r) + s^4(36R^2r - 21Rr^2 - 12r^3) + s^2r^2(64R^3 - 8R^2r - 21Rr^2 - 6r^3) +$$
$$+Rr^3(4R + r)^3 \geq 0$$

*We distinguish the following cases:*

*Case 1). If* $(R - 6r) \geq 0$ *the inequality is obvious.*

*Case 2). If* $(R - 6r) < 0$ *the inequality rewrites itself:*

$$\Leftrightarrow s^4(36R^2r - 21Rr^2 - 12r^3) + s^2r^2(64R^3 - 8R^2r - 21Rr^2 - 6r^3) +$$
$$+Rr^3(4R + r)^3 \geq s^6(6r - R), \text{ which follows from Gerretsen's inequality}$$

$$16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$$

*It remains to prove that:*

$$\Leftrightarrow (16Rr - 5r^2)^2(36R^2r - 21Rr^2 - 12r^3) +$$
$$+(16Rr - 5r^2)r^2(64R^3 - 8R^2r - 21Rr^2 - 6r^3) + Rr^3(4R + r)^3 \geq$$
$$\geq (4R^2 + 4Rr + 3r^2)^3(6r - R) \Leftrightarrow$$

$$\Leftrightarrow 10304R^4r^3 - 11536R^3r^4 + 904R^2r^5 + 1405Rr^6 - 270r^7 \geq$$
$$\geq -64R^7 + 192R^6r + 816R^5r^2 + 1664R^4r^3 + 1860R^3r^4 + 1404R^2r^5 +$$
$$+621Rr^6 + 162r^7 \Leftrightarrow$$

$$\Leftrightarrow 64R^7 - 192R^6r - 816R^5r^2 + 8640R^4r^3 - 13396R^3r^4 - 500R^2r^5 -$$
$$-784Rr^6 - 432r^7 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 16R^7 - 48R^6r - 204R^5R^2 + 2160R^4r^3 - 3349R^3r^4 - 125R^2r^5 + 196Rr^6 -$$
$$-108r^7 \geq 0$$

$$(R - 2r)(16R^6 - 12R^5r - 236R^4r^2 + 1688r^3R^3 + 27R^2r^4 - 71Rr^5 + 54r^6) \geq 0,$$

*obviously from Euler's inequality* $R \geq 2r$.

*Equality holds if and only if the triangle the equilateral.*

***Reference:***

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