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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-III

By Marin Chirciu – Romania

1) In ΔABC the following inequality holds:

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\le 3\cdot\frac{m_am_bm_c}{h_ah_bh_c}$$

Proposed by Adil Abdullayev –Azerbaijan

Solution: *We prove the following lemmas:*

Lemma 1.

2) In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\le\frac{2R+5r}{3r}$$

Proof. *Using the means inequality we obtain:*

$$\sum\sqrt[3]{\frac{r_a}{h_a}}=\sum\sqrt[3]{\frac{r_a}{h_a}\cdot 1\cdot 1}\le\sum\frac{\frac{r_a}{h_a}+1+1}{3}=\frac{1}{3}\left(\sum\frac{r_a}{h_a}+6\right)=\frac{1}{3}\left(\frac{2R-r}{r}+6\right)=$$

$=\frac{2R+5r}{3r}$ *which follows from* $\sum\frac{r_a}{h_a}=\frac{2R-r}{r}$. *Equality holds if and only if the triangle is equilateral.*

Lemma 2.

3) In ΔABC the following inequality holds:

$$\frac{m_am_bm_c}{h_ah_bh_c}\ge\frac{R}{2r}$$

Proof. *From* $m_a\ge\sqrt{s(s-a)}$ *it follows that* $m_am_bm_c\ge sS=rs^2$ *(1)*

From $h_a=\frac{2S}{a}$ *it follows* $h_ah_bh_c=\frac{2r^2s^2}{R}$ *(2)*

From (1) and (2) we obtain $\frac{m_am_bm_c}{h_ah_bh_c}\ge\frac{rs^2}{\frac{2r^2s^2}{R}}=\frac{R}{2r}$.

Equality holds if and only if the triangle is equilateral.

Let's get back to the main problem.

Using the above Lemmas we obtain:

$$\sqrt[3]{\frac{r_a}{h_a}}+\sqrt[3]{\frac{r_b}{h_b}}+\sqrt[3]{\frac{r_c}{h_c}}\overset{Lemma\ 1}{\leq}\frac{2R+5r}{3r}\overset{Lemma\ 2}{\leq}3\cdot\frac{m_am_bm_c}{h_ah_bh_c},\ where\ (3)\Leftrightarrow$$

$$\Leftrightarrow\frac{2R+5r}{3r}\leq\frac{3R}{2r}\Leftrightarrow R\geq 2r,$$

(Euler's inequality)

Equality holds if and only if the triangle is equilateral.

4) In ΔABC the following inequality holds:

$$\sqrt[3]{\frac{h_a}{r_a}}+\sqrt[3]{\frac{h_b}{r_b}}+\sqrt[3]{\frac{h_c}{r_c}}\leq 3\cdot\frac{m_am_bm_c}{h_ah_bh_c}$$

Proposed by Marin Chirciu – Romania

Solution: *We prove the following lemmas:*

Lemma 1.

5) In ΔABC the following relationship holds:

$$\sqrt[3]{\frac{h_a}{r_a}}+\sqrt[3]{\frac{h_b}{r_b}}+\sqrt[3]{\frac{h_c}{r_c}}\leq\frac{2(R+r)^2}{3Rr}$$

Proof.

Using the means inequality we obtain:

$$\sum\sqrt[3]{\frac{h_a}{r_a}}=\sum\sqrt[3]{\frac{h_a}{r_a}\cdot 1\cdot 1}\leq\sum\frac{\frac{h_a}{r_a}+1+1}{3}=\frac{1}{3}\left(\sum\frac{h_a}{r_a}+6\right)=\frac{1}{3}\left(\frac{s^2+r^2-8Rr}{2Rr}+6\right)=$$

$$=\frac{s^2+r^2+4Rr}{6Rr}\overset{Gerretsen}{\leq}\frac{(4R^2+4Rr+3r^2)+r^2+4Rr}{6Rr}=\frac{4(R+r)^2}{6Rr}=\frac{2(R+r)^2}{3Rr}=$$

which follows from Gerretsen's inequality $s^2\leq 4R^2+4Rr+3r^2$.

Equality holds if and only if the triangle is equilateral.

Lemma 2.

6) In ΔABC the following inequality holds:

$$\frac{m_am_bm_c}{h_ah_bh_c}\geq\frac{R}{2r}$$

Proof.

$$\text{From } m_a \geq \sqrt{s(s-a)} \text{ it follows } m_a m_b m_c \geq sS = rs^2 \quad (1)$$

$$\text{From } h_a = \frac{2S}{a} \text{ it follows } h_a h_b h_c = \frac{2r^2 s^2}{R} \quad (2)$$

$$\text{From (1) and (2) we obtain } \frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{rs^2}{\frac{2r^2 s^2}{R}} = \frac{R}{2r}$$

Equality holds if and only if the triangle is equilateral.
Let's get back to the main problem:
Using the above Lemmas, we obtain:

$$\sqrt[3]{\frac{r_a}{h_a}} + \sqrt[3]{\frac{r_b}{h_b}} + \sqrt[3]{\frac{r_c}{h_c}} \overset{Lemma\ 1}{\leq} \frac{2(R+r)^2}{3Rr} \overset{(3)}{\leq} 3 \cdot \frac{m_a m_b m_c}{h_a h_b h_c}, \text{ where} \Leftrightarrow \frac{2(R+r)^2}{3Rr} \leq \frac{3R}{2r} \Leftrightarrow$$

$$\Leftrightarrow 5R^2 - 8Rr - 4r^2 \geq 0 \Leftrightarrow (R-2r)(5R+2r) \geq 0, \text{ obviously from Euler's inequality,}$$

$$R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

Reference:

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