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ABOUT AN INEQUALITY BY ADIL ABDULLAYEV-II

By Marin Chirciu – Romania

1) In ΔABC the following relationship holds:

$$3+\sum\cos(B-C)\ge 6\prod\frac{h_a}{m_a}$$

Proposed by Adil Abdullayev – Azerbaijan

Solution

We prove the following lemma:

Lemma.

2) In ΔABC, holds:

$$\prod\frac{h_a}{m_a}\le\frac{32r^2}{s^2+r^2+2Rr}$$

Proof.

Using $m_a\ge\frac{b+c}{2}\cos\frac{A}{2}$ *we obtain:*

$$\prod m_a\ge\prod\frac{b+c}{2}\cos\frac{A}{2}=\frac{\prod(b+c)}{8}\prod\cos\frac{A}{2}=\frac{2s(s^2+r^2+2Rr)}{8}\cdot\frac{s}{4R}=\frac{s^2(s^2+r^2+2Rr)}{16R}\ (1)$$

Using the known inequality in triangle $\prod h_a=\frac{2s^2r^2}{R}$ *(2)*

From (1) and (2) we obtain $\prod\frac{h_a}{m_a}\le\frac{\frac{2s^2r^2}{R}}{\frac{s^2(s^2+r^2+2Rr)}{16R}}=\frac{32r^2}{s^2+r^2+2Rr}$

Equality holds if and only if the triangle is equilateral.

Let's get back to the main problem.

Using Lemma and the identity $\sum\cos(B-C)=\frac{s^2+r^2+2Rr}{2R^2}-1$ *it suffices to prove that:*

$$3+\frac{s^2+r^2+2Rr}{2R^2}-1\ge 6\cdot\frac{32r^2}{s^2+r^2+2Rr}\Leftrightarrow(s^2+r^2+2Rr)(s^2+r^2+2Rr+4R^2)\ge 384R^2r^2,$$

which follows from Gerretsen's inequaity $s^2\ge 16Rr-5r^2$*. It remains to prove that:*

$$(16Rr-5r^2+r^2+2Rr)(16Rr-5r^2+r^2+2Rr+4R^2)\ge 384R^2r^2\Leftrightarrow$$

$$\Leftrightarrow 18R^3-19R^2r-36Rr^2+4r^3\ge 0\Leftrightarrow(R-2r)(18R^2+17Rr-2r^2)\ge 0$$

obviously from Euler's inequality $R \geq 2r$. Equality holds if and only if the triangle is equilateral.

Remark. *If we replace h_a with r_a we propse that:*

3) In ΔABC the following inequality holds:

$$3+\sum\cos(B-C)\geq\frac{12r}{R}\prod\frac{r_a}{m_a}$$

Proposed by Marin Chirciu – Romania

Solution *We prove the following lemma:*

Lemma.

4) In ΔABC the following relationship holds:

$$\prod\frac{r_a}{m_a}\leq\frac{16Rr}{s^2+r^2+2Rr}$$

Proof. *Using $m_a\geq\frac{b+c}{2}\cos\frac{A}{2}$ we obtain:*

$$\prod m_a\geq\prod\frac{b+c}{2}\cos\frac{A}{2}=\frac{\prod(b+c)}{8}\prod\cos\frac{A}{2}=\frac{2s(s^2+r^2+2Rr)}{8}\cdot\frac{s}{4R}=\frac{s^2(s^2+r^2+2Rr)}{16R}\ (1)$$

Using the known identity in triangle $\prod r_a=rs^2$ (2)

From (1) and (2) we obtain $\prod\frac{r_a}{m_a}\leq\frac{rs^2}{\frac{s^2(s^2+r^2+2Rr)}{16R}}=\frac{16Rr}{s^2+r^2+2Rr}$

Equality holds if and only if the triangle is equilateral.

Let's get back to the main problem.

Using Lemma and the identity $\sum\cos(B-C)=\frac{s^2+r^2+2Rr}{2R^2}-1$ it suffices to prove that:

$$3+\frac{s^2+r^2+2Rr}{2R^2}-1\geq\frac{12r}{R}\cdot\frac{16Rr}{s^2+r^2+2Rr}\Leftrightarrow(s^2+r^2+2Rr)(s^2+r^2+2Rr+4R^2)\geq384R^2r^2$$

which follows from $(16Rr-5r^2+r^2+2Rr)(16Rr-5r^2+r^2+2Rr+4R^2)\geq384R^2r^2$

$$\Leftrightarrow 18R^3-19R^2r-36Rr^2+4r^3\geq0\Leftrightarrow(R-2r)(18R^2+17Rr-2r^2)\geq0$$

Obviously from Euler's inequality $R\geq2r$. Equality holds if and only if the triangle is equilateral.

Reference:

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