The Semi-Infinite Integral of the $\log(x)\,\text{sech}^2(x)$ function

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5 Dhul-Qada 1441AH (26 June 2020)

Abstract

The closed form of the fast converging semi-infinite integral of $\log(x)\,\text{sech}^2(x)$ is what this paper is centered upon. Generalizations of the integrals used to obtain its closed form from other defined integrals and results obtained from the manipulations of the integral are also included.

In this paper, γ represents the Euler-Mascheroni constant, ψ_0 denotes the Digamma function, Γ denotes the Gamma function and H_n denotes the harmonic number. It is also important to note that the type of logarithm function used here is the complex logarithm.

I defined an integral from one of the published problems of Sir Cornel Ioan Valean with the Greek letter 'kappa' as $\kappa(y) = \int_0^\infty \tanh\left(\frac{\pi x}{2}\right)\left(\frac{1}{x} - \frac{x}{x^2+y^2}\right) \mathrm{d}x$.

Evaluating $\kappa(y)$, I start by writing the integrand as a semi-infinite integral by the Laplace transform [2], thus $\kappa(y)$ becoming a double integral.

$$\kappa(y) = \int_0^\infty \tanh\left(\frac{\pi x}{2}\right)\left(\frac{1}{x} - \frac{x}{x^2+y^2}\right) \mathrm{d}x$$

$$\therefore \kappa(y) = \int_0^\infty \int_0^\infty e^{-xs}(1-\cos(ys))\tanh\left(\frac{\pi x}{2}\right) \mathrm{d}s\,\mathrm{d}x$$

$$\kappa(y) \overset{Fubini\ [3]}{=} \int_0^\infty (1-\cos(ys)) \underbrace{\int_0^\infty e^{-xs}\tanh\left(\frac{\pi x}{2}\right)\mathrm{d}x}_{v}\,\mathrm{d}s$$

$$v = \int_0^\infty e^{-xs}\left(\frac{1-e^{-\pi x}}{1+e^{-\pi x}}\right)\mathrm{d}x = \sum_{k=0}^\infty (-1)^k \int_0^\infty e^{-(s+k\pi)x}(1-e^{-\pi x})\,\mathrm{d}x$$

$$= \sum_{k=0}^\infty (-1)^k \left(\frac{1}{s+k\pi} - \frac{1}{s+(k+1)\pi}\right) = \frac{1}{s} - 2\sum_{k=0}^\infty \frac{(-1)^k}{s+(k+1)\pi}$$

$$=\frac{2}{\pi}\left(\psi_0\left(1+\frac{s}{2\pi}\right)-\psi_0\left(\frac{1}{2}+\frac{s}{2\pi}\right)\right)-\frac{1}{s}$$

$$\Rightarrow \kappa(y)=\int_0^\infty(1-\cos(ys))\left(\frac{1}{s}-2\sum_{k=0}^{\infty}\frac{(-1)^k}{s+(k+1)\pi}\right)\mathrm{d}s$$

{Again, by Laplace transform}

$$\kappa(y)=\int_0^\infty(1-\cos(ys))\left(\int_0^\infty e^{-st}\mathrm{d}t-2\sum_{k=0}^{\infty}\int_0^\infty(-1)^k e^{-(s+(k+1)\pi)t}\mathrm{d}t\right)\mathrm{d}s$$

{Interchanging the summation for integration [1]}

$$=\int_0^\infty(1-\cos(ys))\left(\int_0^\infty e^{-st}\mathrm{d}t-2\int_0^\infty e^{-(s+\pi)t}\sum_{k=0}^{\infty}(-1)^k e^{-k\pi t}\mathrm{d}t\right)\mathrm{d}s$$

$$=\int_0^\infty(1-\cos(ys))\left(\int_0^\infty e^{-st}\mathrm{d}t-2\int_0^\infty \frac{e^{-(s+\pi)t}}{1+e^{-\pi t}}\,\mathrm{d}t\right)\mathrm{d}s$$

$$=\int_0^\infty\int_0^\infty(1-\cos(ys))\left(e^{-st}-2\frac{e^{-(s+\pi)t}}{1+e^{-\pi t}}\right)\mathrm{d}t\;\mathrm{d}s$$

$$=\int_0^\infty\int_0^\infty e^{-st}(1-\cos(ys))\left(1+\left(\tanh\left(\frac{\pi t}{2}\right)-1\right)\right)\mathrm{d}s\;\mathrm{d}t$$

$$=\int_0^\infty\left(\frac{1}{t}-\frac{t}{t^2+y^2}\right)\left(1+\left(\tanh\left(\frac{\pi t}{2}\right)-1\right)\right)\mathrm{d}t$$

$$=\int_0^\infty\left(\frac{\tanh\left(\frac{\pi t}{2}\right)}{t}-\frac{t}{t^2+y^2}-\frac{t}{t^2+y^2}\tanh\left(\frac{\pi t}{2}\right)+\frac{t}{t^2+y^2}-\cancel{\frac{1}{t}+\frac{1}{t}}\right)\mathrm{d}t$$

κ can be split to two convergent parts

$$\therefore\ \kappa(y)=\underbrace{\int_0^\infty\left(\frac{\tanh\left(\frac{\pi t}{2}\right)}{t}-\frac{t}{t^2+y^2}\right)\mathrm{d}t}_{\kappa_2}+\underbrace{\int_0^\infty\left(\frac{t}{t^2+y^2}-\frac{t}{t^2+y^2}\tanh\left(\frac{\pi t}{2}\right)\right)\mathrm{d}t}_{\kappa_1}$$

1 Evaluating κ_1

$$\kappa_1 = \int_0^\infty \left(\frac{t}{t^2+y^2} - \frac{t}{t^2+y^2}\tanh\left(\frac{\pi t}{2}\right) \right) \mathrm{d}t$$

$$\kappa_1 = \lim_{N\to\infty} \int_0^{C_1 N} \left(\frac{t}{t^2+y^2} - \frac{t}{t^2+y^2}\tanh\left(\frac{\pi t}{2}\right) \right) \mathrm{d}t = \lim_{N\to\infty} \kappa_1(N) \quad \forall\, C_1 \in \Re$$

By the use of contours, it can be proven that $\displaystyle\int_0^\infty \frac{\sin(ty)}{\sinh(y)}\,\mathrm{d}y = \frac{\pi}{2}\tanh\left(\frac{\pi t}{2}\right)$

Also, by Laplace transform $\displaystyle\int_0^\infty \frac{\sin(ty)}{\sinh(y)}\,\mathrm{d}y = 2\sum_{k=0}^{\infty}\int_0^\infty e^{-(2k+1)y}\sin(ty)\,\mathrm{d}y$

$$= 2t\sum_{k=0}^{\infty}\frac{1}{t^2+(2k+1)^2}$$

$$\therefore \tanh\left(\frac{\pi t}{2}\right) = \frac{4t}{\pi}\sum_{k=0}^{\infty}\frac{1}{t^2+(2k+1)^2}$$

$$\kappa_1 = \lim_{N\to\infty}\int_0^{C_1 N}\left(\frac{t}{t^2+y^2} - \frac{4t}{\pi}\sum_{k=0}^{N}\frac{t^2}{(t^2+(2k+1)^2)(t^2+y^2)} \right)\mathrm{d}t$$

{Evaluating the indefinite integral of the summand of the above, taking $M = 2k+1$}

$$\rho(M) = \int \frac{t^2\,\mathrm{d}t}{(t^2+M^2)(t^2+y^2)} = \frac{1}{2}\int\frac{\mathrm{d}t}{t^2+y^2} + \frac{1}{2}\int\frac{\mathrm{d}t}{t^2+M^2} - \frac{M^2+y^2}{2}\int\frac{\mathrm{d}t}{(t^2+M^2)(t^2+y^2)}$$

$$\rho(M) = \frac{1}{2y}\arctan\left(\frac{t}{y}\right) + \frac{1}{2M}\arctan\left(\frac{t}{M}\right) - \frac{M^2+y^2}{2(M^2-y^2)}\left(\frac{\arctan\left(\frac{t}{y}\right)}{y} - \frac{\arctan\left(\frac{t}{M}\right)}{M} \right)$$

$$\rho(M) = \frac{-y}{M^2-y^2}\arctan\left(\frac{t}{y}\right) + \frac{M}{M^2-y^2}\arctan\left(\frac{t}{M}\right)$$

$$\left\{ \text{It is important to note that } \frac{1}{2y}\left(1 - \frac{M^2+y^2}{M^2-y^2}\right) = \frac{-y}{M^2-y^2} \,\&\, \frac{1}{2M}\left(1 + \frac{M^2+y^2}{M^2-y^2}\right) = \frac{M}{M^2-y^2} \right\}$$

$$\kappa_1(N) = \frac{\log(t^2+y^2)}{2}\Bigg|_0^{C_1N} - \frac{4}{\pi}\sum_{k=0}^{N}\rho(2k+1)|_0^{C_1N}$$
$$= \frac{4}{\pi}\sum_{k=0}^{N}\left(\frac{y}{(2k+1)^2-y^2}\arctan\left(\frac{t}{y}\right) - \frac{2k+1}{(2k+1)^2-y^2}\arctan\left(\frac{t}{2k+1}\right)\right)\Bigg|_0^{C_1N} +$$
$$\frac{1}{2}\log\left(\left(\frac{C_1N}{y}\right)^2+1\right)$$
$$= \frac{4}{\pi}\sum_{k=0}^{N}\left(\frac{y}{(2k+1)^2-y^2}\arctan\left(\frac{C_1N}{y}\right) - \frac{2k+1}{(2k+1)^2-y^2}\arctan\left(\frac{C_1N}{2k+1}\right)\right) +$$
$$\frac{1}{2}\log\left(\left(\frac{C_1N}{y}\right)^2+1\right)$$

$$\kappa_1(y) = \lim_{N\to\infty}\left(\frac{4}{\pi}\sum_{k=0}^{N}\left(\frac{y}{(2k+1)^2-y^2}\underbrace{\arctan\left(\frac{C_1N}{y}\right)}_{\to\frac{\pi}{2}} - \frac{2k+1}{(2k+1)^2-y^2}\underbrace{\arctan\left(\frac{C_1N}{2k+1}\right)}_{\to\frac{\pi}{2}}\right) +\right.$$
$$\left.\frac{1}{2}\log\left(\left(\frac{C_1N}{y}\right)^2+1\right)\right)$$
$$\kappa_1(y) = \lim_{N\to\infty}\left(-2\sum_{k=0}^{N}\frac{1}{2k+1+y} + \frac{1}{2}\log\left(\left(\frac{C_1N}{y}\right)^2+1\right)\right)$$
$$= \lim_{N\to\infty}\left(\sum_{k=0}^{N}\left(\frac{1}{k+1} - \frac{1}{k+\frac{1+y}{2}}\right) - \sum_{k=1}^{N+1}\frac{1}{k} + \frac{1}{2}\log\left(\left(\frac{C_1N}{y}\right)^2+1\right)\right)$$
$$= \lim_{N\to\infty}\left(\sum_{k=0}^{N}\left(\frac{1}{k+1} - \frac{1}{k+\frac{1+y}{2}}\right)\underbrace{-\frac{1}{N+1} + \frac{\log\left(1+\left(\frac{y}{C_1N}\right)^2\right)}{2}}_{\to 0} - \sum_{k=1}^{N}\frac{1}{k} + \log\left(\frac{C_1N}{y}\right)\right)$$

$$= \lim_{N\to\infty}\left(\sum_{k=0}^{N}\left(\frac{1}{k+1} - \frac{1}{k+\frac{1+y}{2}}\right) - \sum_{k=1}^{N}\frac{1}{k} + \log(N) + \log\left(\frac{C_1}{y}\right)\right)$$
$$= -\gamma + \sum_{k=0}^{\infty}\left(\frac{1}{k+1} - \frac{1}{k+\frac{1+y}{2}}\right) - \log(y) + \log(C_1) = \psi_0\left(\frac{1+y}{2}\right) - \log(y) + \log(C_1)$$

$$\text{It can be proven that } \lim_{y\to\infty} \kappa_1(y) = 0 \neq \lim_{y\to\infty} \left(\psi_0\left(\frac{1+y}{2}\right) - \log(y)\right) = \mu \qquad (1)$$

Proof

$$\log\left(\frac{-1+y}{2}\right) < \log(y) \ \ \forall\, y > -1$$

$$-\log\left(\frac{-1+y}{2}\right) > -\log(y)$$

$$\mu = \lim_{y\to\infty}\left(-\gamma + H_{\frac{-1+y}{2}} - \log(y)\right) < \lim_{y\to\infty}\left(-\gamma + H_{\frac{-1+y}{2}} - \log\left(\frac{-1+y}{2}\right)\right) = 0$$

$\mu < 0 \Rightarrow \mu$ converges to a value less than 0, $\therefore$ the inequality (1) is true. ▪

The above proof shows that the arbitrary constant (C_1) has a significance which implies that its value is not compulsorily equal to 1.

$$\lim_{y\to\infty} \kappa_1(y) = \lim_{y\to\infty}\left(\psi_0\left(\frac{1+y}{2}\right) - \log(y) + \log(C_1)\right)$$

$$0 = \lim_{y\to\infty}\left(-\gamma + H_{\frac{-1+y}{2}} - \log\left(\frac{-1+y}{2}\right) + \log\left(\frac{-1+y}{2}.\frac{1}{y}\right) + \log(C_1)\right)$$

$$0 = -\gamma + \gamma - \log(2) + \log(C_1), C_1 = 2$$

$$\therefore \kappa_1(y) = \int_0^{\infty}\left(\frac{t}{t^2+y^2} - \frac{t}{t^2+y^2}\tanh\left(\frac{\pi t}{2}\right)\right) \mathrm{d}t = \psi_0\left(\frac{1+y}{2}\right) + \log\left(\frac{2}{y}\right) \qquad (2)$$

2 Evaluating κ_2

$$
\begin{aligned}
\kappa_2(y) &= \int_0^{\infty}\left(\frac{\tanh\left(\frac{\pi t}{2}\right)}{t}-\frac{t}{t^2+y^2}\right)\mathrm{d}t \\
&= \lim_{N\to\infty}\int_0^{C_2N}\left(\frac{4}{\pi}\sum_{k=0}^{N}\frac{1}{t^2+(2k+1)^2}-\frac{t}{t^2+y^2}\right)\mathrm{d}t \quad \forall\, C_2\in\Re \\
&= \lim_{N\to\infty}\left(\frac{4}{\pi}\sum_{k=0}^{N}\arctan\left(\frac{C_2N}{2k+1}\right)\cdot\frac{1}{2k+1}-\frac{\log((C_2N)^2+y^2)}{2}\right)+\log(y) \\
&= \lim_{N\to\infty}\left(\frac{4}{\pi}\sum_{k=0}^{N}\underbrace{\arctan\left(\frac{C_2N}{2k+1}\right)}_{\to\frac{\pi}{2}}\cdot\frac{1}{2k+1}-\log(C_2N)-\underbrace{\frac{\log\left(1+\frac{y^2}{(C_2N)^2}\right)}{2}}_{\to 0}\right)+\log(y) \\
&= \lim_{N\to\infty}\left(2\sum_{k=0}^{N}\frac{1}{2k+1}-\log(C_2N)\right)+\log(y) \\
&= \lim_{N\to\infty}\left(2\sum_{k=1}^{2N+1}\frac{1}{k}-2\sum_{k=1}^{N}\frac{1}{2k}-\log(N)\right)+\log\left(\frac{y}{C_2}\right) \\
&= \lim_{N\to\infty}\left(2H_{2N+1}-H_N-\log(N)\right)+\log\left(\frac{y}{C_2}\right) \\
&= \lim_{N\to\infty}\left(2H_{2N+1}-2\log(2N+1)-H_N+\log(N)+2\log\left(\frac{2N+1}{N}\right)\right)+\log\left(\frac{y}{C_2}\right) \\
&= 2\gamma-\gamma+\log\left(\frac{4y}{C_2}\right)=\gamma+\log\left(\frac{4y}{C_2}\right)
\end{aligned}
$$

Determining the value of C_2 by taking the limits as y tends to 0 on both sides of $\kappa(y)$

$$
\begin{aligned}
\lim_{y\to 0}\kappa(y) &= \lim_{y\to 0}\int_0^{\infty}\tanh\left(\frac{\pi x}{2}\right)\left(\frac{1}{x}-\frac{x}{x^2+y^2}\right)\mathrm{d}x \\
0 &= \lim_{y\to 0}\left(\gamma+\log\left(\frac{4y}{C_2}\right)-\gamma+H_{\frac{-1+y}{2}}+\log\left(\frac{2}{y}\right)\right) \\
&= \log\left(\frac{8}{C_2}\right)+H_{\frac{-1}{2}}
\end{aligned}
$$

$$H_{\frac{-1}{2}} = \psi_0\left(\frac{1}{2}\right) + \gamma = -\log(4) \Rightarrow \log\left(\frac{8}{C_2}\right) = \log(4), C_2 = 2$$

$$\therefore \kappa_2(y) = \int_0^\infty \left(\frac{\tanh\left(\frac{\pi t}{2}\right)}{t} - \frac{t}{t^2+y^2}\right) \mathrm{d}t = \gamma + \log(2y) \tag{3}$$

Adding (2) and (3), we can conclude that

$$\int_0^\infty \tanh\left(\frac{\pi x}{2}\right)\left(\frac{1}{x} - \frac{x}{x^2+y^2}\right) \mathrm{d}x = -\psi_0\left(\frac{1}{2}\right) + \psi_0\left(\frac{1+y}{2}\right) \tag{4}$$

3 A more direct approach

$$\begin{aligned}
\kappa(y) &= \int_0^\infty \tanh\left(\frac{\pi t}{2}\right)\left(\frac{1}{t}-\frac{t}{t^2+y^2}\right)\mathrm{d}t\\
&= \int_0^\infty \frac{4}{\pi}\sum_{k=0}^\infty\left(\frac{1}{t^2+(2k+1)^2}-\frac{1}{t^2+(2k+1)^2}\cdot\frac{t^2}{t^2+y^2}\right)\mathrm{d}t\\
&= \frac{4}{\pi}\sum_{k=0}^\infty\int_0^\infty\left(\frac{1}{t^2+(2k+1)^2}-\frac{1}{t^2+(2k+1)^2}\cdot\frac{t^2}{t^2+y^2}\right)\mathrm{d}t\\
&= \frac{4}{\pi}\sum_{k=0}^\infty\int_0^\infty\left(\frac{1}{t^2+(2k+1)^2}-\frac{1}{(2k+1)^2-y^2}\cdot\frac{y^2}{t^2+y^2}+\frac{(2k+1)^2}{(2k+1)^2-y^2}\cdot\right.\\
&\qquad\left.\frac{1}{t^2+(2k+1)^2}\right)\mathrm{d}t\\
&= \frac{4}{\pi}\sum_{k=0}^\infty\left(\frac{1}{2k+1}\arctan\left(\frac{t}{2k+1}\right)+\frac{y}{(2k+1)^2-y^2}\arctan\left(\frac{t}{y}\right)-\frac{2k+1}{(2k+1)^2-y^2}\cdot\right.\\
&\qquad\left.\left.\arctan\left(\frac{t}{2k+1}\right)\right)\right|_0^\infty\\
&= 2\sum_{k=0}^\infty\left(\frac{1}{2k+1}+\frac{y}{(2k+1)^2-y^2}-\frac{2k+1}{(2k+1)^2-y^2}\right)\\
&= 2\sum_{k=0}^\infty\left(\frac{1}{2k+1}-\frac{1}{2k+1+y}\right)\\
&= 2\sum_{k=0}^\infty\left(-\frac{1}{2(k+1)}+\frac{1}{2k+1}+\frac{1}{2(k+1)}-\frac{1}{2k+1+y}\right)\\
&= 2\sum_{k=0}^\infty\left(-\frac{1}{2(k+1)}+\frac{1}{2\left(k+\frac{1}{2}\right)}+\frac{1}{2(k+1)}-\frac{1}{2\left(k+\frac{1+y}{2}\right)}\right)\\
&= -\left(-\gamma+\sum_{k=0}^\infty\left(\frac{1}{k+1}-\frac{1}{k+\frac{1}{2}}\right)\right)-\gamma+\sum_{k=0}^\infty\left(\frac{1}{k+1}-\frac{1}{k+\frac{1+y}{2}}\right)
\end{aligned}$$

$$= -\psi_0\left(\frac{1}{2}\right)+\psi_0\left(\frac{1+y}{2}\right)$$

4 The $\log(x)\,\mathrm{sech}^2(x)$ integral

The semi-infinite integral of $\log(x)\,\mathrm{sech}^2(x)$ can be derived by integrating κ_2 by parts.

$$\int_0^\infty \left(\frac{\tanh\left(\frac{\pi t}{2}\right)}{t} - \frac{t}{t^2+y^2}\right) \mathrm{d}t = \gamma + \log(2y) = \lim_{N\to\infty} \int_0^N \left(\frac{\tanh(t)}{t} - \frac{t}{\left(t^2+\frac{\pi^2 y^2}{4}\right)}\right) \mathrm{d}t$$

$$\lim_{N\to\infty} \int_0^N \left(\frac{\tanh(t)}{t} - \frac{t}{\left(t^2+\frac{\pi^2 y^2}{4}\right)}\right) \mathrm{d}t = \lim_{N\to\infty} \int_0^N \left(\tanh(t)\ \mathrm{d}(\log(t)) - \mathrm{d}\left(\frac{\log\left(t^2+\frac{\pi^2y^2}{4}\right)}{2}\right)\right)$$

$$= \lim_{N\to\infty} \left[\log(t)\tanh(t) - \frac{1}{2}\log\left(t^2+\frac{\pi^2y^2}{4}\right)\Big|_0^N - \int_0^N \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t\right]$$

$$= \lim_{N\to\infty} \left[\log(N)\tanh(N) - \frac{1}{2}\log\left(N^2+\frac{\pi^2y^2}{4}\right)\right] + \frac{\log\left(\frac{\pi^2y^2}{4}\right)}{2} - \int_0^\infty \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t$$

$$= \underbrace{\lim_{N\to\infty} \left[\log(N)(\tanh(N)-1) - \frac{1}{2}\log\left(1+\frac{\pi^2y^2}{4N^2}\right)\right]}_{0} + \log\left(\frac{\pi y}{2}\right) - \int_0^\infty \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t$$

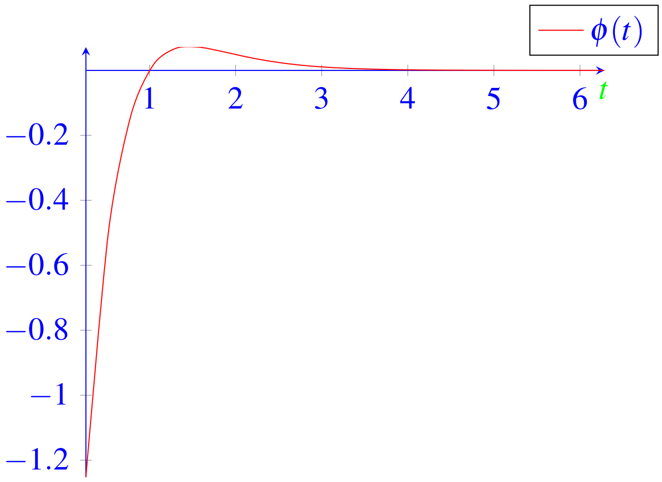
$$= \log\left(\frac{\pi y}{2}\right) - \int_0^\infty \underbrace{\log(t)\,\mathrm{sech}^2(t)}_{\phi(t)}\,\mathrm{d}t$$

$$\gamma + \log(2y) = \log\left(\frac{\pi y}{2}\right) - \int_0^\infty \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t$$

$$-\int_0^\infty \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t = \gamma + \log\left(\frac{4}{\pi}\right)$$

$$\int_0^\infty \phi(t)\ \mathrm{d}t = \int_0^\infty \log(t)\,\mathrm{sech}^2(t)\ \mathrm{d}t = -\gamma + \log\left(\frac{\pi}{4}\right) \tag{5}$$

4.1 Decay of $\phi(t)$ as t increases



5 Generalizations of (2), (3), (4) and (5) for all $\omega, y, \varpi \in \Re - \{0\}$

$$\int_0^\infty \tanh|\omega t| \left(\frac{1}{t} - \frac{t}{t^2+y^2}\right) \mathrm{d}t = -\psi_0\left(\frac{1}{2}\right) + \psi_0\left(\frac{1}{2} + \frac{|\omega y|}{\pi}\right)$$

$$\int_0^\infty \left(\frac{t}{t^2+y^2} - \frac{t}{t^2+y^2}\tanh|\omega t|\right) \mathrm{d}t = \psi_0\left(\frac{1}{2} + \frac{|\omega y|}{\pi}\right) + \log\left|\frac{\pi}{\omega y}\right|$$

$$\int_0^\infty \log|\omega t| \operatorname{sech}^2(\varpi t)\, \mathrm{d}t = \frac{\log|\omega| - \gamma}{|\varpi|} + \frac{1}{|\varpi|}\log\left|\frac{\pi}{4\varpi}\right|$$

$$\int_0^\infty \left(\frac{\tanh|\omega t|}{t} - \frac{t}{t^2+y^2}\right) \mathrm{d}t = \gamma + \log\left|\frac{4\omega y}{\pi}\right|$$

6 More generalizations obtained from manipulating the integrals in section (5) for all $\omega, y, \varpi \in \Re - \{0\}$

$$\int_0^\infty \frac{\log(\cosh(\omega t))}{t^2(t^2+y^2)}\,\mathrm{d}t = \frac{-|\omega|}{y^2}\psi_0\left(\frac{1}{2}\right) + \frac{\pi}{y^3}\log\left(\frac{\Gamma\left(\frac{1}{2}+\frac{|\omega y|}{\pi}\right)}{\sqrt{\pi}}\right)$$

$$\int_0^\infty \arctan\left|\frac{y}{t}\right|(1-\tanh|\omega t|)\,\mathrm{d}t = \frac{\pi}{|\omega|}\log\Gamma\left(\frac{1}{2}+\frac{|\omega y|}{\pi}\right) + |y|\left(\log\left|\frac{\pi}{y\omega}\right|+1\right)$$

$$\int_0^\infty \left(\frac{\log(\cosh(\omega t))}{\omega t^2} - \frac{t}{t^2+y^2}\right)\mathrm{d}t = \gamma - 1 + \log\left|\frac{4y\omega}{\pi}\right|$$

$$\int_0^\infty \left(\log(\cosh(\varpi t)^{-1})^{\varpi^2} - \frac{\arctan|\varpi t|}{t^2}\right)\mathrm{d}t = \log\left(\frac{4}{\pi}\right)^{|\varpi|} + (\gamma-2)|\varpi|$$

$$\int_0^\infty \left(\log\cosh\left(\frac{\varpi}{t}\right) - \frac{1}{yt^2}\arctan|yt|\right)\mathrm{d}t = |\varpi|\left(\gamma - \frac{1}{|y|}\right) - 1 + |\varpi|\log\left|\frac{4y\varpi}{\pi}\right|$$

$$\int_0^\infty \left(\frac{\tanh|\omega t|}{t} - \frac{1}{y}\arctan\left|\frac{y}{t}\right|\right)\mathrm{d}t = \gamma - 1 + \log\left|\frac{4\omega y}{\pi}\right|$$

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