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ABOUT A SURJECTIF SEQUENCE

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ABSTRACT: *In this paper its gived an innovative way to prove a classical result*

Let be $0 < b < a$ irrational numbers having the same greatest integer part. We define the sequence $(x_n)_{n\geq 0}$ by natural numbers such that

$x_n = [an] + [bn], n \geq 0$. Prove that $(x_n)_{n\geq 0}$ is surjectif.

Proof.

In the following we consider that the statement of surjectif sequence assumes that the sequence so named is that sequence for which its terms are same with whole set of natural numbers.

We will use definitions and properties related to the integer part and the fractional part of a real number, in the usual notations: $\forall t \in \mathbb{R}, t = [t] + \{t\}$; (1)

First we notice that the general term of the sequence $(x_n)_{n\geq 0}$ can be rewritten like this:

$$x_n = [an] - [an] = [([a] + \{a\})n] - [([b] + \{b\}n] =$$

$= [n[a] + n\{a\}] - [n[b] + n\{b\}] = n[a] + [n\{a\}] - n[b] - [n\{b\}] = [n\{a\}] - \{n\{b\}]$; (2), where $[a] = [b]$ from hypothesis.

If in (2) denote $x = \{a\}, y = \{b\}$ with $0 < y < x < 1$, the general term of the sequence is reduced to the form

$$x_n = [nx] - [ny], n \geq 0,0 < y < x < 1; \quad (3)$$

We want to proof the following property by the sequence $(x_n)_{n\geq 1}$.

Proposition 1.

Sequence $(x_n)_{n\geq 1}$ with the general term:

$x_n = [nx] - [ny], n \geq, 0 < y < x < 1$ is upper bounded.

Proof.

Let's suppose that sequence $(x_n)_{n\geq 1}$ is upper bounded: $\exists M \in (0, \infty)$ such that $x_n \leq M, n \geq 0$, then it follows that:

$x_n \leq M \Leftrightarrow [nx]-[ny] \leq M \Leftrightarrow n(x-y)-2 < [nx]-[ny] < M \Leftrightarrow n(x-y) < M+2$

$\Leftrightarrow n < \frac{M+2}{x-y}$ impossible, in contrary we was prove that the set $\mathbb{N}$ is finite.

In conclusion, the sequence $(x_n)_{n\geq 1}$ defined from (1) is upper bounded.

Proposition 2.

The sequence $(x_n)_{n\geq 1}$ defined by $x_n = [nx]-[ny], n \geq 0, 0 < y < x < 1$ take all the values k, where $k \in \mathbb{N}$.

Proof.

First, is obviously that $x_n \in \mathbb{N}$, from definition mode by sequence and conditions to x, y.

We observe that for $0 \in \mathbb{N}, \exists n = 0$ such that $x_0 = 0$, hence among the terms of the sequence there is 0.

Next step is to show that 1 among the terms of the sequence, hence we show that $\exists n \in \mathbb{N}$ such that $x_n = 1$ meaning $[nx]-n[y] = 1$, then $[nx] = 1+[ny] \Leftrightarrow$

$$nx-1 \leq [nx] = 1+[ny] \leq 1+[ny] \Leftrightarrow n \leq \frac{2}{x-y}; \quad (4)$$

From (4) we find that exist on last value for $n, n = \left[\frac{1}{x-y}\right]$ such that $x_n = 1$.

Now, for prove that $\{x_n/n \in \mathbb{N}\} = \mathbb{N}$ we use the method of reduction to absurdum: suppose that $\exists k \in \mathbb{N}$ for that we find among terms of sequence.

How the sequence $(x_n)_{n\geq 1}$ is upper bounded $\exists n \in \mathbb{N}$ such that $x_n \leq k-1$ and $x_{n+1} \geq k+1$ respectively, hence among terms of sequence we are not find value of k, then we have:

$[nx]-[ny] \leq k-1; \quad (5)$ and $[(n+1)x]-[(n+1)y] \geq k+1; \quad (6)$ respectively.

From $(5), (6)$ it follows that: $[(n+1)x]-[(n+1)y] \geq [nx]-[ny]+2; \quad (7)$

Next, we want to prove that (7) is not possible, hence $\exists n \in \mathbb{N}$ such that $x_n = k$:

Lemma 1.

For $0 < t < 1$ and $\forall n \in \mathbb{N}$ then $E = [(n+1)t]-[nt] \in \{0,1\}$

Proof.

Using the whole property, we get:

$$(n+1)t-1<[(n+1)t]\leq(n+t)t;\ \ (8)$$

$$nt-1<[nt]\leq nt;\ \ (9)$$

From $(8),(9)$ we get: $-1<t-1<[n+1]-[nt]<t+1<2;\ \ (10)$, therefore

$$E\in\mathbb{N}\cap(-1,2)=\{0,1\}$$

We have: $[(n+1)x]-[(n+1)y]\geq[nx]-[ny]+2\Leftrightarrow[(n+1)x]-[nx]\geq$ $[(n+1)y]-[ny]+2;\ \ (11)$ impossible from Lemma 1.

Reference:

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