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A PROOF FOR A VARIANT OF A CLASSICAL RESULT

Below, I present a proof for a variant of a classical result, proposed by Narendra Bhandari in the Romanian Mathematical Magazine.

To prove that:

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{\phi^n}\right)^{\frac{\mu(n)-\varphi(n)+\lambda(n)}{n}} = \frac{\sqrt{e^3}}{\sqrt{e^{\vartheta_3(0,\phi^{-1})}}}$$

where ϕ is the golden ratio, $\mu(n)$ is the Mobius function, $\varphi(n)$ is the Euler totient function, $\lambda(n)$ is Liouville function, $\vartheta_a(x,q)$ is the Jacobi theta function and e is the Euler number

Proof

$$let\ P = \prod_{n=1}^{\infty}\left(1-\frac{1}{\phi^n}\right)^{\frac{\mu(n)-\varphi(n)+\lambda(n)}{n}} = e^{\sum_{n=1}^{\infty}\log\left(1-\frac{1}{\phi^n}\right)^{\frac{\mu(n)-\varphi(n)+\lambda(n)}{n}}} = e^{\underbrace{\sum_{n=1}^{\infty}\frac{\mu(n)-\varphi(n)+\lambda(n)}{n}\log\left(1-\frac{1}{\phi^n}\right)}_{S}}$$

$$\therefore P = e^S$$

$$S = \sum_{n=1}^{\infty}\frac{\mu(n)-\varphi(n)+\lambda(n)}{n}\log\left(1-\frac{1}{\phi^n}\right) = \underbrace{\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\left(1-\frac{1}{\phi^n}\right)}_{S_1} - \underbrace{\sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\log\left(1-\frac{1}{\phi^n}\right)}_{S_2} + \underbrace{\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\log\left(1-\frac{1}{\phi^n}\right)}_{S_3}$$

$$S = S_1 - S_2 + S_3$$

$$for\ \ S_1 = \sum_{n=1}^{\infty}\frac{\mu(n)}{n}\log\left(1-\frac{1}{\phi^n}\right) = -\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{kn}}{k}$$

consider the sum over divisors by roots of unity filter

$$\therefore for\ any\ m \geq 0\ the\ coefficient\ of\ \left(\frac{1}{\phi}\right)^n\ is\ \frac{\sum_{n|m}\mu(n)}{m} = \begin{cases}1\ if\ n=1\\0\ if\ n>1\end{cases}$$

$$\therefore S_1 = -\sum_{n=1}^{\infty}\left(\frac{1}{\phi}\right)^n = -\left(\frac{1}{\phi}+\underbrace{\sum_{n=2}^{\infty}\left(\frac{1}{\phi}\right)^n}_{0}\right) = -\frac{1}{\phi}$$

$$considering\ S_2 = \sum_{n=1}^{\infty}\frac{\varphi(n)}{n}\log\left(1-\frac{1}{\phi^n}\right) = -\sum_{n=1}^{\infty}\frac{\phi(n)}{n}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{kn}}{k}$$

$$\therefore for\ any\ m \geq 0\ the\ coefficient\ of\ \left(\frac{1}{\phi}\right)^n\ is\ \frac{\sum_{n|m}\phi(n)}{m} = 1$$

$$\therefore S_2 = -\sum_{n=1}^{\infty}\left(\frac{1}{\phi}\right)^n = -\frac{\frac{1}{\phi}}{1-\frac{1}{\phi}} = -\frac{1}{\phi-1} \quad considering\ S_3 = \sum_{n=1}^{\infty}\frac{\lambda(n)}{n} log\left(1-\frac{1}{\phi^n}\right) = -\sum_{n=1}^{\infty}\frac{\lambda(n)}{n}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\phi}\right)^{kn}}{k}$$

$$\therefore for\ any\ m \geq 0\ the\ coefficient\ of\ \left(\frac{1}{\phi}\right)^n\ is\ \frac{\sum_{n|m}\lambda(n)}{m} = \begin{cases}1\ if\ n\ is\ a\ perfect\ square\\ 0\ otherwise\end{cases}$$

$$\therefore S_3 = -\sum_{n=1}^{\infty}\left(\frac{1}{\phi}\right)^{n^2} = -\frac{1}{2}\left(\vartheta_3\left(0,\frac{1}{\phi}\right)-1\right)$$

$$recall\ that\ \boldsymbol{S} = \boldsymbol{S_1} - \boldsymbol{S_2} + \boldsymbol{S_3} = -\frac{1}{\phi}+\frac{1}{\phi-1}+\frac{1}{2}-\frac{1}{2}\vartheta_3\left(0,\frac{1}{\phi}\right) = \frac{3}{2}-\frac{1}{2}\vartheta_3\left(0,\frac{1}{\phi}\right)$$

$$\therefore S = \frac{3}{2}-\frac{1}{2}\vartheta_3\left(0,\frac{1}{\phi}\right)$$

$$recall\ P = e^S = e^{\frac{3}{2}-\frac{1}{2}\vartheta_3\left(0,\frac{1}{\phi}\right)} = e^{\frac{3}{2}}e^{-\frac{1}{2}\vartheta_3\left(0,\frac{1}{\phi}\right)} = \frac{\sqrt{e^3}}{\sqrt{e^{\vartheta_3\left(0,\frac{1}{\phi}\right)}}} = \frac{\sqrt{e^3}}{\sqrt{e^{\vartheta_3(0,\phi^{-1})}}}$$

$$\boxed{\therefore \boldsymbol{P} = \prod_{n=1}^{\infty}\left(\boldsymbol{1}-\frac{\boldsymbol{1}}{\boldsymbol{\phi^n}}\right)^{\frac{\boldsymbol{\mu(n)-\varphi(n)+\lambda(n)}}{\boldsymbol{n}}} = \frac{\sqrt{\boldsymbol{e^3}}}{\sqrt{\boldsymbol{e^{\vartheta_3(0,\phi^{-1})}}}}}$$