A dazzling identity

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Abstract: In this paper we shall be deriving an identity (*main result*) which was an inspiration from **Prof. Daniel Sitaru's** proposal.The identity is in fact the variant version of inspired problem that uses the *Faulhaber's Formula* and elementary algebraic identities.

Theorem(*Main result*): For positive integers M, N and p such that $M \leq N$ the following holds

$$\lim_{n\to\infty}\sum_{j=M}^{N}\left(\sum_{k=1}^{j}\left(\sum_{l=1}^{k}\frac{1}{k}\sqrt[nl]{G(n,l)}\right)^{n}\right)^{p}\binom{N-M+1}{1}^{-1}=\frac{1}{2}\sum_{q=1}^{p}\frac{N^{p}(M-1)^{q}}{N^{q}(M-1)}$$

$$+\frac{1}{p+1}\sum_{q=1}^{p+1}N^{p+1-q}(M-1)^{q-1}+\sum_{j=2}^{p}\sum_{q=1}^{p-j+1}\frac{B_j}{j}\binom{p}{j-1}N^{p-j+1-q}(M-1)^{q-1}$$

where $G(n,l) = 1 + (-1)^{l+1}\frac{\sqrt[nl]{nl}}{nl}$ and B_n is nt Bernoulli number.

Motivation : Before we get to derive the main result, let's shed the light on the original motivating proposed problem of **Prof. Daniel Sitaru** published by *Romanian Mathematical Magazine*. The problem was to find

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{2}\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}+\frac{1}{2}\left(1-\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}\right)^{n}$$

The limit for the above problem is 1. To crack the problem we can observe that limit has 1^{∞} form.

We shall derive the following lemma.

Lemma: For all $n, l \in \mathbf{N}$ if $\overline{G}(n,l) = \left(1 \pm \frac{\sqrt[nl]{nl}}{nl}\right)$, then

$$\lim_{n\to\infty} \sqrt[nl]{\overline{G}(n,l)} = 1$$

Proof for lemma

Proof

It is trival to show that for all $n, l \geq 1$, $0 < \frac{1}{nl} \leq 1$ so by **Bernoulli inequality** we deduce that

$$0 < \sqrt[nl]{\overline{G}(n,l)} = \sqrt[nl]{1 \pm \frac{\sqrt[nl]{nl}}{nl}} \leq 1 \pm \frac{\sqrt[nl]{nl}}{(nl)^2}$$

$$\sqrt[nl+1]{nl+1} \leq \sqrt[nl]{nl} \leq 1 \overset{Bernoulli}{\widehat{\Rightarrow}} 1 + \frac{1}{nl(nl+1)} \leq \sqrt[nl]{nl} \leq 1$$

Now as $n \to \infty$ our sequence $\sqrt[nl]{nl}$ converges to 1 by **Squeeze theorem** and hence

$$\lim_{n\to\infty} \sqrt[nl]{\overline{G}(n,l)} = 1 + \lim_{n\to\infty} \frac{\sqrt[nl]{nl}}{nl} = 1 + 0 = 1$$

this completes the proof for the **Lemma**.

Using the lemma and setting $l = 1$ we have then the case of proposed problem giving us

$$\Omega = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{2}\right)^n = 1^\infty$$

But then the 1^∞ is an indeterminate form so we are not done to show that the limit is 1.

Since we showed that $\lim_{n\to\infty} \frac{\sqrt[n]{n}}{n} = 0 < 1$ or for $n \in \mathbf{N}$ by induction it is easy to verify that $\sqrt[n]{n} \leq \sqrt[n]{n!} \leq n$ which implies $\frac{\sqrt[n]{n}}{n} < 1, \forall n > 1$ and by fractional binomial theorem

$$\frac{1}{2}\left(\sum_{r=0}^{\infty} \binom{\frac{1}{n}}{r} \left(\frac{\sqrt[n]{n}}{n}\right)^r + \left(-\frac{\sqrt[n]{n}}{n}\right)^r\right) = \frac{1}{2}\left(2 + 2\sum_{r=0}^{\infty} \binom{\frac{1}{n}}{2r} \left(\frac{\sqrt[n]{n}}{n}\right)^{2r}\right)$$

$= \frac{1}{2}(2+2R(n)) = 1 + R(n)$ where $R(n)$ is the latter infinite sum. Therefore,

$$\Omega = \lim_{n\to\infty}(1+R(n))^n = \exp(\lim_{n\to\infty} R(n)) \overbrace{=}^{R(n)\to 0} e^0 = 1$$

we are done. Now what if we are interested to determine the limits of

$$\left(\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}}\right)^n, \left(\frac{1}{2}\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{n}} + \frac{1}{2}\left(1-\frac{\sqrt[2n]{2n}}{2n}\right)^{\frac{1}{2n}}\right)^n,$$

$$\left(\frac{1}{3}\left(1+\frac{\sqrt[n]{n}}{n}\right)^{\frac{1}{2n}} + \frac{1}{3}\left(1-\frac{\sqrt[2n]{2n}}{2n}\right)^{\frac{1}{2n}} + \frac{1}{3}\left(1+\frac{\sqrt[3n]{3n}}{3n}\right)^{\frac{1}{3n}}\right)^n$$

and soon. We shall show the limits for the above sequences converges to 1 in the following proof of the *main result.*

Proof for the theorem

Proof. For all $n, l \in \mathbf{N}$ we define $F(n,l) = (-1)^{l+1}\frac{\sqrt[nl]{nl}}{nl}$ and $G(n,l) = 1 + F(n,l)$ then

$$\lim_{n\to\infty}\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)} = \lim_{n\to\infty}\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{1+F(n,l)} \overbrace{=}^{\textbf{Lemma}} \sum_{l=1}^{k} k^{-1} = 1$$

which follows that

$$\lim_{n\to\infty}\left(\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)}\right)^n = \lim_{n\to\infty} 1^n = 1^\infty$$

giving us the 1^∞ form. Though we have 1^∞ but to deal with it we are unknown with function $J(n)$(say) such that $\exp\left(\lim_{n\to\infty} J(n)\right)$ is properly handled(like that of motivated problem resolution).

To obtain the function $J(n)$ we express $J(n)$ as

$$\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)} = \frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{1+F(n,l)} = \frac{1}{k}\sum_{l=1}^{k}\sum_{r=0}^{\infty}\binom{\frac{1}{nl}}{r}F^r(n,l)$$

in the latter expression we uses the **fractional binomial theorem** for $\sqrt[nl]{1+F(n,l)}$. It's valid to use as $|F(n,l)|<1$. Further on expansion of the latter binomial series we can note that

$$\frac{1}{k}\sum_{l=1}^{k}\sum_{r=0}^{\infty}\binom{\frac{1}{nl}}{r}F^r(n,l)=\frac{1}{k}\sum_{l=1}^{k}\Big(F^0(n,l)+F^1(n,l)+O(n^{-3})\Big)$$

$$=\frac{1}{k}\sum_{l=1}^{k}\Big(1+F^1(n,l)\Big)=1+\frac{1}{k}\sum_{l=1}^{k}(-1)^{l+1}\frac{\sqrt[nl]{nl}}{(nl)^2}=1+y(n,k)$$

where $O(.)$ is *Big O notation* and since we are interested in limit guving us the $J(n)=y(n,k)$ in terms of $n\to\infty$. Therefore, we deduce the limit

$$\lim_{n\to\infty}\left(\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,k)}\right)^n=\lim_{n\to\infty}(1+y(n,k))^n=\exp\Big(\lim_{n\to\infty}y(n,k)\Big)=$$

$$\exp\left(\lim_{n\to\infty}\frac{1}{kn^2}\sum_{l=1}^{k}\frac{(-1)^{l+1}\sqrt[nl]{nl}}{l^2}\right)=e^0=1.$$

and hence for $p\geq 1$, we deduce that

$$\lim_{n\to\infty}\sum_{j=M}^{N}\left(\sum_{k=1}^{j}\left(\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)}\right)^n\right)^p=\sum_{J=M}^{N}\left(\sum_{k=1}^{j}k^{-1}\times k\right)^p=\sum_{j=M}^{N}j^p.$$

It is easy to observe the latter sum appears as the *Faulhaber's formula* however, the sum begins from any $1\leq j=M\leq N$ so the direct application formula is not appropriate.

Call $X(M,N,p)=\sum_{j=M}^{N}j^p$ then we can notice that

$$X(M,N,p)=X(1,N,p)-X(1,M-1,p)=\sum_{j=1}^{N}j^p-\sum_{j=1}^{M-1}j^p$$

Recall that (*Faulhaber's formula*)

$$\sum_{j=1}^{N}j^p=\frac{N^{p+1}}{p+1}+\frac{N^p}{2}+\sum_{j=2}^{p}\frac{B_j p!}{j!}\frac{N^{p-j+1}}{(p-j+1)!}\cdots(i)$$

and now for $M-1$

$$\sum_{j=1}^{M-1} j^p = \frac{(M-1)^{p+1}}{p+1} + \frac{(M-1)^p}{2} + \sum_{j=2}^{p} \frac{B_j p!}{j!} \frac{(M-1)^{p-j+1}}{(p-j+1)!} \cdots (ii)$$

subtracting $(i)-(ii)$ it's yields the following

$$X(M,N,p) = \frac{N^{p+1}-(M-1)^{p+1}}{p+1} + \frac{N^p-(M-1)^p}{2} +$$

$$+\sum_{j=2}^{p} \frac{B_j}{j!p!(p-j+1)!} \left(N^{p-j+1}-(M-1)^{p-j+1}\right)$$

for the simplification of the above identity we have elementary algebraic identity

$$a^n - b^n = (a-b)\sum_{q=0}^{N} a^{N-q-1}b^q = (a-b)\sum_{q=1}^{n} a^{n-q}b^{q-1}$$

and thus $X(M,N,p) =$

$$\frac{(N-M+1)}{p+1}\sum_{q=1}^{p+1} N^{p+1-q}(M-1)^{q-1} + \frac{N-M+1}{2}\sum_{q=1}^{p} N^{p-q}(M-1)^{q-1}$$

$$+(N-M+1)\sum_{j=2}^{p}\sum_{q=1}^{p-j+1} \frac{B_j p! N^{p-j+1-q}(M-1)^{q-1}}{j!(p-j+1)}$$

$$X(M,N,p)\binom{N-M+1}{1}^{-1} = \frac{1}{2}\sum_{q=1}^{p} N^{p-q}(M-1)^{q-1} +$$

$$\frac{1}{p+1}\sum_{q=1}^{p+1} N^{p+1-q}(M-1)^{q-1} + \sum_{j=2}^{p}\sum_{q=1}^{p-j+1} \frac{B_j}{j}\binom{p}{j-1} N^{p-j+1-q}(M-1)^{q-1}$$

which completes the proof.

Corollaries

1.Show that

$$\lim_{n\to\infty} \sum_{j=M}^{N}\sum_{k=1}^{j}\left(\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)}\right)^n = \frac{N+M}{2}(N-M+1)$$

this result is immediately followed by setting $p=1$

2. Prove that

$$\lim_{n\to\infty}\sum_{j=M}^{N}\left(\sum_{k=1}^{j}\left(\frac{1}{k}\sum_{l=1}^{k}\sqrt[nl]{G(n,l)}\right)^{n}\right)^{2}=X(M,N,2)$$

$=\dfrac{N-M+1}{3}\left(N^2+NM+\dfrac{N}{2}+M^2-\dfrac{M}{2}\right)$ is the particular case of the *main result* for $p=2$

The particular case for $p=2$ is can be found as proposed problem which is published by *Romanian Mathematical Magazine.*

References

[1] Faulhaber's formula,*https://en.m.wikipedia.org/wiki/Faulhaber.*
[2] Fractional binomial theorem,*https://brilliant.org/wiki/fractional-binomial-theorem/.*
[3] Big O notation,*https://en.m.wikipedia.org/wiki/Big O notation.*