# Infinite Series

## 1 Introduction

$Let f(a,b) = \sum_{1\le m\le n<\infty} \frac{1}{m^a n^{a+b}(m^b+n^b)}$

**Given sum is symmetrical** $f(a,b) = \frac{f(a,b)+f(b,a)}{2}$

$$\sum_{1\le m\le n<\infty} \frac{1}{m^a n^{a+b}(m^b+n^b)} = \frac{1}{2}\sum_{1\le n\le n<\infty}\left(\frac{1}{m^a n^{a+b}(m^b+n^b)} + \frac{1}{m^{a+b} n^a (m^b+n^b)}\right)$$

$$= \frac{1}{2}\sum_{1\le n\le m<\infty} \frac{m^b+n^b}{m^{a+b}n^{a+b}(m^b+n^b)} = \left[\sum_{n=1}^{\infty}\frac{1}{n^{a+b}}\right]^2 = \frac{(\zeta(a+b))^2}{2}$$

**Let's Discuss Some cases**

$b=1, a=1$**We get**

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm^2(m+n)} = \frac{\zeta^2(2)}{2} = \frac{\pi^4}{72}$$

**Since we know**

$$\sum_{n=1}^{\infty}\frac{m}{n(n+m)} = H_m$$

*Hence*

$$\sum_{n=1}^{\infty}\frac{H_n}{n^3} = \frac{\pi^4}{72}$$

**Set b=1,a=2**

$$\sum_{1\le m\le n<\infty}\frac{1}{n^2m^3(m+n)} = \frac{\zeta^2(3)}{2}$$

*Since*

$$\frac{1}{n^2m^3} = \frac{1}{m^4n^2} - \frac{1}{m^4n(n+m)}$$

$$\sum_{1\le n\le m<\infty} \frac{1}{n^2m^3(n+m)} = \sum_{1\le m\le n<\infty} \frac{1}{m^4n^2} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{1}{m^4n(m+n)}$$

$$\zeta(4)\zeta(2) - \sum_{n=1}^{\infty} \frac{H_n}{n^5} = \frac{\zeta^2(3)}{2}$$

$$\sum_{n=1}^{\infty} \frac{H_n}{n^5} = \zeta(2)\zeta(4) - \frac{\zeta^2(3)}{2}$$

**Series Involving Coth(z)**

**Set a=0 ,b=2**

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2(n^2+m^2)} = \frac{\zeta^2(2)}{2}$$

*We know That* $\sum_{n=1}^{\infty} \frac{1}{m^2+n^2} = \frac{\pi\coth(\pi m)}{2m} - \frac{1}{2m^2}$

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{1}{m^2(n^2+m^2)} = \sum_{m=1}^{\infty} \frac{1}{m^2}\left(\frac{\pi\coth(\pi m)}{2m} - \frac{1}{2m^2}\right) = \frac{\pi^4}{72}$$

$$\frac{\pi}{2}\sum_{n=1}^{\infty} \frac{\coth(\pi m)}{m^3} - \frac{\pi^2}{180} = \frac{\pi^4}{72}$$

$$\sum_{m=1}^{\infty} \frac{\coth(m\pi)}{m^3} = \frac{7\pi^3}{180}$$

## 2 Evaluation of Ramanujan Series

Put a=2 ,b=2

$$\sum_{m,n\in N} \frac{1}{m^2n^4(m^2+n^2)} = \frac{\zeta^2(4)}{2}$$

**By partial fraction**

$$\frac{1}{n^4m^2(n^2+m^2)} = \frac{1}{n^6m^2} - \frac{1}{m^6(m^2+n^2)}$$

$$\sum_{n,m\in N} \frac{1}{n^4m^2(m^2+n^2)} = \zeta(2)\zeta(6) - \sum_{m,n\in N} \frac{1}{m^6(m^2+n^2)} = \frac{\pi^8}{16200}$$

**We know** $\sum_{n=1}^{\infty} \frac{1}{m^2+n^2} = \frac{\pi\coth(\pi m)}{2m} - \frac{1}{2m^2}$

$$\zeta(2)\zeta(6) - \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^7} + \zeta(8)/2 = \pi^8/16200$$

$$\sum_{n=1}^{\infty}\frac{\coth(n\pi)}{n^7} = \frac{2\zeta(2)\zeta(6)+\zeta(8)}{\pi} - \frac{\pi^7}{8100} = \frac{19\pi^7}{56700}$$

**By Surjeet Singhania**