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OPPENHEIM'S INEQUALITY REVISITED

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NOTE BY EDITOR: *In his unmistakable style with which the author has accustomed us in the previous articles in this paper is approached the famous Oppenheim inequality in triangle. The author provides an interesting generalization and numerous applications with inequalities and identities that result from it. I notice once again the way in which the author presents his thought process in its entire evolution for obtaining the results-fact that makes the article extremely useful also from a methodical perspective.*

Let M –be any point in euclidian space and $x, y, z \in \mathbb{R}$. In ΔABC holds:

$$(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2$$

Proof: We have: $x\overrightarrow{MA}+y\overrightarrow{MB}+z\overrightarrow{MC} \geq 0$ then:

$$x^2MA^2+y^2MB^2+z^2MC^2+2xy\overrightarrow{MA}\cdot\overrightarrow{MB}+2yz\overrightarrow{MB}\cdot\overrightarrow{MC}+2zx\overrightarrow{MC}\cdot\overrightarrow{MA} \geq 0$$

$$x^2MA^2+y^2MB^2+z^2MC^2+xy(MA^2+MB^2-AB^2)+yz(MB^2+MC^2-BC^2)$$

$$+zx(MC^2+MA^2-AC^2) \geq 0$$

$$x(x+y+z)MA^2+y(x+y+z)MB^2+z(x+y+z)MC^2 \geq yza^2+zxb^2+xyc^2$$

$$(x+y+z)(xMA^2+yMB^2+zMC^2) \geq yza^2+zxb^2+xyc^2$$

If we take $M=0$ then $OA=OB=OC=R$ then we have that:

$$(x+y+z)^2R^2 \geq yza^2+zxb^2+xyc^2 \ (Bottema's\ Inequality)$$

Let be $x, y, z \in \mathbb{R}_+$; $x=a^2x_1, y=b^2y_1, z=c^2z_1$; $x_1, y_1, z_1 \in \mathbb{R}_+$ then follows:

$(a^2x_1+b^2y_1+c^2z_1)^2R^2 \geq a^2b^2c^2(x_1y_1+y_1z_1+z_1x_1)$ and from $abc=4RS$ we get:

$a^2x_1+b^2y_1+c^2z_1 \geq 4S\sqrt{x_1y_1+y_1z_1+z_1x_1}$ (Oppenheim-Bădilă-Chiriță)

In $A_1B_1C_1$ with sides $a_1=\sqrt{a}, b_1=\sqrt{b}, c_1=\sqrt{c}$ the following relationship holds:

$$ax_1+by_1+cz_1 \geq 4S_1\sqrt{x_1y_1+y_1z_1+z_1x_1}$$

$$16S^2 = 2\sum_{cyc} a^2b^2 - (a^4+b^4+c^4) \Rightarrow 16S_1^2 = 2(ab+bc+ca)-(a^2+b^2+c^2)$$

$ab+bc+ca = s^2+4Rr+r^2;\ a^2+b^2+c^2 = 2(s^2-4Rr-r^2)$ *it follows*

$$16S_1^2 = 4r(4R+r) \Rightarrow 4S_1^2 = r(4R+r) \Rightarrow S_1 = \frac{1}{2}\sqrt{r(4R+r)}$$

So, we get: $ax_1+by_1+cz_1 \geq 2\sqrt{r(4R+r)(x_1y_1+y_1z_1+z_1x_1)}$

But $4R+r \geq s\sqrt{4-\frac{2r}{R}}$ *which follows from* $s^2 \leq \frac{R(4R+r)^2}{2(2R-r)}$ $(Blundon)$.

Therefore,

$$ax_1+by_1+cz_1 \geq 2\sqrt{S\sqrt{4-\frac{2r}{R}}(x_1y_1+y_1z_1+z_1x_1)}$$

Let be: $x_1 = \frac{x}{a}, y_1 = \frac{y}{b}, z_1 = \frac{z}{c}; x,y,z > 0$ *then follows*

$$x+y+z \geq 2\sqrt{S\sqrt{4-\frac{2r}{R}}\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{xz}{ac}\right)}$$

$$S = \frac{absinC}{2} = \frac{bcsinA}{2} = \frac{acsinB}{2}$$

$$(x+y+z)^2 \geq 4S\sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc}\frac{xy}{ab} = \sqrt{4-\frac{2r}{R}}\cdot\sum_{cyc}\frac{xy}{ab}\cdot 2absinC$$

$$= \sqrt{4-\frac{2r}{R}}\cdot 2\sum_{cyc} xysinC$$

In $\Delta ABC; x,y,z>0$ the following relationship holds:

$$(x+y+z)^2\geq 2\sqrt{4-\frac{2r}{R}}(yzsinA+zxsinB+xysinC)\ (Oppenheimer\ generalization)$$

Proof.

We prove it that: $ax_1+by_1+cz_1\geq 2\sqrt{r(4R+r)(x_1y_1+y_1z_1+z_1x_1)};x_1,y_1,z_1>0$

Let be: $x_1=\frac{x}{a},y_1=\frac{y}{b},z_1=\frac{z}{c};x,y,z>0$ *then*

$$\frac{(x+y+z)^2}{r_a+r_b+r_c}\geq 4r\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$$

But $tan\frac{A}{2}=\frac{r_a}{s}$ $(and\ analogs)$; $r_a+r_b+r_c=4R+r$, *so we get:*

$(x+y+z)^2\geq 4r(R+r)\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$ *hence* $(x+y+z)^2\geq\frac{4S(4R+r)}{s}\left(\frac{xy}{ab}+\frac{yz}{bc}+\frac{zx}{ca}\right)$ *and*

$$(x+y+z)^2\geq\sum_{cyc}tan\frac{A}{2}\cdot\sum_{cyc}2absinC\cdot\frac{xy}{ab}$$

Therefore,

$$(x+y+z)^2\geq 2(xysinC+xzsinB+yzsinA)\sum_{cyc}tan\frac{A}{2}$$

For $x,y,z>0$ *we have* $ax+by+cz\geq 2\sqrt{r(4R+r)(xy+yz+zx)}$ *and put* $x=\frac{b}{a}$,

$y=\frac{c}{b},z=\frac{a}{c}$ *we get:*

$$a\cdot\frac{b}{a}+b\cdot\frac{c}{b}+c\cdot\frac{a}{c}\geq 2\sqrt{r(4R+r)\left(\frac{b}{a}\cdot\frac{c}{b}+\frac{b}{a}\cdot\frac{a}{c}+\frac{c}{b}\cdot\frac{a}{c}\right)}\Leftrightarrow$$

$$2s\geq 2\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}\Leftrightarrow s\geq\sqrt{r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$$

But $4R+r\geq s\sqrt{3}$ *then* $s\geq\sqrt{rs\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)}$ *and hence*

$$s\geq r\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)(Ion\ Cristian\ Miu-refinement\ Mitrinovic)$$

How $4R+r\geq s\sqrt{4-\frac{2r}{R}}$ *then we have:* $s\geq r\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *and from*

$s^2\geq r(4R+r)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *we get:* $\frac{s}{r}\geq\frac{4R+r}{s}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *and with* $r_a=s\cdot\tan\frac{A}{2}$ *follows*

that: $\frac{s}{r}\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}\right)$.

From $\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s}{r}$ *we get a new inequality:*

$$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}}\geq\frac{a}{b}+\frac{b}{c}+\frac{c}{a}$$

From $ax+by+cz\geq 2\sqrt{r(4R+r)(xy+yz+zx)};x,y,z>0$ *and for* $x=\frac{c}{a},y=\frac{a}{b},$

$z=\frac{b}{c}$ *we have:* $a\cdot\frac{c}{a}+b\cdot\frac{a}{b}+c\cdot\frac{b}{c}\geq 2\sqrt{r(4R+r)\left(\frac{c}{a}\cdot\frac{a}{b}+\frac{c}{a}\cdot\frac{b}{c}+\frac{a}{b}\cdot\frac{b}{c}\right)}$ *therefore,*

$s\geq\sqrt{r(4R+r)\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)}$ *and from* $4R+r\geq s\sqrt{3}$ *we get a new inequality:*

$$s\geq\sqrt{sr\sqrt{3}\left(\frac{a}{c}+\frac{b}{a}+\frac{c}{b}\right)}$$

From $s^2\geq 3r^2\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$ *follows:* $\frac{s^2}{3r^2}\geq\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$

But $4R+r\geq s\sqrt{4-\frac{2r}{R}}$ *then:* $s\geq\sqrt{4-\frac{2r}{R}}\left(\frac{a}{c}+\frac{c}{b}+\frac{b}{a}\right)$ *and from* $s\geq r\sqrt{4-\frac{2r}{R}}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$

it follows: $\frac{s^2}{r^2}\geq\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$.

From $ax+by+cz\geq 2\sqrt{r(4R+r)(xy+yz+zx)}; x,y,z>0$ *and for* $x=\frac{c}{a}, y=\frac{a}{b},$

$z=\frac{b}{c}$ *we get* $s\geq\sqrt{r(4R+r)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$ *and hence*

$$s^2\geq r(4R+r)\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}\Leftrightarrow\frac{s}{r}\geq\frac{4R+r}{s}\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

From $r_a=s\cdot tan\frac{A}{2}(and\ analogs);\ r_a+r_b+r_c=4R+r;\ \sum_{cyc}cot\frac{A}{2}=\frac{s}{r}$ *we get:*

$$\frac{cot\frac{A}{2}+cot\frac{B}{2}+cot\frac{C}{2}}{tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}}\geq\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

Similarly:

$$\frac{cot\frac{A}{2}+cot\frac{B}{2}+cot\frac{C}{2}}{tan\frac{A}{2}+tan\frac{B}{2}+tan\frac{C}{2}}\geq\frac{c}{b}+\frac{b}{a}+\frac{a}{c}$$

Now, m_a,m_b,m_c *it can be the lengths sides of on triangle by area* $S_m=\frac{3}{4}S,$

semiperimeter $s_m=\frac{m_a+m_b+m_c}{2};\ S_m=r_ms_m;\frac{3}{4}S_m=\frac{m_a+m_b+m_c}{2}\cdot r_m\Rightarrow r_m=\frac{3S}{2(m_a+m_b+m_c)}$

We have that: $s\geq r\sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)$ *and* $s\geq r\sqrt{3}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$ *so, for* $\Delta m_am_bm_c$ *we have:*

$$\frac{m_a+m_b+m_c}{2}\geq\frac{3S}{2(m_a+m_b+m_c)}\sqrt{3}\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)\Leftrightarrow$$

$$(m_a+m_b+m_c)^2\geq 3\sqrt{3}S\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)$$

Similarly:

$$(m_a+m_b+m_c)^2\geq 3\sqrt{3}S\left(\frac{m_c}{m_b}+\frac{m_b}{m_a}+\frac{m_a}{m_c}\right)$$

Therefore, we get a new inequality:

$$(m_a+m_b+m_c)^4 \geq (3\sqrt{3}S)^2\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)\left(\frac{m_c}{m_b}+\frac{m_b}{m_a}+\frac{m_a}{m_c}\right)$$

and hence

$$\frac{(m_a+m_b+m_c)^2}{3\sqrt{3}S} \geq \sqrt{\left(\frac{m_a}{m_b}+\frac{m_b}{m_c}+\frac{m_c}{m_a}\right)\left(\frac{m_c}{m_b}+\frac{m_b}{m_a}+\frac{m_a}{m_c}\right)}$$

We know that:

$$s^2 = n_a^2 + 2r_a h_a \Leftrightarrow s^2 - n_a^2 = 2r_a h_a \Leftrightarrow$$

$$(s-n_a)(n+n_a) = 2r_a h_a \Leftrightarrow s - n_a = \frac{2r_a h_a}{s+n_a} \Leftrightarrow \frac{s-n_a}{h_a} = \frac{2r_a}{s+n_a} (and\ analogs)$$

$$\frac{1}{h_a}+\frac{1}{h_b}+\frac{1}{h_c} = \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{1}{r}$$

$$\frac{s}{r} = \sum_{cyc}\frac{n_a}{h_a} + 2\sum_{cyc}\frac{r_a}{s+n_a} \text{ and } \frac{s}{r} = \sum_{cyc}\frac{n_a}{r_a} + 2\sum_{cyc}\frac{h_a}{s+n_a}$$

$$\frac{s}{r} \geq \sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) and \frac{s}{r} \geq \sqrt{3}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)$$

So, it follows that:

$$\sum_{cyc}\frac{n_a}{h_a} \geq \sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 2\sum_{cyc}\frac{r_a}{s+n_a} \text{ and } \sum_{cyc}\frac{n_a}{h_a} \geq \sqrt{3}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right) - 2\sum_{cyc}\frac{r_a}{s+n_a}$$

$$\sum_{cyc}\frac{n_a}{r_a} \geq \sqrt{3}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) - 2\sum_{cyc}\frac{h_a}{s+n_a} \text{ and } \sum_{cyc}\frac{n_a}{r_a} \geq \sqrt{3}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right) - 2\sum_{cyc}\frac{h_a}{s+n_a}$$

$$\frac{s}{r} \geq \sqrt{4 - \frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)} and \frac{s}{r} \geq \sqrt{4 - \frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

Therefore,

$$\sum_{cyc}\frac{n_a}{h_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-2\sum_{cyc}\frac{r_a}{s+n_a}}\ and$$

$$\sum_{cyc}\frac{n_a}{h_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)-2\sum_{cyc}\frac{r_a}{s+n_a}}$$

$$\sum_{cyc}\frac{n_a}{r_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)-2\sum_{cyc}\frac{h_a}{s+n_a}}\ and$$

$$\sum_{cyc}\frac{n_a}{r_a}\ge\sqrt{4-\frac{2r}{R}\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)-2\sum_{cyc}\frac{h_a}{s+n_a}}$$

Applying Jensen Inequality for function convex $f(x)=\tan\frac{x}{4}, x\in\left(0,\frac{\pi}{4}\right)$ *we have:*

$$\tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}\ge 3\tan\left(\frac{\frac{A}{4}+\frac{B}{4}+\frac{C}{4}}{3}\right)\Leftrightarrow \tan\frac{A}{4}+\tan\frac{B}{4}+\tan\frac{C}{4}\ge 3\tan\frac{\pi}{12}$$

$$\tan\frac{\pi}{12}=\tan\frac{1}{2}\left(\frac{\pi}{6}\right)=2-\sqrt{3};\tan\frac{x}{2}=\frac{1-\cos x}{\sin x};\cot\frac{x}{2}=\frac{1+\cos x}{\sin x}$$

$$\cos\frac{A}{2}=\frac{s-a}{AI}(and\ analogs);\sin\frac{A}{2}=\frac{r}{AI}(and\ analogs)$$

$$\tan\frac{A}{4}=\frac{1-\cos\frac{A}{2}}{\sin\frac{A}{2}}=\frac{1-\frac{s-a}{AI}}{\frac{r}{AI}}=\frac{AI+a-s}{AI}\cdot\frac{AI}{r}=\frac{AI+a-s}{r}$$

So, we have: $\tan\frac{A}{4}=\frac{AI+a-s}{r}(and\ analogs)$

$$\sum_{cyc}\tan\frac{A}{4}=\frac{AI+BI+CI+a+b+c-3s}{r}$$

$$\sum_{cyc}\tan\frac{A}{4}=\frac{AI+BI+CI-s}{r}\Rightarrow\frac{AI+BI+CI-s}{r}\ge\frac{s}{r}+3(2-\sqrt{3})$$

$$\cos\frac{B-C}{2}=\left(\frac{b+c}{a}\right)\sin\frac{A}{2}\ (\text{and analogs});\cos\frac{B-C}{2}=\frac{h_a}{w_a}(\text{and analogs})$$

$$\frac{AI}{r}=\frac{1}{\sin\frac{A}{2}}(\text{and analogs})\Rightarrow\frac{h_a}{w_a}=\frac{b+c}{a}\cdot\frac{r}{AI}$$

$$\frac{b+c}{a}=\frac{ab+ac}{a^2};bc=2Rh_a(\text{and analogs});\ a^2=2R\cdot\frac{h_bh_c}{h_a}(\text{and anaolgs})$$

$$\frac{b+c}{a}=\frac{2R(h_b+h_c)}{2R\cdot\frac{h_bh_c}{h_a}}=h_a\cdot\frac{h_b+h_c}{h_bh_c}(\text{and analogs})$$

$$\frac{h_a}{w_a}=h_a\cdot\frac{h_b+h_c}{h_bh_c}\cdot\frac{r}{AI}\Rightarrow\frac{AI}{r}=\frac{w_a}{h_b}+\frac{w_a}{h_c}(\text{and analogs})$$

Therefore,

$$\frac{AI+BI+CI}{r}=\sum_{cyc}\frac{w_b+w_c}{h_a}\geq\frac{s}{r}+3(2-\sqrt{3})$$

$$\frac{s}{r}=\frac{m_a}{h_a}+\frac{m_b}{h_b}+\frac{m_c}{h_c}+2\sum_{cyc}\frac{r_a}{s+n_a}$$

So, we get a new inequality:

$$\sum_{cyc}\frac{w_b+w_c-n_a}{h_a}\geq3(2-\sqrt{3})+2\sum_{cyc}\frac{r_a}{s+n_a}$$

Now,

$$\sum_{cyc}\tan\frac{A}{4}=\frac{AI+BI+CI-s}{r}=\sum_{cyc}\frac{w_b+w_c-n_a}{h_a}-2\sum_{cyc}\frac{r_a}{s+n_a}$$

and from

$$\frac{s}{r}\geq\frac{4R+r}{s}\sqrt{\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)};4R+r\geq s\sqrt{4-\frac{2r}{R}}$$

So, we get a new inequality:

$$\frac{s}{r}\geq\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}$$

Denote:

$$Q=\sqrt{\left(4-\frac{2r}{R}\right)\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{b}{a}+\frac{a}{c}\right)}\Rightarrow\sum_{cyc}tan\frac{A}{4}\leq\sum_{cyc}\frac{w_b+w_c}{h_a}-Q$$

$$cot\frac{A}{4}=\frac{1+cos\frac{A}{2}}{sin\frac{A}{2}}=\frac{1+\frac{s-a}{AI}}{sin\frac{A}{2}}=\frac{AI+s-a}{AI}\cdot\frac{AI}{r}=\frac{AI+s-a}{r}$$

Hence,

$$\sum_{cyc}cot\frac{A}{4}=\frac{AI+BI+CI+s}{r}$$

So, we have the identity:

$$\sum_{cyc}cot\frac{A}{4}=\frac{AI+BI+CI+s}{r}=\sum_{cyc}\frac{w_b+w_c}{h_a}+\frac{s}{r}$$

Therefore,

$$\sum_{cyc}cot\frac{A}{4}\geq Q+\sum_{cyc}\frac{w_b+w_c}{h_a}+\frac{s}{r}=\sum_{cyc}\frac{n_a}{h_a}+2\sum_{cyc}\frac{r_a}{s+n_a}$$

$$\sum_{cyc}cot\frac{A}{4}=2\sum_{cyc}\frac{r_a}{s+n_a}+\sum_{cyc}\frac{w_b+w_c-n_a}{h_a}$$

Denote N_a –Nagel point. Applying Van Aubel theorem, we have:

$$\frac{AN_a}{n_a-AN_a}=\frac{s-c}{s-a}+\frac{s-b}{s-a}=\frac{a}{s-a}\Rightarrow\frac{n_a-AN_a}{AN_a}=\frac{s-a}{a}\Rightarrow\frac{n_a}{AN_a}=\frac{s}{a}(and\ analogs)$$

$$AN_a = \frac{an_a}{s}; 2s = a + b + c \Rightarrow (a + b + c)AN_a = 2an_a \Rightarrow$$

$$1 + \frac{b + c}{a} = \frac{2n_a}{AN_a}(and\ analogs)$$

$$2S = ah_a; 2S = 2sr \Rightarrow ah_a = 2sr \Rightarrow \frac{h_a}{r} = \frac{a + b + c}{a} = 1 + \frac{b + c}{a}$$

Then, we have: $\frac{h_a}{2r} = \frac{n_a}{AN_a}(and\ analogs)$ *and hence*

$$\frac{AN_a + BN_a + CN_a}{2r} = \frac{n_a}{h_a} + \frac{n_b}{h_b} + \frac{n_c}{h_c}$$

$$\frac{AN_a}{a} = \frac{n_a}{s}(and\ analogs) \Rightarrow \sum_{cyc} \frac{AN_a}{a} = \frac{n_a + n_b + n_c}{s}$$

In $\Delta ABC: \frac{MA}{a} + \frac{MB}{b} + \frac{MC}{c} \geq \sqrt{3};\ \forall M \subset (ABC).$

For $M = N_a \Rightarrow \frac{AN_a}{a} + \frac{BN_a}{b} + \frac{CN_a}{c} \geq \sqrt{3} \Rightarrow n_a + n_b + n_c \geq s\sqrt{3}$

Now,

$\frac{b+c}{a} = \frac{r_a+h_a}{r_a} = 1 + \frac{h_a}{r_a}(and\ analogs)$ *and hence* $2 + \frac{h_a}{r_a} = \frac{2n_a}{AN_a}$

So, we have a new inequality:

$$2\sum_{cyc} \frac{n_a}{AN_a} = 6 + \frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c} = \frac{h_a + h_b + h_c}{r}$$

$$bc = 2Rh_a(and\ analogs);\ h_a + h_b + h_c = \frac{ab + bc + ca}{2R};$$

$$ab + bc + ca = s^2 + 4Rr + r^2$$

So, it follows that:

$$\sum_{cyc} \frac{n_a}{AN_a} = \frac{s^2 + 4Rr + r^2}{4Rr};\ s^2 \geq 16Rr - 5r^2(Gerretsen)$$

$$\sum_{cyc} \frac{n_a}{AN_a} \geq 5 - \frac{r}{R}$$

From $\frac{h_a}{2r} = \frac{n_a}{AN_a}$ (*and analogs*) *and from* $h_a \leq s_a$ (*and analogs*)

we get: $\frac{s_a}{2r} \geq \frac{n_a}{AN_a}$ (*and analogs*) *then*

$$\frac{s_a + s_b + s_c}{2r} \geq \sum_{cyc} \frac{n_a}{AN_a}$$

Now, we prove it that: $\frac{R}{r} - 1 \geq \frac{n_a}{h_a}$ (*and analogs*)

$$\frac{R-r}{r} \geq \frac{AN_a}{2r} \Rightarrow 2(R-r) \geq AN_a (\text{and analogs})$$

Therefore,

$$6(R-r) \geq AN_a + BN_a + CN_a$$

$$\frac{AN_a}{a} = \frac{n_a}{s}; \frac{AI}{w_a} = \frac{b+c}{2s} \Rightarrow \frac{2AI}{(b+c)w_a} = \frac{1}{s}$$

$$\frac{AN_a}{a} = \frac{2AI}{(b+c)w_a} \cdot n_a \Rightarrow \frac{AN_a}{AI} = \frac{a}{b+c} \cdot \frac{n_a}{w_a} (\text{and analogs})$$

$$\frac{1}{\sin\frac{A}{2}} = \frac{b+c}{a} \cdot \frac{w_a}{h_a} (\text{and analogs}) \Rightarrow \frac{2a}{b+c} = 2\frac{w_a}{h_a} \sin\frac{A}{2}$$

Hence

$$\frac{AN_a}{AI} = 2\frac{n_a}{h_a} \sin\frac{A}{2} (\text{and analogs}) \Rightarrow \frac{AN_a}{AI} \leq 2\left(\frac{R}{r} - 1\right) \sin\frac{A}{2} (\text{and analogs})$$

$$\sum_{cyc} \frac{AN_a}{AI} < 2\left(\frac{R}{r} - 1\right) \sum_{cyc} \sin\frac{A}{2}$$

$$\sin\frac{A}{2} = \sqrt{\frac{r}{2R}} \cdot \sqrt{\frac{r_a}{h_a}} \Rightarrow 2\sin\frac{A}{2} = \sqrt{\frac{2r}{R}} \cdot \sqrt{\frac{r_a}{h_a}} \Rightarrow$$

$$\frac{AN_a}{AI}=\frac{n_a}{h_a}\sqrt{\frac{2r}{R}}\cdot\sqrt{\frac{r_a}{h_a}}(and\ analogs)$$

From $\frac{R}{2r}\geq\frac{m_a}{h_a}$ $(Panaitopol\ inequality)$ **we get:**

$$\frac{AN_a}{AI}=\frac{n_a}{h_a}\sqrt{\frac{2r}{R}}\cdot\sqrt{\frac{r_a}{h_a}}\leq\frac{n_a}{h_a}\sqrt{\frac{h_a}{m_a}}\cdot\sqrt{\frac{r_a}{h_a}}=\frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}$$

So, we have: $\frac{AN_a}{AI}\leq\frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}(and\ analogs)$ **and summing, it follows that:**

$$\sum_{cyc}\frac{AN_a}{AI}\leq\sum_{cyc}\frac{n_a}{h_a}\sqrt{\frac{r_a}{m_a}}$$

Now, N_a **−Nagel's point**

$$AN_a=\frac{an_a}{s}(and\ analogs);\ n_a^2=s^2-2r_ah_a(and\ analogs)$$

$$AN_a^2=\frac{(s^2-2r_ah_a)a^2}{s^2}=\left(1-\frac{2r_ah_a}{s^2}\right)a^2$$

$$r_a=\frac{S}{s-a};\ h_a=\frac{2S}{a}\Rightarrow 2r_ah_a=\frac{4S^2}{a(s-a)}$$

$$AN_a^2=a^2-\frac{4S^2}{a(s-a)}\cdot\frac{a^2}{s^2};\ S^2=s(s-a)(s-b)(s-c)(Heron)$$

$$AN_a^2=a^2-\frac{4s(s-a)(s-b)(s-c)}{s^2}\cdot\frac{a}{s-a}$$

$$AN_a^2=a\left(a-\frac{4(s-b)(s-c)}{s}\right)(and\ analogs)$$

But $a^2=(b-c)^2+b(s-b)(s-c)(and\ analogs)$

$$AN_a^2=(b-c)^2+4(s-b)(s-c)-\frac{4a(s-b)(s-c)}{s}$$

$$AN_a^2=(b-c)^2+4(s-b)(s-c)\left(1-\frac{a}{s}\right)$$

$$AN_a^2=(b-c)^2+\frac{4s(s-a)(s-b)(s-c)}{s^2}$$

$$AN_a^2=(b-c)^2+\frac{4S^2}{s^2};\ S=sr$$

So, we get a new identity:

$$AN_a^2=(b-c)^2+4r^2(and\ analogs);AN_a^2=\frac{a^2n_a^2}{s^2}\Rightarrow$$

$$\frac{n_a^2}{s^2}=\frac{(b-c)^2+4r^2}{a^2}(and\ analogs)$$

Using AM-QM Inequality, we have:

$$\sqrt{\frac{(b-c)^2+4r^2}{2}}\ge\frac{|b-c|+2r}{2}\Rightarrow\frac{an_a}{s}\ge\frac{|b-c|+2r}{\sqrt{2}}$$

So, we get a new inequality:

$$\frac{n_a}{s}\ge\frac{|b-c|+2r}{a\sqrt{2}}(and\ analogs);abc=4RS=4Rrs$$

$$n_an_bn_c\ge\frac{s^3(|b-c|+2r)(|c-a|+2r)(|a-b|+2r)}{2\sqrt{2}\cdot4Rrs}\Leftrightarrow$$

$$n_an_bn_c\ge\frac{s^2(|b-c|+2r)(|c-a|+2r)(|a-b|+2r)}{8\sqrt{2}\cdot Rr}$$

But

$$n_a\ge\frac{|b-c|+2r}{a\sqrt{2}}\cdot s;\frac{s}{a}=\frac{h_a}{2r}(and\ analogs);n_a\ge\frac{h_a}{2r}\cdot\frac{|b-c|+2r}{\sqrt{2}}$$

So, we get inequality:

$$\frac{n_a}{h_a} \geq \frac{|b-c|+2r}{2\sqrt{2}r}\ (and\ analogs)$$

$$\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c} \geq \frac{|a-b|+|b-c|+|c-a|+2r}{2\sqrt{2}r}+\frac{3\sqrt{3}}{2}$$

$$a\sqrt{2} \geq \frac{s(|b-c|+2r)}{n_a}(and\ analogs)$$

We have prove it that:

$$\frac{AN_a}{2r}=\frac{n_a}{h_a}(and\ analogs)\ and\ AN_a^2=(b-c)^2+4r^2(and\ analogs)$$

So, we get:

$$\frac{n_a^2}{h_a^2}=\frac{(b-c)^2+4r^2}{4r^2}=1+\frac{(b-c)^2}{4r^2}(and\ analogs)$$

Now,

$$s^2=n_a^2+2r_ah_a(and\ analogs)$$

$$s^2-n_a^2=2r_ah_a \Leftrightarrow (s-n_a)(s+n_a)=2r_ah_a \Leftrightarrow \frac{s-n_a}{h_a}=\frac{2r_a}{s+n_a}(and\ analogs)$$

$$2S=ah_a=2sr \Rightarrow \frac{s}{h_a}=\frac{a}{2r}(and\ analogs)$$

So, we get:

$$\frac{a}{2r}=\frac{n_a}{h_a}+\frac{2r_a}{s+n_a}(and\ analogs)$$

Now,

$$cot\frac{B}{2}+cot\frac{C}{2}=\sqrt{\frac{r_ar_c}{rr_b}}+\sqrt{\frac{r_ar_b}{rr_c}}=\frac{\sqrt{r_ar_br_c}}{r_b\sqrt{r}}+\frac{\sqrt{r_ar_br_c}}{r_c\sqrt{r}};\ r_ar_br_c=s^2r$$

So, it follows that:

$$cos\frac{B}{2}+cot\frac{C}{2}=\frac{s}{r_b}+\frac{s}{r_c};\ h_a=\frac{2r_br_c}{r_b+r_c}(and\ analogs)$$

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{2s}{h_a}(and\ analogs)$$

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{a}{r}(and\ analogs)$$

So, we get a new inequality:

$$cot\frac{B}{2}+cot\frac{C}{2}=\frac{2n_a}{h_a}+\frac{4r_a}{s+n_a}\ (and\ analogs)$$

$$cot\frac{A}{4}=\frac{AI+s-a}{r}=\frac{AI}{r}+\frac{n_a}{h_a}+\frac{n_b}{h_b}+\frac{n_c}{h_c}+2\sum_{cyc}\frac{r_a}{s+n_a}-\frac{2n_a}{h_a}-\frac{4r_a}{s+n_a}=$$

$$=\frac{AI}{r}+\frac{n_b}{h_b}+\frac{n_c}{h_c}-\frac{n_a}{h_a}+2\left(\frac{r_b}{s+n_b}+\frac{r_c}{s+n_c}-\frac{r_a}{s+n_a}\right)$$

Finally, we get:

$$cot\frac{A}{4}=\frac{AI}{r}+\frac{n_b}{h_b}+\frac{n_c}{h_c}-\frac{n_a}{h_a}+2\left(\frac{r_b}{s+n_b}+\frac{r_c}{s+n_c}-\frac{r_a}{s+n_a}\right)$$

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