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$\varphi(n) = \varphi\left(p_1^{\alpha_1} \cdot p_2^{\alpha_2} \cdot \ldots \cdot p_n^{\alpha_n}\right)$ –Euler's totient function, $n \in \mathbb{N}, n \geq 2$.

Prove that:

$\sum_{k|n} \varphi^2(k) \geq \frac{n^2}{(1+\alpha_1)(1+\alpha_2)\cdot\ldots\cdot(1+\alpha_n)}, p_1, p_2, \ldots$ –prime numbers.

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Solution by Abdul Hannan-Kolkata-India

Let $u(n) = 1, \forall n \in \mathbb{N}$

Also, let $\sigma(n)$ be the number of divisors of n. Then the inequality can be rewritten as

$(\varphi^2 * u)(n) \geq \frac{n^2}{\sigma(n)}$ where $*$ denotes Diriclet convolution.

Since σ is multiplicative, RHS is multiplicative function of n.

Since φ^2 and u are both multiplicative, so is their convolution.

Therefore, it is enough to prove the inequality for $n = p^\alpha$ where $\alpha \geq 1$.i.e.

$$\sum_{k|n} \varphi^2(k) \geq \frac{p^{2\alpha}}{\sigma(p^\alpha)}$$

$$\Leftrightarrow \varphi^2(1) + \varphi^2(p) + \varphi^2(p^2) + \cdots + \varphi^2(p^\alpha) \geq \frac{p^{2\alpha}}{\alpha + 1}$$

$$\Leftrightarrow 1+(p-1)^2+p^2(p-1)^2+\cdots+(p^2)^{\alpha-1}(p-1)^2 \ge \frac{p^{2\alpha}}{\alpha+1}$$

$$\Leftrightarrow 1+(p-1)^2\cdot\frac{p^{2\alpha}-1}{p^2-1} \ge \frac{p^{2\alpha}}{\alpha+1}$$

$$1+\frac{(p-1)(p^{2\alpha}-1)}{p+1} \ge \frac{p^{2\alpha}}{\alpha+1}$$

If $\alpha=1$, then $4 \ge (3-p)p^2 \Leftrightarrow p^3-3p^2+4 \ge 0 \Leftrightarrow (p-2)^2(p+1) \ge 0\ (true)$.

If $\alpha>1$, then $\alpha(p-1) \ge 2(2-1)=2 \Rightarrow$

$$\alpha+2-p\alpha \le 0 \Rightarrow 2(\alpha+1)>0 \ge (\alpha+2-p\alpha)p^{2\alpha}.$$

Note by editor:
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