# Infinite Series

$$\sum_{k=1}^{\infty}\frac{cos(\frac{2\pi k}{3})}{k(k+1)^2}=\frac{\gamma}{2}+\frac{\pi^2}{12}+\frac{\psi(\frac{4}{3})}{2}-\frac{\psi^{'}(\frac{4}{3})}{6}-1$$

*Proof:* Consider the integral,

$$\int_0^1 ln(x)ln(1-x^3)dx$$

This can be solved in two ways, first by expanding $ln(1-x^3)$ and then integrating term by term, hence finding its closed form. Second way is to use logarithmic and complex properties of cube root of unity and then form an infinite series.

First way:

$$\int_0^1 ln(x)ln(1-x^3)dx=\int_0^1 ln(x)\sum_{k=1}^{\infty}\frac{-x^{3k}}{k}dx=-\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{3k}ln(x)dx=\sum_{k=0}^{\infty}\frac{1}{k(3k+1)^2}$$

Now using the method of partial fractions, observe that,

$$\frac{1}{k(3k+1)^2}=\frac{1}{k}-\frac{3}{3k+1}-\frac{3}{(3k+1)^2}=\frac{1}{k}-\frac{1}{k+\frac{1}{3}}-\frac{3}{(3k+1)^2}$$

Now summing both the sides, we obtain,

$$\int_0^1 ln(x)ln(1-x^3)dx=\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+\frac{1}{3}}\right)-\sum_{k=1}^{\infty}\frac{3}{(3k+1)^2}$$

Since,

$$\psi(1+z)=-\gamma+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{k+z}\right)=-\gamma+\sum_{k=1}^{\infty}\frac{z}{k(k+z)}$$

and

$$\psi^{'}(1+z)=\sum_{k=1}^{\infty}\frac{1}{(k+z)^2}$$

Hence,

$$\int_0^1 ln(x)ln(1-x^3)dx = \gamma + \psi(1+\frac{1}{3}) - \frac{\psi^{'}(1+\frac{1}{3})}{3} = \gamma + \psi\Big(\frac{4}{3}\Big) - \frac{\psi^{'}(\frac{4}{3})}{3} \quad (1)$$

Second way:

$$\int_0^1 ln(x)ln(1-x^3)dx = \int_0^1 ln(x)ln((1-x)(1+x+x^2))dx$$

$$= 2 - \frac{\pi^2}{6} + \int_0^1 ln(x)ln\Big(\Big(1-\frac{x}{\omega}\Big)\Big)dx + \int_0^1 ln(x)ln\Big(\Big(1-\frac{x}{\omega^2}\Big)\Big)dx$$

$$= 2 - \frac{\pi^2}{6} - \sum_{k=1}^{\infty}\Big(\int_0^1 \frac{(x/\omega)^k}{k} ln(x)dx\Big) - \sum_{k=1}^{\infty}\Big(\int_0^1 \frac{(x/\omega^2)^k}{k} ln(x)dx\Big)$$

$$= 2 - \frac{\pi^2}{6} + \sum_{k=1}^{\infty}\Big(\frac{\omega^{-k}+\omega^{-2k}}{k(k+1)^2}\Big)$$

Now,

$$\omega^{-k} + \omega^{-2k} = \omega^{-k} + \omega^{k} = 2cos\Big(\frac{2\pi k}{3}\Big)$$

Thus,

$$\int_0^1 ln(x)ln(1-x^3)dx = 2 - \frac{\pi^2}{6} + \sum_{k=1}^{\infty}\left(\frac{2cos\Big(\frac{2\pi k}{3}\Big)}{k(k+1)^2}\right) \quad (2)$$

Hence, using (1) and (2), we obtain,

$$2 - \frac{\pi^2}{6} + \sum_{k=1}^{\infty}\left(\frac{2cos\Big(\frac{2\pi k}{3}\Big)}{k(k+1)^2}\right) = \gamma + \psi\Big(\frac{4}{3}\Big) - \frac{\psi^{'}(\frac{4}{3})}{3}$$

Finally,

$$\sum_{k=1}^{\infty}\frac{cos(\frac{2\pi k}{3})}{k(k+1)^2} = \frac{\gamma}{2} + \frac{\pi^2}{12} + \frac{\psi(\frac{4}{3})}{2} - \frac{\psi^{'}(\frac{4}{3})}{6} - 1$$

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