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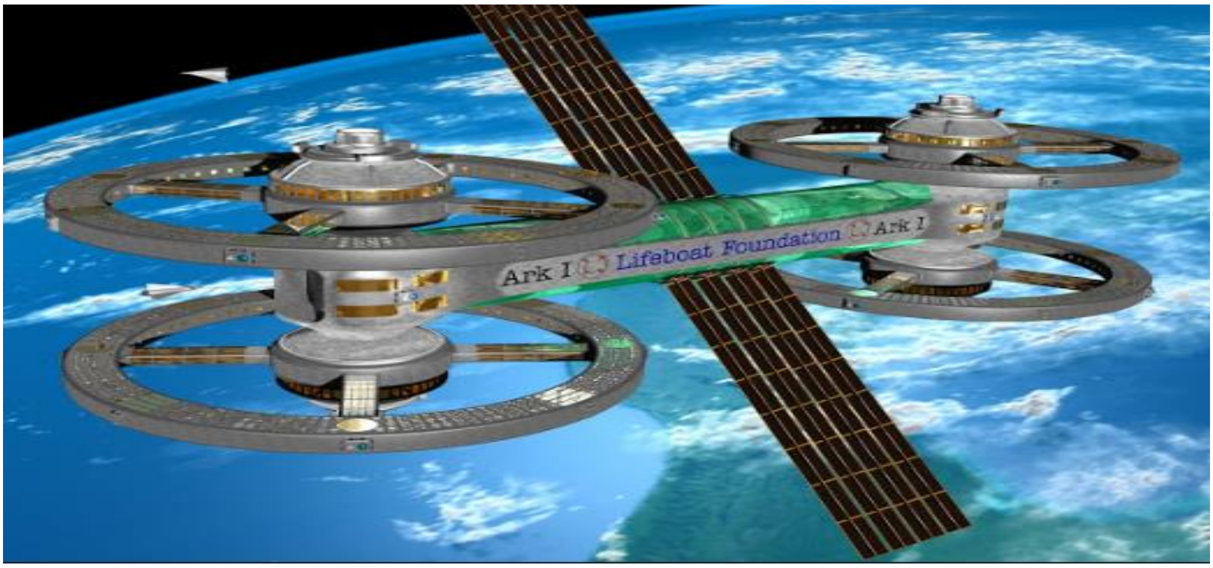

In ΔABC the following relationship holds:

$$\frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)}\le\frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Bogdan Fuştei-Romania,Solution 2 by Soumava Chakraborty-Kolkata-India,Solution 3 by Tran Hong-Dong Thap-Vietnam

Solution 1 by Bogdan Fuştei-Romania

Using Goldstone Inequality:

$$16r^2s^2\le a^2b^2+b^2c^2+c^2a^2\le 4R^2s^2$$

We have: $abc=4RS; a+b+c=2s\Rightarrow$

$$\frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)}\le\frac{4R^2s^2}{4RS\cdot 2s}=\frac{Rs}{2S}=\frac{R}{2r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)}=\frac{(s^2+4Rr+r^2)^2}{8Rrs^2}-2\le\frac{R}{2r}\Leftrightarrow\frac{(s^2+4Rr+r^2)^2}{8Rrs^2}\le\frac{R+4r}{2r}$$

$$\Leftrightarrow 4R(R+4r)s^2\ge(s^2+4Rr+r^2)^2$$

$$\Leftrightarrow s^4-s^2(4R^2+8Rr-2r^2)+r^2(4R+r)^2\overset{(1)}{\lessapprox}0$$

Now, Rouche $\Rightarrow s^2-(m-n)\ge 0$ and $s^2-(m+n)\le 0$, where $m=2R^2+10Rr-r^2$ and $n=2(R-2r)\sqrt{R^2-2Rr}$

$$\therefore\left(s^2-(m+n)\right)\left(s^2-(m-n)\right)\le 0\Rightarrow s^4-s^2(2m)+m^2-n^2\le 0$$

$$\Rightarrow s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3\overset{(i)}{\lessapprox}0$$

$\therefore$ (i) $\Rightarrow$ in order to prove (1), it suffices to prove :

$$s^4-s^2(4R^2+8Rr-2r^2)+r^2(4R+r)^2\le s^4-s^2(4R^2+20Rr-2r^2)+r(4R+r)^3$$

$$\Leftrightarrow 3s^2\le(4R+r)^2\rightarrow\text{true (Trucht)}$$

$$\Rightarrow (1)\text{ is true }\therefore\frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)}\le\frac{R}{2r}\text{ (Proved)}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

In any triangle we have: $\frac{a^3+b^3+c^3+abc}{abc}\le\frac{2R}{r}\Leftrightarrow\frac{1}{4}\cdot\frac{a^3+b^3+c^3+abc}{abc}\le\frac{R}{2r}$

We need to prove:

$$\frac{1}{4}\cdot\frac{a^3+b^3+c^3+abc}{abc}\ge\frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)}\Leftrightarrow$$

$$(a+b+c)(a^3+b^3+c^3+abc)\ge 4(a^2b^2+b^2c^2+c^2a^2)\Leftrightarrow$$

$$a^4+b^4+c^4+abc(a+b+c)+ab(a^2+b^2)+bc(b^2+c^2)$$
$$+ca(c^2+a^2)\overset{(*)}{\ge}4(a^2b^2+b^2c^2+c^2a^2)$$

But: $a^4+b^4+c^4+abc(a+b+c)\overset{Schur}{\ge}ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\Rightarrow$

$$a^4+b^4+c^4+abc(a+b+c)+ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ge$$
$$\ge 2[ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)]\overset{(**)}{\ge}$$
$$\ge 4(a^2b^2+b^2c^2+c^2a^2)\Leftrightarrow$$

$$ab(a^2+b^2)+bc(b^2+c^2)+ca(c^2+a^2)\ge 2(a^2b^2+b^2c^2+c^2a^2)\Leftrightarrow$$

$$ab(a-b)^2+bc(b-c)^2+ca(c-a)^2\ge 0$$

Which is clearly true$\Rightarrow$ $(**)$true$\Rightarrow$ $(*)$ true.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solutions.