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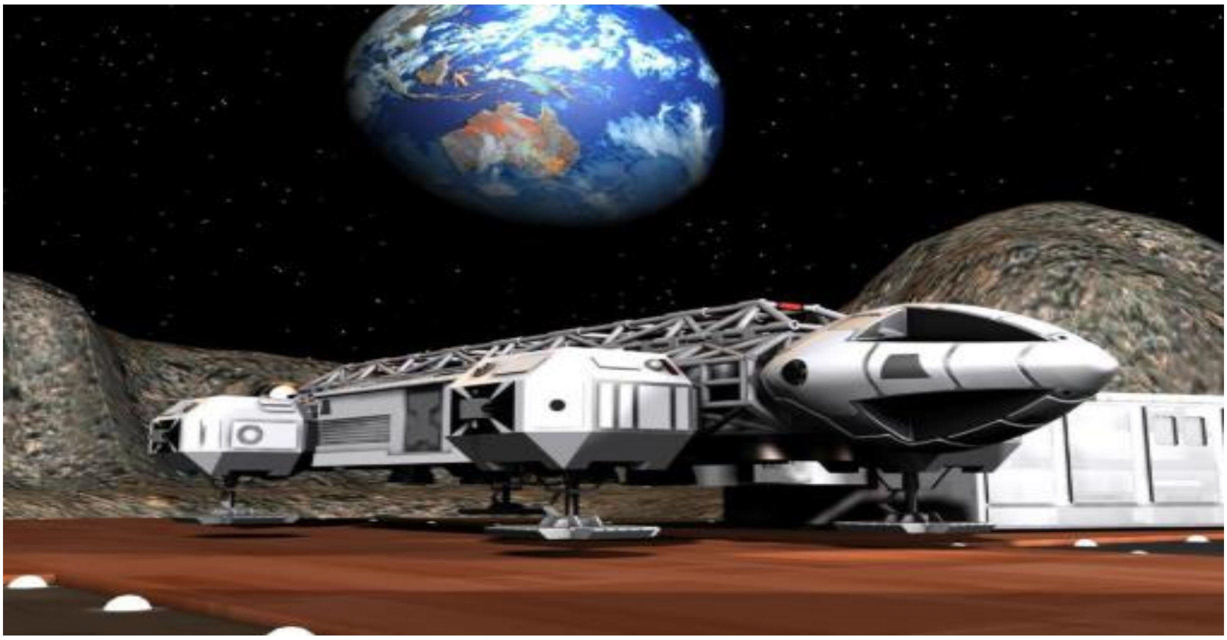

In $\Delta ABC, n_a$ –Nagel's cevian, g_a –Gergonne's cevian the following relationship holds:

$$\frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \overset{(1)}{\geq} \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^2$$

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$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$$

$$\therefore an_a^2 . ag_a^2 \geq a^2 s^2 (s-a)^2$$

$$\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\}\{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \overset{(a)}{\geq} a^2 s^2 (s-a)^2$$

Let $s-a = x, s-b = y$ and $s-c = z \therefore s = x+y+z \Rightarrow a = y+z, b = z+x$ and $c = x+y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \geq s(s-a) \text{ and analogs}$$

$$\Rightarrow \sum n_a g_a \ge s\sum(s-a) = s^2 \Rightarrow \left(\sum n_a g_a\right)\left(\sum h_a h_b\right)^{-1} \ge s^2\left\{\sum\left(\frac{bc}{2R}\right)\left(\frac{ca}{2R}\right)\right\}^{-1}$$

$$= s^2\left\{\left(\frac{4Rrs}{4R^2}\right)^{-1}\right\}\left\{\left(\sum a\right)^{-1}\right\} = \frac{Rs^2}{2s.rs} = \frac{R}{2r}$$

$$\Rightarrow \frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \overset{(m)}{\gtrsim} \frac{R}{2r}$$

$$\text{Now, } r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{\cong} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \ge 32Rr\cos^2\frac{A}{2} \overset{\text{by (i)}}{\cong} 8r(r_b + r_c) = 8r^2 s\left(\frac{1}{s-b} + \frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \ge 0 \Leftrightarrow (b+c-2a)^2 \ge 0 \to \text{true} \therefore b+c$$

$$\ge 4\sqrt{2Rr}\cos\frac{A}{2} \Rightarrow \sum m_a \overset{\text{Ioscu}}{\gtrsim} \sum\left(\frac{b+c}{2}\cos\frac{A}{2}\right)$$

$$\ge \sqrt{2Rr}\sum 2\cos^2\frac{A}{2} = \sqrt{2Rr}\sum(1+\cos A) = \sqrt{2Rr}\left(4 + \frac{r}{R}\right) = \sqrt{\frac{2r}{R}}\left(\sum r_a\right)$$

$$\Rightarrow \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c}\right)^2 \overset{(n)}{\lesssim} \frac{R}{2r} (m), (n) \Rightarrow (1) \text{ is true (QED)}$$