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In $\Delta ABC, I$ –incenter, the following relationship holds:

$$\frac{r_a r_b r_c}{h_a h_b h_c} \geq \left(\frac{r_a + r_b + r_c + AI + BI + CI}{m_a + m_b + m_c + h_a + h_b + h_c - 3r}\right)^2$$

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$$r_b + r_c = s\left(\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}} + \frac{\sin\frac{C}{2}}{\cos\frac{C}{2}}\right) = \frac{s\sin\left(\frac{B+C}{2}\right)\cos\frac{A}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}} = \frac{s\cos^2\frac{A}{2}}{\left(\frac{s}{4R}\right)} = 4R\cos^2\frac{A}{2}$$

$$\therefore r_b + r_c \overset{(i)}{=} 4R\cos^2\frac{A}{2}$$

$$\text{Now, } (b+c)^2 \geq 32Rr\cos^2\frac{A}{2} \overset{\text{by (i)}}{=} 8r(r_b + r_c) = 8r^2 s\left(\frac{1}{s-b} + \frac{1}{s-c}\right)$$

$$= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \to \text{true} \therefore b+c$$

$$\geq 4\sqrt{2Rr}\cos\frac{A}{2} \text{ and analogs} \Rightarrow \sum m_a \overset{\text{Ioscu}}{\geq} \sum \frac{b+c}{2}\cos\frac{A}{2}$$

$$\geq \sqrt{2Rr}\sum\left(2\cos^2\frac{A}{2}\right) = \sqrt{2Rr}\sum(1+\cos A) = \sqrt{2Rr}\left(\frac{4R+r}{R}\right) = \sqrt{\frac{2r}{R}}(4R+r)$$

$$\Rightarrow \sqrt{\frac{R}{2r}}\sum m_a \overset{(ii)}{\geq} \sum r_a$$

Again, $(s^2-2Rr+r^2)^2 \geq 8Rr(s^2+4Rr+r^2)$

$$\Leftrightarrow s^4 \overset{(a)}{\geq} (12Rr-2r^2)s^2 + 8Rr(4Rr+r^2) - (2Rr-r^2)^2$$

Now, LHS of $(a) \overset{\text{Gerretsen}}{\geq} (16Rr-5r^2)s^2 \overset{?}{\geq} (12Rr-2r^2)s^2 + 8Rr(4Rr+r^2) - (2Rr-r^2)^2$

$$\Leftrightarrow (4Rr-3r^2)s^2 \underset{(b)}{\overset{?}{\geq}} 8Rr(4Rr+r^2) - (2Rr-r^2)^2$$

Again, LHS of (b) $\overset{\text{Gerretsen}}{\geq} (4Rr-3r^2)(16Rr-5r^2) \overset{?}{\geq} 8Rr(4Rr+r^2) - (2Rr-r^2)^2$

$$\Leftrightarrow 9R^2 - 20Rr + 4r^2 \overset{?}{\geq} 0$$

$\Leftrightarrow (R-2r)(9R-2r) \overset{?}{\geq} 0 \to$ true $\because R \overset{\text{Euler}}{\geq} 2r \Rightarrow$ (b) $\Rightarrow (a)$ is true

$$\therefore s^2 - 2Rr + r^2 \overset{(iii)}{\geq} 2\sqrt{2Rr}\sqrt{s^2+4Rr+r^2}$$

Now, $\sum AI = r\sum\frac{1}{\sin\frac{A}{2}} = r\sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}} \overset{\text{CBS}}{\leq} \frac{r}{r\sqrt{s}}\sqrt{\sum bc}\sqrt{\sum(s-a)}$

$$= \sqrt{s^2+4Rr+r^2} \overset{\text{by (iii)}}{\leq} \frac{s^2-2Rr+r^2}{2\sqrt{2Rr}}$$

$$\Rightarrow s^2 - 2Rr + r^2 \overset{(iv)}{\geq} 2\sqrt{2Rr}\sum AI$$

Also, $h_a + h_b + h_c - 3r = \frac{s^2+4Rr+r^2}{2R} - 3r = \frac{s^2-2Rr+r^2}{2R} \overset{\text{by (iv)}}{\geq} \frac{2\sqrt{2Rr}}{2R}\sum AI$

$$\Rightarrow \sqrt{\frac{R}{2r}}(h_a+h_b+h_c-3r) \overset{(v)}{\geq} \sum AI$$

(ii) + (v) $\Rightarrow \sqrt{\frac{R}{2r}}(m_a+m_b+m_c+h_a+h_b+h_c-3r) \geq r_a+r_b+r_c+AI+BI+CI$

$$\Rightarrow \frac{R}{2r} \geq \left(\frac{r_a+r_b+r_c+AI+BI+CI}{m_a+m_b+m_c+h_a+h_b+h_c-3r}\right)^2 \Rightarrow \frac{r_ar_br_c}{h_ah_bh_c} = \frac{rs^2}{\left(\frac{8r^3s^3}{4Rrs}\right)} = \frac{R}{2r}$$

$$\geq \left(\frac{r_a+r_b+r_c+AI+BI+CI}{m_a+m_b+m_c+h_a+h_b+h_c-3r}\right)^2 \text{ (Proved)}$$