Generalised Ramanujan's Master Theorem

1 Abstract

Ramanujan's master theorem is a technique that provides an analytic expression for the Mellin transform of an analytic function. In this paper, I prove a generalised version of Ramanujan's master theorem by following the same procedure as was followed by Ramanujan in his First Quarterly Report.

2 Main Proof

Theorem 2.1. Let $f(x)$ and $F(x)$ be two analytic functions on $(0,\infty)$ such that the Mellin transform of $f(x)$ is $\phi(n)$, where n is a positive integer and the series expansion of $F(x)$ is given by

$$F(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} \eta(k) x^k$$

then, the Mellin transform of $F(x)$ is given by,

$$\int_0^\infty x^{n-1} F(x) dx = \phi(n)\eta(-n)$$

Proof: Since the Mellin transform of $f(x)$ is $\phi(n)$, we have,

$$\int_0^\infty x^{n-1} f(x) dx = \phi(n)$$

Substituting $x = az$, we obtain,

$$a^n \int_0^\infty z^{n-1} f(az) dz = \phi(n) \Longrightarrow \int_0^\infty z^{n-1} f(az) dz = a^{-n}\phi(n)$$

Put $a = y^k$ and multiply both the sides with $\frac{f^{(k)}(b)t^k}{k!}$, we obtain,

$$\frac{f^{(k)}(b)t^k}{k!} \int_0^\infty z^{n-1} f(y^k z) dz = y^{-kn}\phi(n) \frac{f^{(k)}(b)t^k}{k!}$$

Summing both the sides from $k = 0$ to $k = \infty$, we have,

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(b)t^k}{k!} \int_0^\infty z^{n-1} f(y^k z) dz = \sum_{k=0}^{\infty} y^{-kn} \phi(n) \frac{f^{(k)}(b)t^k}{k!}$$

Inverting the order of summation and integration and simplifying,

$$\int_0^\infty z^{n-1} \sum_{k=0}^{\infty} \frac{f^{(k)}(b)t^k}{k!} f(y^k z) dz = \phi(n) \sum_{k=0}^{\infty} \frac{f^{(k)}(b)(ty^{-n})^k}{k!} \tag{1}$$

We know from Taylor's theorem for the series expansion of $f(x)$ that,

$$f(x) = \sum_{m=0}^{\infty} \frac{f^{(m)}(r)(x-r)^m}{m!}$$

Thus,

$$f(y^k z) = \sum_{m=0}^{\infty} \frac{f^{(m)}(r)(y^k z - r)^m}{m!} \tag{2}$$

Using (2) in (1), we obtain,

$$\int_0^\infty z^{n-1} \sum_{k=0}^{\infty} \frac{f^{(k)}(b)t^k}{k!} \sum_{m=0}^{\infty} \frac{f^{(m)}(r)(y^k z - r)^m}{m!} dz = \phi(n) f(ty^{-n} + b)$$

Again inverting the order of summation, we have,

$$\int_0^\infty z^{n-1} \sum_{m=0}^{\infty} \frac{f^{(m)}(r)}{m!} \sum_{k=0}^{\infty} \frac{f^{(k)}(b)t^k(y^k z - r)^m}{k!} dz = \phi(n) f(ty^{-n} + b)$$

Put $r = 0$ and substituting $z = x$, we obtain,

$$\int_0^\infty x^{n-1} \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} \sum_{k=0}^{\infty} \frac{f^{(k)}(b)t^k(y^k x)^m}{k!} dx = \phi(n) f(ty^{-n} + b)$$

Again re-arranging the terms and employing Taylor's theorem, we have,

$$\int_0^\infty x^{n-1} \sum_{m=0}^{\infty} \frac{f^{(m)}(0)}{m!} f(ty^m + b) x^m dx = \phi(n) f(ty^{-n} + b)$$

Since t,y and b are constants, we can replace $f(ty^m+b)$ by $\eta(m)$ and $f(ty^{-n}+b)$ by $\eta(-n)$, also we can replace m by k since m is just a dummy variable, thus we obtain,

$$\int_0^\infty x^{n-1}\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\eta(k)x^k dx = \phi(n)\eta(-n)$$

Since, the summation $\sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}\eta(k)x^k$ is just the series expansion of $F(x)$, we finally have,

$$\int_0^\infty x^{n-1}F(x)dx = \phi(n)\eta(-n)$$

This completes the proof.

Corollary 2.1. If we put $f(x) = e^{-x}$, then we get Ramanujan's master theorem as a special case, since $\phi(n) = \Gamma(n)$ and $f^{(k)}(0) = (-1)^k$.

Corollary 2.2. If we put $f(x) = \frac{x}{e^x-1}$, then $F(x)$ is given by,

$$F(x) = \sum_{k=0}^{\infty}\frac{B_k}{k!}\eta(k)x^k = \sum_{k=0}^{\infty}\frac{(-1)^k B_k}{k!}\eta(k)(-x)^k \tag{3}$$

where B_k is the k^{th} Bernoulli number, then we have,

$$\int_0^\infty x^{n-1}F(x)dx = \Gamma(n+1)\zeta(n+1)\eta(-n)$$

Thus,

$$\int_0^\infty F(x)dx = \frac{\pi^2\eta(-1)}{6}$$

Now,

$$\Gamma(n+1)\zeta(n+1)\eta(-n) = \Gamma(n)(n\zeta(n+1)\eta(-n)) \tag{4}$$

Thus by comparing the coefficients of $(-x)^k$ in (3) with (4), we have,

$$(-1)^n B_n\eta(n) = (-n)\zeta(-n+1)\eta(n)$$

Since B_n vanishes for odd values of n, we have,

$$\zeta(-n) = \frac{-B_{n+1}}{n+1}$$

for $n = 1, 3, 5...$

Corollary 2.3. It is easy to show that if $f(x) = \frac{1}{(1+x^p)^q}$, then,

$$\phi(n) = \frac{\Gamma(\frac{n}{p})\Gamma(q - \frac{n}{p})}{p\Gamma(q)}$$

Hence,

$$\int_0^\infty x^{n-1}F(x)dx = \frac{\eta(-n)\Gamma(\frac{n}{p})\Gamma(q - \frac{n}{p})}{p\Gamma(q)}$$

3 References

[1] B.C. Berndt, Ramanujans Notebooks: Part I. New York: SpringerVerlag, p. 298, 1985.

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