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If $A, B \in M_n(\mathbb{R}), AB + BA = O_n$ then:

$$det(A^2 + B^2) \geq det\big((A - B)(A + B)\big)$$

Proposed by Marian Ursărescu-Romania

Solution by Santos Martins Junior-Brusels-Belgium

Consider the following product's:

1) $(A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + B^2$

2) $(A - B)(A - B) = A^2 - (AB + BA) + B^2 = A^2 + B^2$

Hence,

$$det(A + B)(A + B) = det(A + B)det(A + B) = det(A^2 + B^2); \quad (3)$$

$$det(A - B)(A - B) = det(A - B)det(A - B) = det(A^2 + B^2); \quad (4)$$

Doing $(3) \cdot (4)$: $det^2(A + B)det^2(A - B) = det^2(A^2 + B^2) \in \mathbb{R}$

Hence, we can rearrange the following:

$$\big(det(A - B)det(A + B)\big)^2 = det^2(A^2 + B^2)$$

$$det^2\big((A - B)(A + B)\big) = det^2(A^2 + B^2)$$

1) $\boldsymbol{det(A^2+B^2)=det\big((A-B)(A+B)\big)}$

2) $\boldsymbol{det(A^2+B^2)=-det\big((A-B)(A+B)\big)}$

Therefore,

$$\boldsymbol{det(A^2+B^2)\geq det\big((A-B)(A+B)\big)}$$

Note by editor:

Many thanks to Florică Anastase-Romania for typed solution.