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$$z_1, z_2, z_3 \in \mathbb{C} - \{0\}, \text{ different in pairs,}$$

$$|z_1| = |z_2| = |z_3| = 1, A(z_1), B(z_2), C(z_3)$$

$$\sum_{cyc} \left| \frac{(z_1 - z_2)(z_1 - z_3)}{2z_1 - z_2 - z_3} \right| = 3 \Rightarrow AB = BC = CA$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan, Solution 2 by Adrian Popa-Romania

Solution 1 by Rovsen Pirguliyev-Sumgait-Azerbaijan

Denote $A(z_1), B(z_2), C(z_3)$, then $AB = |z_1 - z_2|, BC = |z_2 - z_3|, AC = |z_1 - z_3|$

$$\sum_{cyc} \left| \frac{(z_1 - z_2)(z_1 - z_3)}{2z_1 - z_2 - z_3} \right| = \sum_{cyc} \left| \frac{(z_1 - z_2)(z_1 - z_3)}{(z_1 - z_2) + (z_1 - z_3)} \right| = \sum_{cyc} \frac{AB \cdot AC}{AB + AC} = 3; \quad (1)$$

Since, $\frac{ab}{a+b} \le \frac{a+b}{4}$, then

$$3 = \sum_{cyc} \frac{AB \cdot AC}{AB + AC} \le \frac{AB + BC + CA}{2} \Rightarrow AB + BC + CA \ge 6 \Rightarrow$$

$$|z_1 - z_2| + |z_2 - z_3| + |z_1 - z_3| \ge 6; \quad (2)$$

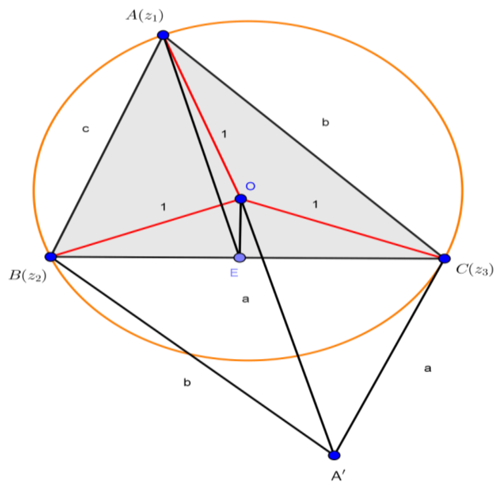
Using: $|z_1 - z_2| \le |z_1| + |z_2|$ we have $\begin{cases} |z_1 - z_2| \le 2 \\ |z_2 - z_3| \le 2 \Rightarrow \\ |z_1 - z_3| \le 2 \end{cases}$

$$|z_1 - z_2| + |z_2 - z_3| + |z_1 - z_3| \le 6; \ \ (3)$$

From (2),(3) we get: $|z_1 - z_2| + |z_2 - z_3| + |z_1 - z_3| = 6$

Equality holds if $|z_1 - z_2| = |z_2 - z_3| = |z_1 - z_3| \Rightarrow AB = BC = CA.$

Solution 2 by Adrian Popa-Romania



$$|z_1| = |z_2| = |z_3| = 1 \Rightarrow \ A(z_1), B(z_2), C(z_3) \in C(O, 1)$$

$$a < 2, b < 2, c < 2 \Rightarrow a + b + c < 6$$

$$|z_1 - z_2| = \left|\overrightarrow{OA} - \overrightarrow{OB}\right| = \left|\overrightarrow{BA}\right| = c$$

Similarly: $|z_2 - z_3| = b, |z_1 - z_3| = c$

$$|2z_1 - z_2 - z_3| = |z_1 - z_2 + z_1 - z_3| = \left|\overrightarrow{OA} - \overrightarrow{OB} + \overrightarrow{OA} - \overrightarrow{OC}\right| =$$

$$= \left|\overrightarrow{BA} + \overrightarrow{CA}\right| = \left|\overrightarrow{AA'}\right| = 2m_a$$

$$\frac{bc}{2m_a} = \frac{ac}{2m_b} = \frac{ab}{2m_c} = 3$$

$$\begin{cases} 2m_a < b + c \\ 2m_b < c + a \\ 2m_c < a + b \end{cases} \Rightarrow 3 \ge \frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b}; \ \ (1)$$

But $\frac{bc}{b+c} < \frac{b+c}{4}$ and analogs$\Rightarrow \sum \frac{bc}{b+c} < \sum \frac{b+c}{4} = \frac{2(a+b+c)}{4} = \frac{a+b+c}{2} \le \frac{6}{3} = 3; \ \ (2)$

From (1),(2) we get:

$$\frac{bc}{b+c} + \frac{ca}{c+a} + \frac{ab}{a+b} = 6$$

$$\frac{bc}{b+c}+\frac{ca}{c+a}+\frac{ab}{a+b}\le\frac{b+c}{4}+\frac{c+a}{4}+\frac{a+b}{4}=\frac{a+b+c}{2}$$

$3<\frac{a+b+c}{2}\Rightarrow a+b+c>6$, but $a+b+c<6$ then $a+b+c=6$.

So,

$$|z_1-z_2|+|z_2-z_3|+|z_1-z_3|=6$$

$$\begin{cases}|z_1-z_2|\le|z_1|+|z_2|\\|z_2-z_3|\le|z_2|+|z_3|\\|z_3-z_1|\le|z_3|+|z_1|\end{cases}$$

Equality holds if $|z_1-z_2|=|z_2-z_3|=|z_1-z_3|\Rightarrow AB=BC=CA$.

Note by editor:

Many thanks to Florică Anastase-Romania for typed solutions.