Interesting Problem

If $G(n)$ is defined as follows,

$$G(n) = \int_0^1 ln(x)ln(1-x^n)dx$$

then find the value of

$$\int_0^{1/2} G\Big(\frac{1}{x}\Big)dx$$

Solution:

$$G(n) = \int_0^1 ln(x)ln(1-x^n)dx = \int_0^1 ln(x)\sum_{k=1}^{\infty}\frac{-x^{nk}}{k}dx$$

$$= -\sum_{k=1}^{\infty}\frac{1}{k}\int_0^1 x^{kn}ln(x)dx = \sum_{k=0}^{\infty}\frac{1}{k(kn+1)^2}$$

Now,

$$\frac{1}{k(kn+1)^2} = \frac{1}{k} - \frac{n}{kn+1} - \frac{n}{(nk+1)^2} = \frac{1}{k} - \frac{1}{k+\frac{1}{n}} - \frac{n}{(nk+1)^2}$$

Hence,

$$G(n) = \sum_{k=1}^{\infty}\Big(\frac{1}{k} - \frac{1}{k+\frac{1}{n}}\Big) - \sum_{k=1}^{\infty}\frac{n}{(nk+1)^2} = \gamma + \psi\Big(1+\frac{1}{n}\Big) - \frac{\psi'(1+\frac{1}{n})}{n}$$

Now, replacing x by t (since x is just a dummy variable) and n by $\frac{1}{x}$ and integrating both the sides from 0 to t, we obtain,

$$\int_0^t G\Big(\frac{1}{x}\Big)dx = \int_0^t \Big(\gamma + \psi(1+x) - x\psi'(1+x)\Big)dx$$

$$= \gamma t + ln(\Gamma(1+t)) - \Big(t\psi(1+t) - \int_0^t \psi(1+x)dx\Big)$$

$$= \gamma t + ln(\Gamma(1+t)) - t\psi(1+t) + \int_0^t \psi(1+x)dx$$

$$= \gamma t + ln(\Gamma(1+t)) - t\psi(1+t) + ln(\Gamma(1+t))$$

$$= \gamma t + 2ln(t\Gamma(t)) - t\psi(1+t)$$

Thus,

$$\int_0^t G\Big(\frac{1}{x}\Big)dx = \gamma t + 2ln(t\Gamma(t)) - t\psi(1+t)$$

Put $t = 1/2$, we get,

$$\int_0^{1/2} G\Big(\frac{1}{x}\Big)dx = \frac{\gamma}{2} + 2ln\Big(\frac{1}{2}\Gamma\Big(\frac{1}{2}\Big)\Big) - \frac{1}{2}\psi\Big(\frac{3}{2}\Big)$$

$$= \frac{\gamma}{2} + 2ln\Big(\frac{\sqrt{\pi}}{2}\Big) - \frac{1}{2}\Big(-\gamma - 2ln(2) + 2\Big)$$

$$= \gamma + ln\Big(\frac{\pi}{2e}\Big)$$

Finally,

$$\int_0^{1/2} G\Big(\frac{1}{x}\Big)dx = \gamma + ln\Big(\frac{\pi}{2e}\Big)$$

ANGAD SINGH

Department of Electronics and Telecommunications, Pune Institute of Computer Technology, Pune, India

email-id: angadsingh1729@gmail.com