Proposed by Narendra Bhandari, Bajura,Nepal and published by *Romanian mathematical Magazine*

Let $\Re(k) > 0$ and if

$$\sum_{1\le n\le m\le\infty} \frac{1}{n^2m + m^2n + kmn} = \frac{(H_{k+1})^2 - \psi^1(k+\alpha)}{k+\beta} + \frac{\pi^\alpha}{\lambda(k+\beta)}\textbf{and}$$

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} \frac{x^n}{(q+b)\sqrt{q+b+1} + (q+b+1)\sqrt{q+b}} = \frac{1}{\sqrt{b}(1-x)} - \Phi\left(x, \frac{1}{\alpha}, b+1\right)$$

where $b \in \mathbf{N}$ and $|x| < 1$ then prove that $\Phi\left(\beta-\alpha, \beta, (\alpha+2\beta+\lambda)^{-1}\right) =$

$$\frac{2\pi}{\sqrt{5}-1} + \sqrt{\phi+2}\log\left(\theta - \frac{8\theta}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right) + 2^{-1}\sqrt{10-2\sqrt{5}}\log\left(\frac{\sqrt{3}\theta-1}{\theta+\sqrt{3}}\right)$$

and

$$\theta = \sqrt{\frac{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{8-\sqrt{10-2\sqrt{5}}-\sqrt{15}-\sqrt{3}}} = \frac{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{36-4\sqrt{5}-4\sqrt{6(5+\sqrt{5})}}$$

where $\Phi(z, s, a)$ is Lerch Transcendent function , H_k is the Kth Harmonic number, $\psi^1(x)$ is the trigamma function and ϕ is the Golden ratio.

Solution by proposer

Before we start with the problem we shall consider the integral of the form $I(n) = \int_0^1 x^{n-1}\ln(1-x)dx = \frac{1}{n}\int_0^1 \frac{d}{dx}(x^n-1)\ln(1-x)dx$ and by integration by parts we yield

$$I(n) = \frac{1}{n}\left[\underbrace{(x^n-1)\ln(1-x)}_{0}\right]_0^1 - \frac{1}{n}\int_0^1 \frac{1-x^n}{1-x}dx$$

$$= -\frac{1}{n}\int_0^1 \sum_{j=1}^{n} x^{j-1} = -\frac{1}{n}\sum_{j=1}^{n}\int_0^1 x^{j-1}dx = -\frac{1}{n}\sum_{j=1}^{n}\frac{1}{j} = -\frac{H_n}{n} \cdots (1)$$

Note that $D(k) = \dfrac{1}{n^2m + m^2n + kmn} = \dfrac{1}{mn}\displaystyle\int_0^1 x^{m+n+k-1}dx = \dfrac{x^{k-1}}{mn}\int_0^1 x^{m+n}dx$ and therefore

$$\sum_{1\le n\le m\le\infty} D(k) = \int_0^1 x^{k-1}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^n}{n}\cdot\frac{x^m}{m}dx = \int_0^1 x^{k-1}\ln^2(1-x)dx$$

Now we consider the latter integral $I(k) = \displaystyle\int_0^1 x^k \ln^2(1-x)dx$ for $k \ge 0$ which further can be written as $\dfrac{1}{k+1}\displaystyle\int_0^1 \frac{d}{dx}(x^{k+1}-1)\ln^2(1-x)dx$ and hence on

Integration by parts we see that

$$I(k) = \frac{1}{k+1}\left[\underbrace{(x^{k+1}-1)\ln^2(1-x)}_{0}\right]_0^1 - \frac{2}{k+1}\int_0^1 \frac{1-x^{k+1}}{1-x}\ln(1-x)dx$$

$$= -\frac{2}{k+1}\int_0^1 \sum_{n=1}^{k+1} x^{n-1}\ln(1-x) = -\frac{2}{k+1}\sum_{n=1}^{k+1}\int_0^1 x^{n-1}\ln(1-x)dx$$

Plugging the result from (1) to last integral we have

$$\frac{2}{k+1}\sum_{n=1}^{k+1}\frac{H_n}{n} = \frac{2}{k+1}\left(\frac{H_{k+1}^2 + H_{k+1}^{(2)}}{2}\right) = \frac{H_{k+1}^2 + H_{k+1}^{(2)}}{k+1}$$

Further note that nth partial sum of $H_{k+1}^{(2)} = \zeta(2) - \psi^1(k+2)$ giving us required result of

$$D(k) = \frac{H_{k+1}^2 - \psi^1(k+2)}{k+1} + \frac{\pi^2}{6(k+1)}$$

Now we notice that

$$\frac{1}{(q+b)\sqrt{q+b++1} + (q+b+1)\sqrt{q+b}} = \frac{(q+b)\sqrt{q+b+1} - (q+b+1)\sqrt{q+b}}{(q+b)\sqrt{q+b+1}(-1)}$$

$$= \frac{1}{\sqrt{q+b}} - \frac{1}{\sqrt{q+b+1}}$$

and hence

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} x^n \left(\underbrace{\frac{1}{\sqrt{q+b}} - \frac{1}{\sqrt{q+b+1}}}_{Telescoping}\right) = \sum_{n=0}^{\infty} x^n\left(\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{n+b+1}}\right)$$

and by hence by algebra of sum and for for $|x| < 1$

$$\sum_{n=0}^{\infty}\frac{x^n}{\sqrt{b}} - \sum_{n=0}^{\infty}\frac{x^n}{\sqrt{n+b+1}} = \frac{1}{\sqrt{b}(1-x)} - \Phi\left(x, \frac{1}{2}, 1\right)$$

and we deduce that $\alpha = 2, \beta = 1, \gamma = 6$. Thus

$$\Phi\left(\beta-\alpha, \beta, (\alpha+2\beta\gamma)^{-1}\right) = \Phi\left(-1, 1, \frac{1}{10}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{\left(n+\frac{1}{10}\right)} = D$$

$$D = \sum_{n=0}^{\infty}\frac{1}{(2n+10^{-1})(2n+10^{-1}+1)} = \frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{(n+20^{-1})(n+20^{-1}+2^{-1})}$$

$$= 2^{-1}\left(\psi\left(\frac{1}{2}+\frac{1}{10}\right)-\psi\left(\frac{1}{20}\right)\right) = 2^{-1}\left(\psi\left(\frac{1}{10}\right)-2\psi\left(\frac{1}{20}\right)-2\ln 2\right)$$

and finally we have $D = \psi\left(\frac{1}{10}\right)-\psi\left(\frac{1}{20}\right)-\ln 2$. Here to evaluate this expression D , we use the **Gauss's-Diagamma theorem**, ie

$$\psi\left(\frac{r}{m}\right) = -\gamma-\ln(2m)-\frac{\pi}{2}\cot\left(\frac{r\pi}{m}\right)+2\sum_{q=1}^{\lfloor\frac{m-1}{2}\rfloor}\cos\left(\frac{2\pi qr}{m}\right)\ln\sin\left(\frac{\pi q}{n}\right)$$

plugging $r=1, m=10,20$ we have

$$I = \psi\left(\frac{1}{10}\right) = -\gamma-\ln(20)-\frac{\pi}{2}\cot\left(\frac{\pi}{10}\right)+2\sum_{q=1}^{4}\cos\left(\frac{\pi q}{5}\right)\ln\sin\left(\frac{\pi q}{10}\right)$$

and

$$K = \psi\left(\frac{1}{20}\right) = -\gamma-\ln(40)-\frac{\pi}{2}\cot\left(\frac{\pi}{20}\right)+2\sum_{q=1}^{9}\cos\left(\frac{\pi q}{10}\right)\ln\sin\left(\frac{\pi q}{20}\right)$$

and also $D = I-k-\ln 2$ and on further simplification of the D we obtain that

$$D = \frac{\pi}{4}\frac{1}{\sin\left(\frac{1}{20}\right)\cos\left(\frac{1}{20}\right)}+A\left(\log\cot\left(\frac{\pi}{20}\right)\right)+B\left(\log\cot\left(\frac{3\pi}{20}\right)\right)$$

with $A=\sqrt{\frac{5+\sqrt{5}}{2}}=\sqrt{\phi+2}$ and $B=\sqrt{\frac{5-\sqrt{5}}{2}}=2^{-1}\sqrt{10-2\sqrt{5}}$ and $\frac{\pi}{4}\frac{1}{\sin\left(\frac{\pi}{20}\right)\cos\left(\frac{\pi}{20}\right)}$ $=\frac{\pi}{2}\frac{1}{\sin(18^\circ)}=\frac{2\pi}{\sqrt{5}-1}$ since $\sin(18^\circ)=4^{-1}\left(\sqrt{5}-1\right)$.

Further we have $\cot\left(\frac{\pi}{20}\right)=\cot(9^\circ)$. To evaluate $\cot(9^\circ$ we shall be using the fact that

$$\cot(3x)=\frac{\cos(3x)}{\sin(3x)}=\frac{4\cos^3 x-3\cos x}{3\sin x-4\sin^3 x}=\frac{\cos x(1-4\sin^2 x)}{\sin x(3-4\sin^2 x)}=\cot x\left(1-\frac{2}{3-4\sin^2 x}\right)$$

setting $x=3$ we have the expression for $\cot 9^\circ$ but then we will show that

$$\sin 3^\circ = \frac{1}{4}\sqrt{8-\sqrt{10-2\sqrt{5}}-\sqrt{15}-\sqrt{3}}$$

Note: This identity is proposed by **Narendra Bhandari** and is proved by **Sergio Esteban**(beautiful geometry work) and **Ahmed Hegazi**(trigonometry work). From half angle formula we have that

$$\sin\frac{\phi}{2}=\sqrt{\frac{1-\cos\phi}{2}}$$

now set $\phi = 6°$. Giving us

$$\sin 3° = \sqrt{\frac{1-\cos 6°}{2}} = \sqrt{\frac{1-\cos(36°-30°)}{2}}$$

Now using compound angle formula we can get $\cos(36°-30°) = \cos 36° \cos 30° + \sin 36° \sin 30° = \left(\frac{\sqrt{5}+1}{4}\right)\frac{\sqrt{3}}{2} + \left(\frac{\sqrt{10-2\sqrt{5}}}{4}\right)\frac{1}{2} = \frac{\sqrt{10-2\sqrt{5}}+\sqrt{3}+\sqrt{15}}{8}$ putting in original equation

$$\sin 3° = \frac{1}{\sqrt{2}}\left(\sqrt{1-\frac{\sqrt{10-2\sqrt{5}}+\sqrt{3}+\sqrt{15}}{8}}\right) = \frac{1}{4}\left(\sqrt{8-\sqrt{10-2\sqrt{5}}-\sqrt{3}-\sqrt{15}}\right)$$

and hence

$$\cos 3° = \sqrt{1-\sin^2 3°} = \frac{1}{4}\sqrt{8+\sqrt{10-2\sqrt{5}}+\sqrt{10}+\sqrt{3}}$$

and hence we deduce easily that

$$\cot 9° = \cot 3° \left(1 - \frac{8}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right)$$

also $\cot\left(\frac{3\pi}{20}\right) = \cot(27° = \cot(30-3)$ and using compound angle formula for $cot(a-b)$ we have

$$\cot 27° = \frac{\cot 3° \cot 30° - 1}{\cot 3° + \cot 30°} = \frac{\sqrt{3}\cot 3° - 1}{\cot 3° + \sqrt{3}}$$

and hence we have $D = \Phi(\beta-\alpha, \beta, (\alpha+2\beta+\gamma)^{-1}$

$$= \frac{2\pi}{\sqrt{5}-1} + \sqrt{\phi+2}\log\left(\theta - \frac{8\theta}{4+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}\right) + 2^{-1}\sqrt{10-2\sqrt{5}}\log\left(\frac{\sqrt{3}\theta-1}{\theta+\sqrt{3}}\right)$$

and thus $\theta = \frac{\sqrt{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}}{\sqrt{8-\sqrt{10-2\sqrt{5}}-\sqrt{15}-\sqrt{3}}}$

we are done

and on simplification(rationalization) we have

$$\theta = \frac{8+\sqrt{10-2\sqrt{5}}+\sqrt{15}+\sqrt{3}}{36-4\sqrt{5}-4\sqrt{6(5+\sqrt{5})}} = \frac{\cos^2 3°}{36-4\sqrt{5}-4\sqrt{6(5+\sqrt{5})}}$$

Surprisingly, it is note worthy that the polynomial equation

$$F(X) = x^4 - 144x^3 + 6656x^2 - 98304x + 65536 = 0$$

has root $36 - 4\sqrt{5} - 4\sqrt{30 + 6\sqrt{5}}$ and

close approximation for $\ln 2$

$$(8+\sqrt{10 - 2\sqrt{5}}+\sqrt{15}+\sqrt{3})(8-\sqrt{10 - 2\sqrt{5}}-\sqrt{15}-\sqrt{3}) = 36-4\sqrt{5}-4\sqrt{30 + 6\sqrt{5}} \sim \ln 2$$