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UP.327. If $(x_n)_{n\ge 1}, x_n \in \mathbb{R}_+^*, \forall n \in \mathbb{N}^*$ satisfy $\lim\limits_{n\to\infty}(x_{n+1}-x_n)=x\in\mathbb{R}_+^*$,

then find:

$$\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_n\sqrt[n]{n}\right)$$

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Solution by Marian Ursărescu-Romania

$$L=\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_n\sqrt[n]{n}\right)=$$

$$=\lim_{n\to\infty}\left(x_{n+1}\sqrt[n+1]{n+1}-x_{n+1}\sqrt[n]{n}+x_{n+1}\sqrt[n]{n}-x_n\sqrt[n]{n}\right)=$$

$$=\lim_{n\to\infty}x_{n+1}\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)+\lim_{n\to\infty}\sqrt[n]{n}(x_{n+1}-x_n);\ \ (1)$$

$$\lim_{n\to\infty}\sqrt[n]{n}(x_{n+1}-x_n)=1\cdot x=x;(2)$$

$$\lim_{n\to\infty}x_{n+1}\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)=\lim_{n\to\infty}\frac{x_{n+1}}{n+1}\cdot(n+1)\cdot\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)$$

$$\lim_{n\to\infty}\frac{x_{n+1}}{n+1}=\lim_{n\to\infty}\frac{x_n}{n}\overset{LC-S}{=}\lim_{n\to\infty}\frac{x_{n+1}-x_n}{n+1-n}=x;\ \ (3)$$

Let be the function $f:[n,n+1]\to\mathbb{R}, f(x)=x^{\frac{1}{x}}$, from MVT we have:

$\exists c\in(n,n+1)$ such that $\frac{f(n+1)-f(n)}{n+1-n}=f'(c)\Leftrightarrow$

$$\sqrt[n+1]{n+1}-\sqrt[n]{n}=\frac{c^{\frac{1}{c}}(1-\log c)}{c^2}\Rightarrow$$

$$\lim_{n\to\infty}(n+1)\cdot\left(\sqrt[n+1]{n+1}-\sqrt[n]{n}\right)=\lim_{n\to\infty}(n+1)\cdot\frac{c^{\frac{1}{c}}(1-\log c)}{c^2}=0;\ \ (4)\text{ because}$$

From $c\in(n,n+1)\Rightarrow\lim_{x\to\infty}\frac{x+1}{x}\cdot\frac{1-\log x}{x}=1\cdot 0=0$

From (1),(2),(3),(4) we get $L=x$.

Note by editor:
Many thanks to Florică Anastase-Romania for typed solution.