SP.322 Let a, b, c be the lengths of the sides of a triangle with circumradius R and iradius r. Prove that:

$$\frac{2r}{R} \le \frac{a^2}{b^2+bc+c^2} + \frac{b^2}{c^2+ca+a^2} + \frac{c^2}{a^2+ab+b^2} \le \frac{R^2}{2r^2} - 1$$

Proposed by George Apostolopoulos-Greece

Solution 1 by Avishek Mitra-West Bengal-India, Solution 2 by Adrian Popa-Romania, Solution 3 by Ertan Yildirim-Turkey, Solution 4 by proposer, Solution 5 and generalization by Marin Chirciu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\sum_{cyc} \frac{a^2}{b^2+bc+c^2} \overset{Bergstrom}{\ge} \frac{(a+b+c)^2}{2(a^2+b^2+c^2)+ab+bc+ca} \overset{\sum x^2 \ge \sum xy}{\ge} \frac{4s^2}{3(a^2+b^2+c^2)} =$$

$$= \frac{4s^2}{6(s^2-4Rr-r^2)}$$

We need to show: $\frac{4s^2}{6(s^2-4Rr-r^2)} \ge \frac{2r}{R} \Leftrightarrow s^2(R-3r) + 12Rr^2 + 3r^3 \ge 0$

From $s^2 \ge 16Rr - 5r^2$ $(Gerretsen)$ we need to show that:

$$(16Rr-5r^2)(R-3r) + 12Rr^2 + 3r^3 \ge 0 \Leftrightarrow 16R^2 - 41Rr + 18r^2 \ge 0 \Leftrightarrow$$

$$(R-2r)(16R-9r) \ge 0 (true)\ R \ge 2r (Euler).$$

Now,

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{AM-GM}{\le}\sum_{cyc}\frac{a^2}{2bc+bc}=\frac{1}{3}\sum_{cyc}\frac{a^2}{bc}=\frac{1}{3abc}\sum_{cyc}a^3=$$

$$=\frac{3abc+\sum a^3-3abc}{3abc}=\frac{3abc+(\sum a)(\sum a^2-\sum ab)}{3abc}=$$

$$=\frac{12Rrs+2s(2s^2-8Rr-2r^2-s^2-r^2-4Rr)}{3abc}=$$

$$=\frac{2s(s^2-6Rr-3r^2)}{12Rrs}=\frac{2(s^2-6Rr-3r^2)}{12Rr}\overset{Gerretsen}{\le}$$

$$\le\frac{4R^2+4Rr+3r^2-6Rr-3r^2}{6Rr}=\frac{2R-r}{3r}$$

Need to show:

$$\frac{2R-r}{3r}\le\frac{R^2}{2r^2}-1=\frac{R^2-2r^2}{2r^2}\Leftrightarrow 4Rr-2r^2\le 3R^2-6r^2\Leftrightarrow$$

$$3R^2-4Rr-4r^2\ge 0\Leftrightarrow (R-2r)(3R+2r)\ge 0\ (true)R\ge 2r\ (Euler).$$

Provd.

Solution 2 by Adrian Popa-Romania

$$b^2+bc+c^2\overset{AM-GM}{\ge}2bc+bc=3bc\Rightarrow\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\le\sum_{cyc}\frac{a^2}{2bc+bc}=$$

$$=\sum_{cyc}\frac{a^3}{3abc}=\frac{a^3+b^3+c^3}{3abc}=\frac{2s(s^2-3r^2-6Rr)}{3abc}=\frac{2s^3-6sr^2}{12Rrs}-1\overset{(1)}{\le}\frac{R^2}{2r^2}-1$$

$$(1)\Leftrightarrow\frac{s^3-3sr^2}{6Rrs}\le\frac{R^2}{2r^2}\Leftrightarrow\frac{s^2-3r^2}{3R}\le\frac{R^2}{r}\Leftrightarrow s^2r-3r^3\le 3R^3$$

$$\therefore s^2\le 4R^2+4Rr+3r^2\ (Gerretsen)$$

$s^2r-3r^3\le(4R^2+4Rr+3r^2)r-3r^3\le 3R^3\Leftrightarrow 3\left(\frac{R}{r}\right)^2-\frac{4R}{r}-4\ge 0$ **true from**

$$R\ge 2r(Euler)$$

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\ge\sum_{cyc}\frac{2a^2}{3(b^2+c^2)}\ge\frac{2}{3}\cdot\frac{(a+b+c)^2}{2(a^2+b^2+c^2)}=\frac{4s^2}{3(a^2+b^2+c^2)}\overset{(2)}{\ge}\frac{2r}{R}$$

$$(2)\Leftrightarrow 2s^2R\ge 3r(2s^2-8Rr-2r^2)\Leftrightarrow$$

$$(3r-R)s^2 \overset{Gerretsen}{\leq} (3r-R)(4R^2+4Rr+3r^2) = 12R^2r - 4R^3 + 12Rr^2 - 4R^2r +$$

$$+9r^3 - 3r^2R \overset{(3)}{\leq} 12Rr^2 + 3r^3s^2$$

$$(3) \Leftrightarrow 4R^3 - 8R^2r + 3Rr^2 - 6r^3 \geq 0 \Leftrightarrow (R-2r)(4R^2+3r^2) \geq 0 \text{ true from}$$

$R \geq 2r$ $(Euler)$. Proved.

Solution 3 by Ertan Yildirim-Turkey

Lemma 1. $a^3+b^3+c^3 = 2s(s^2-3r^2-6Rr)$

Lemma 2. $2s^2 \geq 27Rr$

Lemma 3. $a^2+b^2+c^2 \leq 9R^2$

$$Rhs: \sum_{cyc}\frac{a^2}{b^2+bc+c^2} \overset{AM-GM}{\leq} \sum_{cyc}\frac{a^2}{2bc+bc} = \sum_{cyc}\frac{a^2}{3bc} = \sum_{cyc}\frac{a^3}{3abc} = \frac{1}{3abc}\sum_{cyc}a^3 =$$

$$= \frac{1}{3\cdot 4Rrs}\cdot 2s(s^2-3r^2-6Rr) = \frac{1}{6Rr}\cdot(s^2-3r^2-6Rr) \overset{(1)}{\leq} \frac{R^2}{2r^2} - 1$$

$$(1) \Leftrightarrow s^2 - 3r^2 - 6Rr \leq \frac{3R^3}{r} - 6Rr \Leftrightarrow s^2 \leq 3r^2 + \frac{3R^3}{r}$$

$$s^2 \overset{Gerretsen}{\leq} 4R^2 + 4Rr + 3r^2 \overset{(2)}{\leq} 3r^2 + \frac{3R^3}{r}$$

$$(2) \Leftrightarrow 4\left(\frac{R}{r}\right)^2 + 4\left(\frac{R}{r}\right) \leq 3\left(\frac{R}{r}\right)^3 \text{ let } t = \frac{R}{r} \Rightarrow 4t^2 + 4t \leq 3t^2 \Leftrightarrow$$

$$0 \leq t(3t^2-4t-4) = t(3t+2)(t-2) \Rightarrow t = \frac{R}{r} \geq 2\ (true) R \geq 2r\ (Euler)$$

$$Lhs:\ \sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{Bergstrom}{\geq}\frac{(a+b+c)^2}{2(a^2+b^2+c^2)+ab+bc+ca}\overset{\sum x^2\geq\sum xy}{\geq}\frac{4s^2}{3(a^2+b^2+c^2)}=$$

$$\geq\frac{4s^2}{3\cdot 9R^2}\overset{(3)}{\geq}\frac{2r}{R}$$

$$(3)\Leftrightarrow 2s^2\geq 27Rr\ (true)$$

Solution 4 by proposer

We have: $(a-b)^2\geq 0\Leftrightarrow a^2+b^2-2ab\geq 0\Leftrightarrow$

$$2a^2+4ab+4b^2-3a^2-3b^2-6ab\geq 0\Leftrightarrow (a^2+ab+b^2)\geq\frac{3}{4}(a+b)^2$$

$$\Leftrightarrow\frac{1}{a^2+ab+b^2}\leq\frac{4}{3}\cdot\frac{1}{(a+b)^2}.$$

Also we know that: $\frac{1}{(a+b)^2}\leq\frac{1}{8}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right).$

So, $\frac{1}{a^2+ab+b^2}\leq\frac{1}{6}\cdot\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\Leftrightarrow\frac{c^2}{a^2+ab+b^2}\leq\frac{1}{6}\cdot\left(\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)$ **similarly** $\frac{b^2}{c^2+ca+a^2}\leq\frac{1}{6}\cdot\left(\frac{b^2}{c^2}+\frac{b^2}{a^2}\right)$

and $\frac{a^2}{b^2+bc+c^2}\leq\frac{1}{6}\cdot\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right).$

Adding up these inequalities, we have:

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}$$
$$\leq\frac{1}{6}\cdot\left(\left(\frac{a^2}{b^2}+\frac{a^2}{c^2}\right)+\left(\frac{b^2}{c^2}+\frac{b^2}{a^2}\right)+\left(\frac{c^2}{a^2}+\frac{c^2}{b^2}\right)\right)$$

Now, will prove that: $\frac{a}{b}+\frac{b}{a}\leq\frac{R}{r}.$

Consider the supstitutions $a=y+z, b=z+x, c=x+y$, **where** x,y,z **are positive real numbers.**

We know that: $\frac{R}{r}=\frac{abc}{4(s-a)(s-b)(s-c)}$, **where** $s=\frac{a+b+c}{2}$ **is the semiperimeter.**

So, $\frac{R}{r}=\frac{(x+y)(y+z)(z+x)}{4xyz}$. **We have:**

$\frac{1}{(z+x)^2}+\frac{1}{(y+z)^2}\leq\frac{1}{4zx}+\frac{1}{4yz}=\frac{x+y}{4xyz}$ **and multiplying by** $(z+x)(y+z)$ **both sides, we have**

$\frac{y+z}{z+x}+\frac{z+x}{y+z}\le\frac{(x+y)(y+z)(z+x)}{4xyz}$, namely $\frac{a}{b}+\frac{b}{a}\le\frac{R}{r}$, similarly $\frac{b}{c}+\frac{c}{b}\le\frac{R}{r}$ and $\frac{a}{c}+\frac{c}{a}\le\frac{R}{r}$.

So, $\frac{a^2}{b^2}+\frac{a^2}{c^2}\le\frac{R^2}{r^2}-2$, then

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}\le\frac{1}{6}\cdot\left(\left(\frac{R^2}{r^2}-2\right)+\left(\frac{R^2}{r^2}-2\right)+\left(\frac{R^2}{r^2}-2\right)\right)$$
$$=\frac{R^2}{2r^2}-1.$$

Now, using Cauchy-Rogers inequality, we have:

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}\ge\frac{(a+b+c)^2}{2(a^2+b^2+c^2)+(ab+bc+ca)}$$
$$\ge\frac{4S^2}{2(a^2+b^2+c^2)+a^2+b^2+c^2}=\frac{2(2S^2)}{3(a^2+b^2+c^2)}$$

We know that: $2S^2\ge 27Rr$ and $a^2+b^2+c^2\ge 9R^2$. So,

$$\frac{a^2}{b^2+bc+c^2}+\frac{b^2}{c^2+ca+a^2}+\frac{c^2}{a^2+ab+b^2}\ge\frac{2r}{R}.$$

Equality holds if the triangle is equilateral.

Solution 5 and generalization by Marin Chirciu-Romania

For LHS using Bergtrom inequality, we have:

$$\sum_{cyc}\frac{a^2}{b^2+bc+c^2}\overset{Bergstrom}{\ge}\frac{(\sum a)^2}{\sum(b^2+bc+c^2)}=\frac{4s^2}{2\cdot 2(s^2-r^2-4Rr)+s^2+r^2+4Rr}=$$
$$=\frac{4s^2}{5s^2-3r^2-12Rr}\overset{(1)}{\ge}\frac{2r}{R}$$

Where (1)$\Leftrightarrow 2Rs^2\ge r(5s^2-3r^2-12Rr)\Leftrightarrow s^2(2R-5r)+3r^2(4R+r)\ge 0$

We distinguish the cases:

(I) If $2R-5r\ge 0$ inequality is obviously.

(II) If $2R-5r<0$ inequality it can be written as: $3r^2(4R+r)\ge s^2(5r-2R)$

Which follows from $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$.

It remains to prove that $3r^2(4R + r) \ge (4R^2 + 4Rr + 3r^2)(5r - 2R) \Leftrightarrow$

$4R^3 - 6R^2r - Rr^2 - 12r^3 \ge 0 \Leftrightarrow (R - 2r)(4R^2 + 2Rr + 3r^2) \ge 0$ which is true from

$R \ge 2r$ $(Euler)$

Equality holds if and only if triangle is equilateral.

For RHD we have:

$$\sum_{cyc} \frac{a^2}{b^2 + bc + c^2} \overset{AM-GM}{\le} \sum_{cyc} \frac{a^2}{2bc + bc} = \frac{1}{3} \cdot \sum_{cyc} \frac{a^2}{bc} = \frac{1}{3} \cdot \frac{s^2 - 3r^2 - 6Rr}{2Rr} =$$

$$= \frac{s^2 - 3r^2 - 6Rr}{6Rr} \overset{(2)}{\le} \frac{R^2}{2r^2} - 1$$

Where (2)$\Leftrightarrow \frac{s^2-3r^2-6Rr}{6Rr} \le \frac{R^2}{2r^2} - 1 \Leftrightarrow r(s^2 - 3r^2 - 6Rr) \le 3R(R^2 - 2r^2)$

Which follows from $s^2 \le 4R^2 + 4Rr + 3r^2$ $(Gerretsen)$.

It remains to prove that

$r(4R^2 + 4Rr + 3r^2 - 3r^2 - 6Rr) \le 3R(R^2 - 2r^2) \Leftrightarrow$

$3R^2 - 4Rr - 4r^2 \ge 0 \Leftrightarrow (R - 2r)(3R + 2r) \ge 0$ which is true from $R \ge 2r(Euler)$

Equality holds if and only if triangle is equilateral.

Remark. The inequality it can be developed.

In ΔABC the following relationship holds:

$$\frac{6r}{(\lambda + 2)R} \le \sum_{cyc} \frac{a^2}{b^2 + \lambda bc + c^2} \le \frac{3}{\lambda + 2}\left(\frac{R^2}{2r^2} - 1\right), \qquad \lambda > -2$$

Proposed by Marin Chirciu-Romania

Solution by proposer

For LHS using Bergtrom inequality, we have:

$$\sum_{cyc} \frac{a^2}{b^2 + \lambda bc + c^2} \overset{Bergstrom}{\ge} \frac{(\sum a)^2}{\sum(b^2 + \lambda bc + c^2)} = \frac{4s^2}{2\sum a^2 + \lambda \sum bc} =$$

$$= \frac{4s^2}{2 \cdot 2(s^2 - r^2 - 4Rr) + \lambda(s^2 + r^2 + 4Rr)} = \frac{4s^2}{(\lambda + 4)s^2 + (\lambda - 4)r^2 + (4\lambda - 16)Rr} \overset{(1)}{\ge}$$

$$\overset{(1)}{\geq} \frac{6r}{(\lambda+2)R} \text{ where (1)} \Leftrightarrow$$

$$\frac{4s^2}{(\lambda+4)s^2+(\lambda-4)r^2+(4\lambda-16)Rr} \geq \frac{6r}{(\lambda+2)R} \Leftrightarrow$$

$$2(\lambda+2)Rs^2 \geq 3r[(\lambda+4)s^2+(\lambda-4)r^2+(4\lambda-16)Rr] \Leftrightarrow$$

$$s^2[2(\lambda+2)R-3(\lambda+4)r]+3r^2[(4-\lambda)r+(16-4\lambda R]\geq 0$$

We distinguish the cases:

(I) If $2(\lambda+2)R-3(\lambda+4)r\geq 0$ inequality is obviously.

(II) If $2(\lambda+2)R-3(\lambda+4)r\leq 0$ the inequality it can be written as:

$3r^2[(4-\lambda)r+(16-4\lambda R]\geq s^2[3(\lambda+4)r-2(\lambda+2)R]$ **which follows from**

$$s^2\leq 4R^2+4Rr+3r^2\ (Gerretsen).$$

It remains to prove that

$$3r^2[(4-\lambda)r+(16-4\lambda R]\geq (4R^2+4Rr+3r^2)[3(\lambda+4)r-2(\lambda+2)R] \Leftrightarrow$$

$$(4\lambda+8)R^3-(2\lambda+16)R^2r+(-9\lambda+6)Rr^2-(6\lambda+12)r^3\geq 0 \Leftrightarrow$$

$(R-2r)[(4\lambda+8)R^2+6\lambda Rr+(3\lambda+6)r^2]\geq 0$ **which follows from** $R\geq 2r(Euler)$

and $[(4\lambda+8)R^2+6\lambda Rr+(3\lambda+6)r^2]>0.$

Equality holds if and only if triangle is equilateral.

For RHD we have:

$$\sum_{cyc}\frac{a^2}{b^2+\lambda bc+c^2} \overset{AM-GM}{\leq} \sum_{cyc}\frac{a^2}{2bc+\lambda bc}=\frac{1}{2+\lambda}\cdot\sum_{cyc}\frac{a^2}{bc}=\frac{1}{2+\lambda}\cdot\frac{s^2-3r^2-6Rr}{2Rr}=$$

$$=\frac{s^2-3r^2-6Rr}{2(2+\lambda)Rr}\overset{(2)}{\leq}\frac{3}{\lambda+2}\left(\frac{R^2}{2r^2}-1\right)$$

Where (2)$\Leftrightarrow \frac{s^2-3r^2-6Rr}{6Rr}\leq\frac{R^2}{2r^2}-1 \Leftrightarrow r(s^2-3r^2-6Rr)\leq 3R(R^2-2r^2)$

Which follows from $s^2\leq 4R^2+4Rr+3r^2\ (Gerretsen)$.

It remains to prove that

$$r(4R^2+4Rr+3r^2-3r^2-6Rr)\leq 3R(R^2-2r^2) \Leftrightarrow$$

$3R^2-4Rr-4r^2\geq 0 \Leftrightarrow (R-2r)(3R+2r)\geq 0$ **which is true from** $R\geq 2r(Euler)$

Equality holds if and only if triangle is equilateral.

Note:

For $\lambda = 1$ we get the Problem SP.322 from 22-RMM-Autumn Edition 2021, proposed by George Apostolopoulos-Greece.

Note by editor:
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