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DANIEL SITARU

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Proposed by

Daniel Sitaru - Romania

Adil Abdullayev-Baku-Azerbaijan

George Florin Șerban – Romania

Bogdan Fuștei-Romania, Marin Chirciu-Romania

Radu Diaconu-Romania

Marian Ursărescu-Romania

Mokhtar Khassani-Mostaganem-Algerie

Rahim Shahbazov-Baku-Azerbaijan

Alex Szoros – Romania, Cristian Miu – Romania

Florentin Vișescu-Romania,

Jalil Hajimir – Canada, Gheorghe Alexe - Romania



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Solutions by

Daniel Sitaru-Romania

Tran Hong-Dong Thap-Vietnam

Soumava Chakraborty-Kolkata-India

Bogdan Fuștei-Romania

Adrian Popa-Romania

George Florin Șerban – Romania

Marian Ursărescu-Romania

Cristian Miu – Romania

Marin Chirciu-Romania

Asmat Quatea-Kabul-Afganistan



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1701. In ΔABC the following relationships holds:

$$R^2 + \frac{4}{a^2 b^2 c^2} \left(\sum_{cyc} bc(s-a)^2 \right)^2 + 11r^2 \geq 21Rr$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \sum_{cyc} bc(s-a)^2 &= bc(s-a)^2 + ca(s-b)^2 + ab(s-c)^2 = \\ &= bc(s^2 - 2sa + a^2) + ca(s^2 - 2sb + b^2) + ab(s^2 - 2sc + c^2) = \\ &= s^2(ab + bc + ca) - 6abcs + abc(a+b+c) = \\ &= s^2(s^2 + 4Rr + r^2) - 16Rrs^2 = s^2(s^2 - 12Rr + r^2) \Rightarrow \\ \frac{4}{a^2 b^2 c^2} \left(\sum_{cyc} bc(s-a)^2 \right)^2 &= \frac{4}{(4Rrs)^2} (s^2(s^2 - 12Rr + r^2))^2 = \\ \frac{s^2(s^2 - 12Rr + r^2)^2}{4R^2 r^2} &\stackrel{s^2 \geq 16Rr - 5r^2}{\geq} \frac{(16Rr - 5r^2)(4Rr - 4r^2)^2}{4R^2 r^2} = \\ \frac{4r(16R - 5r)(R - r)^2}{R^2} &= 4(16Rr - 5r^2) \left(1 - \frac{r}{R}\right)^2 \end{aligned}$$

Let: $\frac{R}{r} = t \geq 2$. We need to prove:

$$t^2 + 4(16t - 5) \left(1 - \frac{1}{t}\right)^2 + 11 \geq 21t \Leftrightarrow$$

$$t^4 + 4(16t - 5)(t - 1)^2 - 21t^3 + 11t^2 \geq 0$$

$$t^4 + (64t - 20)(t - 1)^2 - 21t^3 + 11t^2 \geq 0$$

$$t^4 + 43t^3 - 137t^2 + 104t - 20 \geq 0$$

$$(t-2)(t^3 - 45t^2 - 47t + 10) \geq 0$$

Which is clearly true, because:

$$t \geq 2 \Rightarrow t - 2 \geq 0 \text{ and}$$

$$t^3 - 45t^2 - 47t + 10 \geq 2(2^3 - 45 \cdot 2^2 - 47 \cdot 2 + 10) = 104 > 0.$$

Proved.



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1702. In ΔABC the following relationship holds:

$$\left(m_a m_b + \frac{b^2 + c^2}{2ca} \right) \left(\frac{1}{h_a h_b} + \frac{c^2 + a^2}{2bc} \right) \geq \left(m_a(c^2 + a^2) + \frac{b^2 + c^2}{h_b} \right) \left(\frac{m_b}{2bc} + \frac{1}{2cah_a} \right)$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} & \left(m_a m_b + \frac{b^2 + c^2}{2ca} \right) \left(\frac{1}{h_a h_b} + \frac{c^2 + a^2}{2bc} \right) \geq \left(m_a(c^2 + a^2) + \frac{b^2 + c^2}{h_b} \right) \left(\frac{m_b}{2bc} + \frac{1}{2cah_a} \right) \\ & \frac{m_a m_b}{h_a h_b} + \frac{m_a m_b(c^2 + a^2)}{2bc} + \frac{b^2 + c^2}{2cah_a h_b} + \frac{(b^2 + c^2)(c^2 + a^2)}{4abc^2} \geq \\ & \geq \frac{m_a m_b(c^2 + a^2)}{2bc} + \frac{m_b(b^2 + c^2)}{2bch_b} + \frac{m_a(c^2 + a^2)}{2cah_a} + \frac{b^2 + c^2}{2cah_a h_b} \\ & \frac{m_a m_b}{h_a h_b} + \frac{(b^2 + c^2)(c^2 + a^2)}{4abc^2} \geq \frac{m_b(b^2 + c^2)}{2bch_b} + \frac{m_a(c^2 + a^2)}{2cah_a} \\ & \frac{m_a m_b}{h_a h_b} + \frac{m_a m_b}{s_a s_b} \geq \frac{m_a m_b}{s_a h_b} + \frac{m_a m_b}{h_a s_b} \text{ because } \left(s_a = \frac{2bc}{b^2 + c^2} \cdot m_a \right) \\ & \Leftrightarrow h_a h_b + s_a s_b \geq h_a s_b + h_b s_a \Leftrightarrow h_a(h_b - s_b) + s_a(s_b - h_b) \geq 0 \\ & \Leftrightarrow (s_b - h_b)(s_a - h_a) \geq 0. \end{aligned}$$

Which is clearly true,because: $s_b \geq h_b$; $s_a \geq h_a$.Proved.

1703. In $\Delta ABC, \Delta A'B'C'$ the following relationship holds:

$$(a + a')(b + b')(c + c') \geq 8(r^2 s + r'^2 s' + 6rr' \sqrt{ss'})$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} & (a + a')(b + b')(c + c') = \\ & = \left((\sqrt[3]{a})^3 + (\sqrt[3]{a'})^3 \right) \left((\sqrt[3]{b})^3 + (\sqrt[3]{b'})^3 \right) \left((\sqrt[3]{c})^3 + (\sqrt[3]{c'})^3 \right) \stackrel{\text{Holder}}{\geq} \\ & \geq \left(\sqrt[3]{abc} + \sqrt[3]{a'b'c'} \right)^3 \geq \left(\sqrt[3]{4Rrs} + \sqrt[3]{4R'r's'} \right)^3 \stackrel{R \geq 2r, R' \geq 2r'}{\geq} \\ & \geq \left(\sqrt[3]{8r^2 s} + \sqrt[3]{8r'^2 s'} \right)^3 = 8 \left(\sqrt[3]{r^2 s} + \sqrt[3]{r'^2 s'} \right)^3 \end{aligned}$$

Let: $x = \sqrt[3]{r^2 s}$; $y = \sqrt[3]{r'^2 s'}$ ($x, y > 0$)



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We need to prove:

$$\begin{aligned}
 8(x+y)^3 &\geq 8(x^3 + y^3 + 6xy\sqrt{xy}) \Leftrightarrow \\
 x^3 + y^3 + 3xy(x+y) &\geq x^3 + y^3 + 6xy\sqrt{xy} \Leftrightarrow \\
 3xy(x+y) &\geq 6xy\sqrt{xy} \Leftrightarrow xy(x+y) \geq 2xy\sqrt{xy}
 \end{aligned}$$

Which is clearly true, because $x, y > 0, x+y \stackrel{AGM}{\geq} 2\sqrt{xy} \Leftrightarrow$

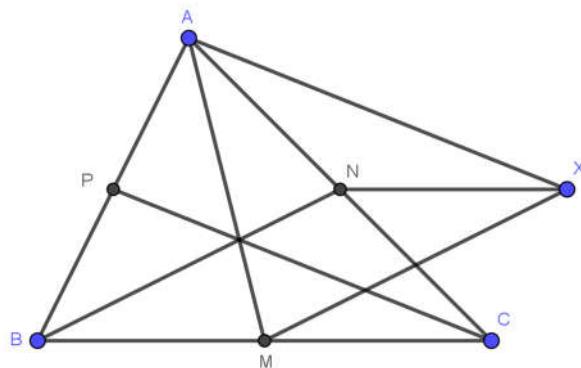
$xy(x+y) \geq 2xy\sqrt{xy}$. Proved.

1704. In ΔABC , $DE = m_b$, $EF = m_c$, $FD = m_a$, R_m , r_m - circumradii and inradii in ΔDEF . Prove that:

$$\left(\frac{R}{r}\right)^2 \geq \frac{2R_m}{r_m}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Tran Hong-Dong Thap-Vietnam



We have: $MB = MC, NA = NC, PA = PB$

$BMXN$ - parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A, D \equiv M, E \equiv X \Rightarrow S_{\Delta DEF} = \frac{3}{4}S_{\Delta ABC}$

$$\Rightarrow R_m = \frac{m_a m_b m_c}{3S_{\Delta ABC}}; r_m = \frac{3S_{\Delta ABC}}{2(m_a + m_b + m_c)}$$

$$\Rightarrow \frac{R_m}{r_m} = \frac{2}{9} \cdot \frac{(m_a + m_b + m_c)m_a m_b m_c}{S^2}$$



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$$\Rightarrow \frac{2R_m}{r_m} = \frac{4(m_a + m_b + m_c)m_a m_b m_c}{9s^2} = \frac{4(m_a + m_b + m_c)m_a m_b m_c}{9(sr)^2} \stackrel{(*)}{\leq} \left(\frac{R}{r}\right)^2$$

$$(*) \Leftrightarrow (m_a + m_b + m_c)m_a m_b m_c \leq \frac{9}{4}(sR)^2$$

Which is clearly true because: $m_a + m_b + m_c \leq 4R + r \stackrel{r \leq \frac{R}{2}}{\leq} \frac{9R}{2}$

$$\frac{m_a m_b m_c}{r_a r_b r_c} \leq \frac{R}{2r} \Rightarrow \frac{m_a m_b m_c}{s^2 r} \leq \frac{R}{2r} \Rightarrow m_a m_b m_c \leq \frac{Rs^2}{2}$$

$$\Rightarrow (m_a + m_b + m_c)m_a m_b m_c \leq \frac{9R^2 s^2}{4} \Rightarrow (*) \text{ is true} \Rightarrow \text{proved.}$$

1705. In acute } \Delta ABC, o_a -\text{circumcevian, the following relationship holds:}

$$\frac{h_a}{o_a} + \frac{h_b}{o_b} + \frac{h_c}{o_c} + 1 \geq \frac{8r}{R}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Daniel Sitaru – Romania

$$o_a = AA_1, A_1 \in (BC), \mu(\triangle BAA_1) = \frac{\pi}{2} - C, \mu(\triangle BA_1A) = \frac{\pi}{2} + C - B$$

$$\Delta BAA_1: \frac{o_a}{\sin B} = \frac{c}{\sin\left(\frac{\pi}{2} + C - B\right)} \Rightarrow o_a = \frac{c \sin B}{\cos(B - C)}$$

$$1 + \sum_{cyc} \frac{h_a}{o_a} = 1 + \sum_{cyc} \frac{c \sin B}{\frac{c \sin B}{\cos(B - C)}} = 1 + \sum_{cyc} \cos(B - C) =$$

$$= 1 + \frac{s^2 + r^2 + 2Rr}{2R^2} - 1 \stackrel{GERRETSEN}{\geq}$$

$$\geq \frac{16Rr - 5r^2 + r^2 + 2Rr}{2R^2} = \frac{18Rr - 4r^2}{2R^2} \geq$$

$$\stackrel{EULER}{\geq} \frac{18Rr - 2r \cdot R}{2R^2} = \frac{16rR}{2R^2} = \frac{8r}{R}$$

1706. In } \Delta ABC \text{ then following relationship holds:}

$$\frac{1}{h_a^2 h_b} + \frac{1}{h_b^2 h_c} + \frac{1}{h_c^2 h_a} \geq \frac{2}{3R} \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} \right)$$

Proposed by George Florin Șerban-Romania



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Solution by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \frac{1}{h_a^2 h_b} + \frac{1}{h_b^2 h_c} + \frac{1}{h_c^2 h_a} &\stackrel{AGM}{\geq} 3 \sqrt[3]{\frac{1}{(h_a h_b h_c)^3}} = \frac{3}{h_a h_b h_c} = \frac{3}{\frac{3R}{2S^2}} = \frac{3R}{2S^2} \\ \frac{2}{3R} \left(\frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} \right) &= \frac{2}{3R} \cdot \frac{a^2 + b^2 + c^2}{4S^2} \end{aligned}$$

We must show that: $\frac{3R}{2S^2} \geq \frac{2}{3R} \cdot \frac{a^2 + b^2 + c^2}{4S^2} \Leftrightarrow 9R^2 \geq a^2 + b^2 + c^2$ which is true by Leibnitz's Inequality. Proved.

1707. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \stackrel{(1)}{\leq} \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c} \right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) \\ &= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= a g_a^2 + a(s - b)(s - c) \\ &\quad \therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2 \\ &\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) \\ &\quad - a(s - b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\begin{aligned} &\Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 \\ &\quad - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2 \\ &\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true} \\ &\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \geq s(s - a) \text{ and analogs} \end{aligned}$$



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$$\Rightarrow \sum n_a g_a \geq s \sum (s - a) = s^2 \Rightarrow \left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1} \geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1}$$

$$= s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\} = \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r}$$

$$\Rightarrow \frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \stackrel{(m)}{\geq} \frac{R}{2r}$$

$$\text{Now, } r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\cong} 4R \cos^2 \frac{A}{2}$$

$$\text{Now, } (b+c)^2 \geq 32Rr \cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\cong} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c \geq 4\sqrt{2Rr} \cos \frac{A}{2} \Rightarrow \sum m_a \stackrel{\text{loscu}}{\geq} \sum \left(\frac{b+c}{2} \cos \frac{A}{2} \right)$$

$$\geq \sqrt{2Rr} \sum 2 \cos^2 \frac{A}{2} = \sqrt{2Rr} \sum (1 + \cos A) = \sqrt{2Rr} \left(4 + \frac{r}{R} \right) = \sqrt{\frac{2r}{R}} \left(\sum r_a \right)$$

$$\Rightarrow \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c} \right)^2 \stackrel{(n)}{\leq} \frac{R}{2r} (m), (n) \Rightarrow (1) \text{ is true (QED)}$$

1708. In any ΔABC , g_a –Gergonne's cevian, holds:

$$\frac{2r}{R} + \frac{8r_a r_b r_c}{g_a g_b g_c} \geq 9$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India



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$$\begin{aligned}
 \text{Triangle inequality} \Rightarrow g_a &\leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abccos \frac{A}{2}}{a(b+c)} \\
 &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rrscos \frac{A}{2}}{4R(b+c)\sin \frac{A}{2}\cos \frac{A}{2}} \\
 &\Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a+b+c}{(b+c)\sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a}{(b+c)\sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c)\sin \frac{A}{2} \stackrel{?}{\leq} a \\
 &\Leftrightarrow 4R\cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leq} 4R\sin \frac{A}{2} \cos \frac{A}{2} \\
 &\Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true} \therefore w_a \geq g_a \text{ and analogs} \Rightarrow \frac{8r_ar_b r_c}{g_a g_b g_c} \stackrel{?}{\geq} \frac{8r_ar_b r_c}{w_a w_b w_c} \\
 &= \frac{8rs^2}{\prod \left(\frac{2bccos \frac{A}{2}}{b+c} \right)} = \frac{8rs^2 \prod (b+c)}{8a^2 b^2 c^2 (\prod \cos \frac{A}{2})} \\
 &= \frac{8rs^2 \cdot 2s(s^2 + 2Rr + r^2)}{128R^2 r^2 s^2 \left(\frac{s}{4R} \right)} = \frac{s^2 + 2Rr + r^2}{2Rr} \Rightarrow \frac{2r}{R} + \frac{8r_ar_b r_c}{g_a g_b g_c} \stackrel{?}{\geq} \frac{2r}{R} + \frac{s^2 + 2Rr + r^2}{2Rr} \\
 &= \frac{s^2 + 2Rr + 5r^2}{2Rr} \stackrel{?}{\geq} 9 \Leftrightarrow s^2 \stackrel{?}{\geq} 16Rr - 5r^2 \\
 &\rightarrow \text{true (Gerretsen)} \therefore \frac{2r}{R} + \frac{8r_ar_b r_c}{g_a g_b g_c} \stackrel{?}{\geq} 9 \text{ (Proved)}
 \end{aligned}$$

1709. In any ΔABC , n_a –Nagel's cevian, holds:

$$\sum \left(a + \sqrt{h_a(h_b - 2r)} \right) (n_a + \sqrt{2r_a h_a}) \leq 3\sqrt{2}s^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) &= a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= a n_a^2 + a(as - s^2)
 \end{aligned}$$



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$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc)$$

$$= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}$$

$$= as^2 - \frac{4as(s-b)(s-c)}{a} \Rightarrow n_a^2 = s^2 - \frac{4s(s-b)(s-c)}{a}$$

$$\therefore n_a + \sqrt{2r_a h_a} \stackrel{\text{CBS}}{\leq} \sqrt{2}\sqrt{n_a^2 + 2h_a r_a} = \sqrt{2} \sqrt{s^2 - \frac{4s(s-b)(s-c)}{a} + 2h_a r_a}$$

$$= \sqrt{2} \sqrt{s^2 - \frac{4s(s-b)(s-c)}{a} + \frac{4(s-a)s(s-b)(s-c)}{a(s-a)}} \Rightarrow n_a + \sqrt{2r_a h_a}$$

$\leq \sqrt{2}s$ and analogs

$$\Rightarrow \sum (a + \sqrt{h_a(h_b - 2r)}) (n_a + \sqrt{2r_a h_a}) \leq \sqrt{2}s \left(2s + \sum \sqrt{h_a(h_b - 2r)} \right) \stackrel{?}{\leq} 3\sqrt{2}s^2$$

$$\Leftrightarrow \sum \sqrt{h_a(h_b - 2r)} \stackrel{?}{\leq}_{(1)} s$$

$$\text{Now, } \sum \sqrt{h_a(h_b - 2r)} = \sum \sqrt{\frac{2rs}{a} 2r \left(\frac{s-b}{b} \right)}$$

$$= 2r \sum \sqrt{\frac{1}{ab}} \sqrt{s(s-b)} \stackrel{\text{CBS}}{\leq} 2r \sqrt{\sum \frac{1}{ab}} \sqrt{\sum s(s-b)} = 2r \sqrt{\frac{2s}{4Rrs}} s$$

$$= \frac{2rs}{\sqrt{2Rr}} \stackrel{\text{Euler}}{\leq} \frac{2rs}{\sqrt{4r^2}}$$

$$\Rightarrow \sum \sqrt{h_a(h_b - 2r)} \leq s \Rightarrow (1) \text{ is true } \therefore \sum (a + \sqrt{h_a(h_b - 2r)}) (n_a + \sqrt{2r_a h_a}) \leq 3\sqrt{2}s^2 \text{ (Proved)}$$

1710. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \stackrel{(1)}{\geq} \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c} \right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India



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$$\begin{aligned}
 & \text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) \\
 & = a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\
 & = a g_a^2 + a(s - b)(s - c) \\
 & \quad \therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2 \\
 & \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s \\
 & \quad - b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2
 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\begin{aligned}
 & \Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 - yz(y \\
 & \quad + z)\} \geq x^2(y + z)^2(x + y + z)^2 \\
 & \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true} \\
 & \Rightarrow (\text{a}) \text{ is true} \Rightarrow n_a g_a \geq s(s - a) \text{ and analogs}
 \end{aligned}$$

$$\begin{aligned}
 & \Rightarrow \sum n_a g_a \geq s \sum (s - a) = s^2 \Rightarrow \left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1} \geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1} \\
 & = s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\} = \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r} \\
 & \Rightarrow \frac{n_a g_a + n_b g_b + n_c g_c}{h_a h_b + h_b h_c + h_c h_a} \stackrel{(m)}{\geq} \frac{R}{2r}
 \end{aligned}$$

$$\begin{aligned}
 \text{Proof : } r_b + r_c &= s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \\
 &\therefore r_b + r_c \stackrel{(i)}{\equiv} 4R \cos^2 \frac{A}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } (b + c)^2 &\geq 32Rr \cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\equiv} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s - b} + \frac{1}{s - c} \right) \\
 &= 8(s - a)(s - b)(s - c) \frac{a}{(s - b)(s - c)} = 4a(b + c - a)
 \end{aligned}$$



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$$\begin{aligned}
 & \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c \\
 & \geq 4\sqrt{2Rr}\cos\frac{A}{2} \Rightarrow \sum m_a \stackrel{\text{loscu}}{\geq} \sum \left(\frac{b+c}{2} \cos\frac{A}{2} \right) \\
 & \geq \sqrt{2Rr} \sum 2\cos^2\frac{A}{2} = \sqrt{2Rr} \sum (1 + \cos A) = \sqrt{2Rr} \left(4 + \frac{r}{R} \right) = \sqrt{\frac{2r}{R}} \left(\sum r_a \right) \\
 & \Rightarrow \left(\frac{r_a + r_b + r_c}{m_a + m_b + m_c} \right)^2 \stackrel{(n)}{\leq} \frac{R}{2r}(m), (n) \Rightarrow (1) \text{ is true (QED)}
 \end{aligned}$$

1711. In any } ABC, n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \frac{n_a g_a}{h_b h_c} \geq \frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Stewart's theorem} & \Rightarrow b^2(s-c) + c^2(s-b) = a n_a^2 + a(s-b)(s-c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s-a) & = a n_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 & = a n_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) & = a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\
 & = as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\
 & = as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s-a} \right) = as^2 - 2a h_a r_a \stackrel{(1)}{\therefore} n_a^2 \geq s^2 - 2h_a r_a
 \end{aligned}$$

$$\begin{aligned}
 \text{Again, Stewart's theorem} & \Rightarrow b^2(s-c) + c^2(s-b) \\
 & = a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 & = a g_a^2 + a(s-b)(s-c) \\
 & \therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s-a)^2 \\
 \Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} & \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2
 \end{aligned}$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$



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Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (\text{a}) \text{ is true} \Rightarrow n_a g_a \geq s(s-a)$$

$$\Rightarrow n_a g_a r_a \geq s(s-a) \left(\frac{rs}{s-a} \right) = rs^2 \Rightarrow n_a g_a r_a \geq rs^2 \Rightarrow \frac{n_a g_a r_a}{h_b h_c} \geq \frac{bc \cdot rs^2}{4r^2 s^2} = \frac{bc}{4r}$$

$$= \left(\frac{R}{2r} \right) h_a \stackrel{\text{Panaitopol}}{\geq} m_a \Rightarrow \frac{n_a g_a}{h_b h_c} \geq \frac{m_a}{r_a} \text{ and analgs}$$

$$\Rightarrow \sum \frac{n_a g_a}{h_b h_c} \geq \frac{m_a}{r_a} + \frac{m_b}{r_b} + \frac{m_c}{r_c} \text{ (Proved)}$$

1712. In ΔABC the following relationship holds:

$$\frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{abc(a+b+c)} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Bogdan Fuștei-Romania

Using Goldstone Inequality: $16r^2s^2 \leq a^2b^2 + b^2c^2 + c^2a^2 \leq 4R^2s^2$

We have: $abc = 4RS; a+b+c = 2s \Rightarrow$

$$\frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{abc(a+b+c)} \leq \frac{4R^2 s^2}{4RS \cdot 2s} = \frac{Rs}{2S} = \frac{R}{2r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{abc(a+b+c)} &= \frac{(s^2 + 4Rr + r^2)^2}{8Rrs^2} - 2 \leq \frac{R}{2r} \Leftrightarrow \frac{(s^2 + 4Rr + r^2)^2}{8Rrs^2} \leq \frac{R + 4r}{2r} \\ \Leftrightarrow 4R(R + 4r)s^2 &\geq (s^2 + 4Rr + r^2)^2 \end{aligned}$$

$$\Leftrightarrow s^4 - s^2(4R^2 + 8Rr - 2r^2) + r^2(4R + r)^2 \stackrel{(1)}{\leq} 0$$

Now, Rouche $\Rightarrow s^2 - (m-n) \geq 0$ and $s^2 - (m+n) \leq 0$, where m

$$= 2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r)\sqrt{R^2 - 2Rr}$$



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$$\therefore (s^2 - (m+n))(s^2 - (m-n)) \leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0$$

$$\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \stackrel{(i)}{\leq} 0$$

$\therefore (i) \Rightarrow$ in order to prove (1), it suffices to prove :

$$s^4 - s^2(4R^2 + 8Rr - 2r^2) + r^2(4R+r)^2 \leq s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3$$

$$\Leftrightarrow 3s^2 \leq (4R+r)^2 \rightarrow \text{true (Trucht)}$$

$$\Rightarrow (1) \text{ is true } \therefore \frac{a^2b^2 + b^2c^2 + c^2a^2}{abc(a+b+c)} \leq \frac{R}{2r} \text{ (Proved)}$$

Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\text{In any triangle we have: } \frac{a^3+b^3+c^3+abc}{abc} \leq \frac{2R}{r} \Leftrightarrow \frac{1}{4} \cdot \frac{a^3+b^3+c^3+abc}{abc} \leq \frac{R}{2r}$$

$$\text{We need to prove: } \frac{1}{4} \cdot \frac{a^3+b^3+c^3+abc}{abc} \geq \frac{a^2b^2+b^2c^2+c^2a^2}{abc(a+b+c)} \Leftrightarrow$$

$$(a+b+c)(a^3+b^3+c^3+abc) \geq 4(a^2b^2+b^2c^2+c^2a^2) \Leftrightarrow$$

$$a^4 + b^4 + c^4 + abc(a+b+c) + ab(a^2+b^2) + bc(b^2+c^2)$$

$$+ ca(c^2+a^2) \stackrel{(*)}{\geq} 4(a^2b^2+b^2c^2+c^2a^2)$$

$$\text{But: } a^4 + b^4 + c^4 + abc(a+b+c) \stackrel{Schur}{\geq} ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \Rightarrow$$

$$a^4 + b^4 + c^4 + abc(a+b+c) + ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \geq$$

$$\geq 2[ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2)] \stackrel{(**)}{\geq}$$

$$\geq 4(a^2b^2+b^2c^2+c^2a^2) \Leftrightarrow$$

$$ab(a^2+b^2) + bc(b^2+c^2) + ca(c^2+a^2) \geq 2(a^2b^2+b^2c^2+c^2a^2) \Leftrightarrow$$

$$ab(a-b)^2 + bc(b-c)^2 + ca(c-a)^2 \geq 0$$

Which is clearly true $\Rightarrow (**)$ true $\Rightarrow (*)$ true.

1713. In ΔABC the following relationship holds:

$$\frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \leq \left(\frac{R}{2r}\right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution 1 by Tran Hong-Dong Thap-Vietnam

Let: $x = m_a, y = m_b, z = m_c \Rightarrow x + y + z \leq 4R + r$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

$$\frac{(x+y)(y+z)(z+x)}{xyz} = \frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{z}{y} + \frac{y}{z} + 2 = (x+y+z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 1 \Rightarrow$$

$$LHS = \frac{1}{8} \left[(x+y+z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 1 \right] \leq \frac{1}{8} \left[(4R+r) \cdot \frac{1}{r} - 1 \right] = \frac{1}{8} \cdot \frac{4R}{r} = \frac{R}{2r} \stackrel{(*)}{\leq} \left(\frac{R}{2r} \right)^2$$

$(*) \Leftrightarrow t^2 \geq t \left(t = \frac{R}{2r} \geq 1 \right) \Leftrightarrow t(t-1) \geq 0$ true by $t \geq 1 \Rightarrow t-1 \geq 0 \Rightarrow t(t-1) \geq 0$. Proved.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum xy &\leq \sum x^2 \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \leq \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \\ &= \frac{\left(\frac{9}{16}\right) \sum a^2 b^2}{9S^2} \stackrel{\text{Goldstone}}{\leq} \frac{\left(\frac{9}{16}\right) 4R^2 s^2}{9r^2 s^2} = \left(\frac{R}{2r}\right)^2 \\ &\therefore \left(\frac{R}{2r}\right)^2 \stackrel{(1)}{\leq} \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{abc(a+b+c)}{S^2} &\geq \frac{2 \prod(a+b)}{abc} \Leftrightarrow \frac{16R^2 r^2 s^2 \cdot 2s}{r^2 s^2} \geq 4s(s^2 + 2Rr + r^2) \\ &\Leftrightarrow 8R^2 \stackrel{(i)}{\geq} s^2 + 2Rr + r^2 \end{aligned}$$

$$\text{Now, RHS of (i)} \stackrel{\text{Gerretsen}}{\leq} 4R^2 + 6Rr + 4r^2 \stackrel{?}{\leq} 8R^2 \Leftrightarrow (R-2r)(2R+r) \stackrel{?}{\geq} 0 \rightarrow \text{true}$$

$$\Rightarrow (i) \text{ is true} \therefore \frac{abc(a+b+c)}{S^2} \geq \frac{2 \prod(a+b)}{abc}$$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course

$= \frac{S}{3}$, we get :

$$\frac{\frac{8}{27} m_a m_b m_c \left(\frac{2}{3}\right) (m_a + m_b + m_c)}{\frac{S^2}{9}} \geq \frac{\frac{16}{27} (m_a + m_b) (m_b + m_c) (m_c + m_a)}{\frac{8}{27} m_a m_b m_c}$$



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$$\begin{aligned} \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} &\stackrel{(2)}{\geq} \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \therefore (1), (2) \\ \Rightarrow \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} &\leq \left(\frac{R}{2r}\right)^2 \text{ (Proved)} \end{aligned}$$

Solution 3 by Bogdan Fuștei-Romania

Let $x = ab, y = bc, z = ca \Rightarrow xy = ab^2c, yz = abc^2, zx = a^2bc \Rightarrow xy + yz + zx = abc(a + b + c) \leq a^2b^2 + b^2c^2 + c^2a^2$ where a, b, c -lengths sides of the triangle ABC and m_a, m_b, m_c – can be the sides of on triangle with area S_m .

We can rewritten the relationship: $abc(a + b + c) \leq a^2b^2 + b^2c^2 + c^2a^2$ for

m_a, m_b, m_c :

$$m_a m_b m_c (m_a + m_b + m_c) \leq m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2$$

$$\text{But: } m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 = \frac{9}{16} (a^2 b^2 + b^2 c^2 + c^2 a^2),$$

$$bc = 2Rh_a \text{ (and analogs)} \Rightarrow a^2 b^2 + b^2 c^2 + c^2 a^2 = 4R^2 (h_a^2 + h_b^2 + h_c^2)$$

But: $w_a \geq h_a$ (and analogs) $\Rightarrow w_a^2 + w_b^2 + w_c^2 \geq h_a^2 + h_b^2 + h_c^2$ and with $w_a^2 \leq s(s - a)$ (and analogs) $\Rightarrow w_a^2 + w_b^2 + w_c^2 \leq s^2 \Rightarrow$

$$h_a^2 + h_b^2 + h_c^2 \leq s^2 \xrightarrow{4R^2} a^2 b^2 + b^2 c^2 + c^2 a^2 \leq 4R^2 s^2 \Rightarrow$$

$$m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 \leq 4R^2 s^2 \cdot \frac{9}{16} = \frac{9}{4} R^2 s^2 \xrightarrow{s^2 = s^2 r^2}$$

$$\frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \leq \frac{9}{4} R^2 s^2 \cdot \frac{1}{9s^2 r^2}$$

$$\frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \leq \frac{R^2}{4r^2} \Leftrightarrow \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \leq \left(\frac{R}{2r}\right)^2$$

$$m_a m_b m_c (m_a + m_b + m_c) \leq \left(\frac{R}{2r}\right)^2 ; (1)$$

We must show that:

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c}$$

$$S_m == \frac{3}{4} S; abc = 4RS; m_a m_b m_c = 4R_m S_m = 4R_m \cdot \frac{3}{4} S = 3SR_m \Rightarrow$$



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$$R_m = \frac{m_a m_b m_c}{3S}$$

$$S_m = s_m r_m; s_m = \frac{m_a + m_b + m_c}{2}$$

$$\frac{3}{4}S = \frac{m_a + m_b + m_c}{2} \cdot r_m \Rightarrow r_m = \frac{3S}{2(m_a + m_b + m_c)}$$

$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c}{3S} \cdot \frac{m_a + m_b + m_c}{3S} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

So, the inequality:

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c}$$

becomes: $\frac{R_m}{2r_m} \geq \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c}$

$$\frac{R}{2r} \geq \frac{(a + b)(b + c)(c + a)}{8abc} \Rightarrow \frac{4R}{r} \geq \frac{(a + b)(b + c)(c + a)}{abc}$$

$$(a + b)(b + c)(c + a) = 2s(s^2 + r^2 + 2Rr);$$

$$ab + bc + ca = s^2 + r^2 + 4Rr$$

$$2R(h_a + h_b + h_c) = s^2 + r^2 + 4Rr; bc = 2Rh_a (\text{and analogs})$$

$$2R(h_a + h_b + h_c - r) = s^2 + r^2 + 2Rr$$

$$(a + b)(b + c)(c + a) = 4Rs(h_a + h_b + h_c - r)$$

$$\frac{4R}{r} \geq \frac{4Rs(h_a + h_b + h_c - r)}{4Rrs} \Rightarrow 1 \geq \frac{h_a + h_b + h_c - r}{4R} \Rightarrow$$

$$4R \geq h_a + h_b + h_c - r \Rightarrow 4R + r \geq h_a + h_b + h_c \Leftrightarrow$$

$$r_a + r_b + r_c \geq h_a + h_b + h_c$$

So, we get: $\frac{R}{2r} \geq \frac{(a+b)(b+c)(c+a)}{8abc} \Rightarrow$

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c}; (2)$$

From (1),(2) we have:

$$\frac{(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \leq \left(\frac{R}{2r}\right)^2$$



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1714. In ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian, the following relationship holds:

$$m_a^2 \geq \frac{1}{2}(n_a g_a + r_b r_c) \geq r_b r_c$$

Proposed by Bogdan Fuștei-Romania

Solution 1 by Adrian Popa-Romania

$$1) \frac{1}{2}(n_a g_a + r_b r_c) \geq r_b r_c \Leftrightarrow n_a g_a + r_b r_c \geq 2r_b r_c \Leftrightarrow n_a g_a \geq r_b r_c$$

$$r_b r_c = \frac{s}{s-a} \cdot \frac{s}{s-c} = \frac{s^2}{(s-a)(s-c)} = \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)(s-c)} = s(s-a) \Rightarrow$$

$$n_a g_a \geq s(s-a) \text{ true.}$$

$$2) m_a^2 \geq \frac{1}{2}(n_a g_a + r_b r_c) \Leftrightarrow 2m_a^2 \geq n_a g_a + r_b r_c$$

$$\begin{cases} n_a^2 + g_a^2 = 4m_a^2 - 2s(s-a) \\ n_a^2 + g_a^2 \stackrel{AGM}{\geq} 2n_a g_a \end{cases} \Rightarrow 2n_a g_a \leq 4m_a^2 - 2s(s-a) \Rightarrow$$

$$n_a g_a \leq 2m_a^2 - s(s-a) \Rightarrow 2m_a^2 \geq n_a g_a + s(s-a) \Rightarrow$$

$$m_a^2 \geq \frac{1}{2}(n_a g_a + s(s-a)) = \frac{1}{2}(n_a g_a + r_b r_c)$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) \\ &= a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\ &= a g_a^2 + a(s-b)(s-c) \\ &\quad \therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s-a)^2 \\ &\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2 \end{aligned}$$

Let $s-a=x$, $s-b=y$ and $s-c=z$ $\therefore s=x+y+z \Rightarrow a=y+z$, $b=z+x$ and $c=x+y$



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Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (\text{a}) \text{ is true} \Rightarrow n_a g_a \stackrel{(1)}{\geq} s(s-a)$$

$$\text{Also, Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b)$$

$$= a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c)$$

$$= a g_a^2 + a(s-b)(s-c)$$

$$\text{Adding the above two, we get : } (b^2 + c^2)(2s - b - c)$$

$$= a n_a^2 + a g_a^2 + 2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2)$$

$$= 2(n_a^2 + g_a^2) + a^2 - (b-c)^2$$

$$\Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2 + g_a^2)$$

$$\Rightarrow 4m_a^2 + (b-c)^2 + 4r_b r_c = 2(n_a^2 + g_a^2) + 4r_b r_c$$

$$\Rightarrow 4m_a^2 + (b-c)^2 + 4s(s-a) = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + 4m_a^2$$

$$= 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow n_a^2 + g_a^2 = 4m_a^2 - 2r_b r_c$$

$$\Rightarrow 4m_a^2 - 2r_b r_c \stackrel{\text{A-G}}{\geq} 2n_a g_a \Rightarrow m_a^2 \stackrel{(i)}{\geq} \frac{1}{2}(n_a g_a + r_b r_c)$$

$$\text{Again, (1)} \Rightarrow 2r_b r_c \leq r_b r_c + n_a g_a \Rightarrow \frac{1}{2}(n_a g_a + r_b r_c) \stackrel{(ii)}{\geq} r_b r_c \therefore (i), (ii) \Rightarrow m_a^2$$

$$\geq \frac{1}{2}(n_a g_a + r_b r_c) \geq r_b r_c \text{ (Proved)}$$

1715. In ΔABC the following relationship holds:

$$8 \prod_{cyc} \frac{m_a^2 + m_b^2}{(m_a + m_b)^2} \leq \left(\frac{R}{2r}\right)^4$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam



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$$(m_a + m_b)^2(m_b + m_c)^2(m_c + m_a)^2 \stackrel{AGM}{\geq} 4m_a m_b \cdot 4m_b m_c \cdot 4m_c m_a = 64(m_a m_b m_c)^2$$

$$\text{Let: } x = m_a^2; y = m_b^2; z = m_c^2 \Rightarrow x + y + z = \frac{3}{4}(a^2 + b^2 + c^2) =$$

$$= \frac{3}{4} \cdot 2(s^2 - 4Rr - r^2)$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{h_a^2} + \frac{1}{h_b^2} + \frac{1}{h_c^2} = \frac{s^2 - 4Rr - r^2}{2s^2r^2} \Rightarrow$$

$$LHS = 8 \prod_{cyc} \frac{m_a^2 + m_b^2}{(m_a + m_b)^2} \leq \frac{8(x+y)(y+z)(z+x)}{64xyz} =$$

$$= \frac{1}{8} \left[(x+y+z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 1 \right] = \frac{1}{8} \left[\frac{3(s^2 - 4Rr - r^2)^2}{4s^2r^2} - 1 \right] \stackrel{(*)}{\leq} \left(\frac{R}{2r} \right)^4$$

$$\Leftrightarrow r^2[3(s^2 - 4Rr - r^2)^2 - 4s^2r^2] \leq 2s^2R^4$$

$$\Leftrightarrow r^2[3(s^4 + 16R^2r^2 + r^4 - 8Rrs^2 - 2s^2r^2 + 8Rr^3) - 4s^2r^2] \leq 2s^2R^4$$

$$\Leftrightarrow r^2(3s^4 + 48R^2r^2 + 3r^4 - 24Rrs^2 - 10s^2r^2 + 24Rr^3) \leq 2s^2R^4$$

$$\Leftrightarrow 2(R^4 + 12Rr^3 + 5r^4)s^2 \geq 3r^2s^4 + r^2(48R^2r^2 + 3r^4 + 24Rr^3)$$

$$\Leftrightarrow [2(R^4 + 12Rr^3 + 5r^4) - 3r^2s^2]s^2 \stackrel{(**)}{\geq} r^2(48R^2r^2 + 3r^4 + 24Rr^3)$$

But: $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ (Gerretsen)

$$\Rightarrow 2(R^4 + 12Rr^3 + 5r^4) - 3r^2s^2 \geq 2(R^4 + 12Rr^3 + 5r^4) - 3r^2(4R^2 + 4Rr + 3r^2)$$

$$= 2R^4 + 12Rr^3 + r^2 - 12R^2r^2 \stackrel{t=\frac{R}{r} \geq 2}{=} t^4 - 12t^2 + 12t + 1 = 2t(t^3 - 6t + 6) + 1 \geq 2 \cdot 2(2^3 - 6 \cdot 2 + 6) = 9 > 0$$

So, we need to prove:

$$(2R^4 + 12Rr^3 + r^2 - 12R^2r^2)(16R - 5r) \geq 3r^3(16R^2 + r^2 + 8Rr)$$

$$\Leftrightarrow (2t^4 - 12t^2 + 12t + 1)(16t - 5) \geq 3(16t^2 + 8t + 1)$$

$$\Leftrightarrow 32t^5 - 10t^4 - 192t^3 + 204t^2 - 68t - 8 \geq 0$$

$$\Leftrightarrow 2(t-2)(16t^4 + 27t^3 - 42t^2 + 18t + 2) \geq 0$$

Which is clearly true, because: $t \geq 2 \Rightarrow t-2 \geq 0$

$$16t^4 + 27t^3 - 42t^2 + 18t + 2 = t[(t(16t^2 + 27t) - 42) + 18] + 2 = 342 > 0 \Rightarrow (**)$$

is true $\Rightarrow (*)$ true. Proved.



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum xy \leq \sum x^2 \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \leq \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2}$$

$$= \frac{\left(\frac{9}{16}\right) \sum a^2 b^2}{9S^2} \stackrel{\text{Goldstone}}{\leq} \frac{\left(\frac{9}{16}\right) 4R^2 s^2}{9r^2 s^2} = \left(\frac{R}{2r}\right)^2$$

$\therefore \left(\frac{R}{2r}\right)^4 \geq \left\{ \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \right\}^2$ and so, it suffices to prove :

$$\left\{ \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \right\}^2 \stackrel{(1)}{\geq} 8 \prod \frac{m_a^2 + m_b^2}{(m_a + m_b)^2}$$

$$\text{Now, } 8 \prod \frac{b^2 + c^2}{(b + c)^2} = 8 \prod \left(\frac{1}{1 + \frac{2bc}{b^2 + c^2}} \right) \stackrel{\text{Bandila}}{\leq} 8 \prod \left(\frac{1}{1 + \frac{2r}{R}} \right)$$

$$= \frac{8R^3}{(R + 2r)^3} \stackrel{\text{A-G}}{\leq} \frac{8R^3}{4R \cdot 2r(R + 2r)} = \frac{R^2}{r(R + 2r)} \stackrel{\text{Euler}}{\leq} \frac{R^2}{r(R + R)} = \frac{R}{2r}$$

$$\stackrel{\text{Euler}}{\leq} \left(\frac{R}{2r}\right)^2 \therefore \left(\frac{R}{2r}\right)^2 \geq 8 \prod \frac{b^2 + c^2}{(b + c)^2} \Rightarrow \left\{ \frac{abc(a + b + c)}{16S^2} \right\}^2$$

$$\geq 8 \prod \frac{b^2 + c^2}{(b + c)^2} \text{ applying which on a triangle with sides } \frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$$

$$\text{whose area of course } = \frac{S}{3}, \text{ we get : } \left\{ \frac{\frac{8}{27} m_a m_b m_c \left(\frac{2}{3}\right) (m_a + m_b + m_c)}{\frac{16S^2}{9}} \right\}^2$$

$$\geq 8 \prod \frac{\left(\frac{4}{9}\right) (m_a^2 + m_b^2)}{\left(\frac{4}{9}\right) (m_a + m_b)^2}$$

$$\Rightarrow \left\{ \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \right\}^2 \geq 8 \prod \frac{m_a^2 + m_b^2}{(m_a + m_b)^2} \Rightarrow (1) \text{ is true } \therefore 8 \prod \frac{m_a^2 + m_b^2}{(m_a + m_b)^2}$$

$$\leq \left(\frac{R}{2r}\right)^4 \text{ (Proved)}$$



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1716. In ΔABC then following relationship holds:

$$\frac{2}{3} \left(\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \right) \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum xy &\leq \sum x^2 \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \leq \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \\ &= \frac{\left(\frac{9}{16}\right) \sum a^2 b^2}{9S^2} \stackrel{\text{Goldstone}}{\leq} \frac{\left(\frac{9}{16}\right) 4R^2 s^2}{9r^2 s^2} = \left(\frac{R}{2r}\right)^2 \\ &\therefore \left(\frac{R}{2r}\right)^2 \stackrel{(1)}{\geq} \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \end{aligned}$$

$$\begin{aligned} \text{Now, } \frac{R}{2r} &\geq \frac{1}{4} + \frac{\sum a^3}{4abc} = \frac{4Rrs + 2s(s^2 - 6Rr - 3r^2)}{16Rrs} = \frac{s^2 - 4Rr - 3r^2}{8Rr} \\ &\Leftrightarrow s^2 - 4Rr - 3r^2 \leq 4R^2 \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \rightarrow \text{true} \end{aligned}$$

$$(\text{Gerretsen}) \therefore \frac{abc(a+b+c)}{16S^2}$$

$\geq \frac{1}{4} + \frac{\sum a^3}{4abc}$ applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course $= \frac{S}{3}$

$$\text{we get : } \frac{\frac{8}{27} m_a m_b m_c \left\{ \frac{2}{3} (m_a + m_b + m_c) \right\}}{16 \left(\frac{S^2}{9} \right)} \geq \frac{1}{4} + \frac{\frac{8}{27} (m_a^3 + m_b^3 + m_c^3)}{4 \left(\frac{8}{27} \right) m_a m_b m_c}$$

$$\Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c}$$

$$\therefore \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \stackrel{(2)}{\geq} \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c}$$

$$\text{Again, } \frac{1}{4} + \frac{\sum a^3}{4abc} \geq \left(\frac{\sum a^2}{\sum ab} \right)^2 \Leftrightarrow \frac{1}{4} + \frac{3abc + (\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \left(\frac{\sum a^2}{\sum ab} \right)^2$$

$$\Leftrightarrow \frac{(\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \left(\frac{\sum a^2}{\sum ab} \right)^2 - 1$$



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$$\Leftrightarrow \frac{(\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \frac{(\sum a^2 - \sum ab)(\sum a^2 + \sum ab)}{(\sum ab)^2}$$

$$\Leftrightarrow \left(\sum a^2 - \sum ab \right) \left\{ \frac{\sum a}{4abc} - \frac{\sum a^2 + \sum ab}{(\sum ab)^2} \right\} \geq 0 \therefore \text{in order to prove :}$$

$$\frac{1}{4} + \frac{\sum a^3}{4abc} \geq \left(\frac{\sum a^2}{\sum ab} \right)^2, \text{ it suffices to prove : } \frac{\sum a}{4abc} > \frac{\sum a^2 + \sum ab}{(\sum ab)^2} \Leftrightarrow (s^2 + 4Rr + r^2)^2 \\ > 8Rr(3s^2 - 4Rr - r^2)$$

$$\Leftrightarrow s^4 - s^2(16Rr - 2r^2) + r^2(48R^2 + 16Rr + r^2) \stackrel{(i)}{\geq} 0$$

Gerretsen

$$\text{Now, LHS of (i)} \stackrel{\geq}{\text{ }} s^2(16Rr - 5r^2) - s^2(16Rr - 2r^2) + r^2(48R^2 + 16Rr + r^2) \\ = r^2(48R^2 + 16Rr + r^2 - 3s^2)$$

Gerretsen

$$\stackrel{\geq}{\text{ }} r^2(48R^2 + 16Rr + r^2 - 12R^2 - 12Rr - 9r^2)$$

$$= 4r^2(9R^2 + r(R - 2r)) \stackrel{\text{Euler}}{\geq} 36R^2r^2 > 0 \Rightarrow (i) \text{ is true} \therefore \frac{1}{4} + \frac{\sum a^3}{4abc}$$

$$\geq \left(\frac{\sum a^2}{\sum ab} \right)^2$$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we get

$$\begin{aligned} & \cdot \frac{1}{4} + \frac{\frac{8}{27}(m_a^3 + m_b^3 + m_c^3)}{4\left(\frac{8}{27}\right)m_a m_b m_c} \geq \left(\frac{\frac{4}{9}(m_a^2 + m_b^2 + m_c^2)}{\frac{4}{9}(m_a m_b + m_b m_c + m_c m_a)} \right)^2 \\ & \Rightarrow \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \stackrel{(3)}{\geq} \left(\frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2 \therefore (1), (2), (3) \Rightarrow \\ & \left(\frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \\ & \geq \left(\frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2 \Rightarrow \frac{R}{2r} \stackrel{(4)}{\geq} \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \end{aligned}$$



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$$\begin{aligned}
 \text{Now, } \frac{2}{3} \sum \frac{a}{b+c} &\leq \frac{\sum a^2}{\sum ab} \Leftrightarrow \left(\frac{2}{3}\right) \frac{\sum a(c+a)(a+b)}{2s(s^2 + 2Rr + r^2)} \leq \frac{2(s^2 - 4Rr - r^2)}{s^2 + 4Rr + r^2} \\
 &\Leftrightarrow \left(\frac{1}{3}\right) \frac{\sum \{a(ab + a^2)\}}{2s(s^2 + 2Rr + r^2)} \leq \frac{s^2 - 4Rr - r^2}{s^2 + 4Rr + r^2} \\
 &\Leftrightarrow \left(\frac{1}{3}\right) \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \leq \frac{s^2 - 4Rr - r^2}{s^2 + 4Rr + r^2} \\
 &\Leftrightarrow 3(s^2 - 4Rr - r^2)(s^2 + 2Rr + r^2) \geq (s^2 + 4Rr + r^2)(2s^2 - 2Rr - 2r^2) \\
 &\Leftrightarrow s^4 - 12Rrs^2 - r^2(4R + r)^2 \stackrel{(ii)}{\geq} 0
 \end{aligned}$$

$$\text{Now, LHS of (ii)} \stackrel{\text{Gerretsen}}{\leq} s^2(4Rr - 5r^2)$$

$$-r^2(4R + r)^2 \stackrel{\text{Gerretsen}}{\geq} r^2\{(16R - 5r)(4R - 5r) - (4R + r)^2\}$$

$$= 12r^2(R - 2r)(4R - r) \stackrel{\text{Euler}}{\geq} 0$$

$$\Rightarrow \text{(ii) is true} \Leftrightarrow \frac{2}{3} \sum \frac{a}{b+c}$$

$\leq \frac{\sum a^2}{\sum ab}$ applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$, we get :

$$\begin{aligned}
 \frac{2}{3} \sum \left\{ \frac{\frac{2}{3}m_a}{\frac{2}{3}(m_b + m_c)} \right\} &\leq \frac{\left(\frac{4}{9}\right) \sum m_a^2}{\left(\frac{4}{9}\right) \sum m_a m_b} \Rightarrow \frac{2}{3} \left(\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \right) \\
 &\leq \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \stackrel{\text{by (4)}}{\geq} \frac{R}{2r} \text{ (Proved)}
 \end{aligned}$$

Solution 2 by Bogdan Fuștei-Romania

$$\frac{2}{3} \left(\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} \right) \leq \frac{R}{r_1}$$

$$\frac{2}{3} \left(\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} \right) \stackrel{CBS}{\leq} \frac{2}{3} \sqrt{a^2 + b^2 + c^2} \cdot \sqrt{\frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2}}$$

$$\begin{cases} a^2 + b^2 + c^2 \leq 9R^2 (\text{Leibniz}) \\ \frac{1}{a_1^2} + \frac{1}{b_1^2} + \frac{1}{c_1^2} \leq \frac{1}{4r_1^2} \end{cases} \Rightarrow \frac{2}{3} \left(\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} \right) \leq \frac{2}{3} \cdot 3R \cdot \frac{1}{2r_1} \Rightarrow$$



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$$\frac{2}{3} \left(\frac{a}{a_1} + \frac{b}{b_1} + \frac{c}{c_1} \right) \leq \frac{R}{r_1}$$

Let be $a_1 = b + c, b_1 = c + a, c_1 = a + b, s_1 = \frac{a_1 + b_1 + c_1}{2} = 2s$

$$2sr_1 = \sqrt{2s(2s - b - c)(2s - c - a)(2s - a - b)} = \sqrt{2sabc} \xrightarrow[=4Rrs]{abc=4RS}$$

$$2sr_1 = \sqrt{2s \cdot 4Rrs} = 2s\sqrt{2Rr} \Rightarrow r_1 = \sqrt{2Rr}$$

So, we have the following relationship holds:

$$\frac{2}{3} \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \leq \sqrt{\frac{R^2}{2Rr}} = \sqrt{\frac{R}{2r}}$$

m_a, m_b, m_c can be the sides of the triangle, then we get:

$$\frac{2}{3} \left(\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \right) \leq \sqrt{\frac{R_m}{2r_m}}; (1)$$

$$S_m = \frac{3}{4}S; abc = 4RS \Rightarrow m_a m_b m_c = 4R_m S_m = 4R_m \cdot \frac{3}{4}S = 3SR_m$$

$$R_m = \frac{m_a m_b m_c}{3S}; S_m = s_m r_m; s_m = \frac{m_a + m_b + m_c}{2}$$

$$\frac{3}{4}S = r_m \cdot \frac{m_a + m_b + m_c}{2} \Rightarrow r_m = \frac{3S}{2(m_a + m_b + m_c)}$$

$$\Rightarrow \frac{R_m}{2r_m} = \frac{m_a m_b m_c}{3S} \cdot \frac{m_a + m_b + m_c}{3S} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

Using $\frac{1}{2}s^2R \geq m_a m_b m_c$

$$m_a + m_b + m_c \leq 4R + r; 4R + r \leq \frac{9R}{2} \Leftrightarrow 8R + 2r \leq 9R \Leftrightarrow R \geq 2r (\text{Euler})$$

$$\Rightarrow m_a + m_b + m_c \leq \frac{9R}{2}$$

$$\frac{1}{2}R_s^2 \cdot \frac{9R}{2} \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{s^2} \xrightarrow{\frac{1}{r^2}}$$

$$\frac{R^2}{4r^2} \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} = \frac{R_m}{2r_m} \Rightarrow \frac{R}{2r} \geq \sqrt{\frac{R_m}{2r_m}}; (2)$$

From (1),(2) we get:



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$$\frac{2}{3} \left(\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \right) \leq \frac{R}{2r}$$

1717. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$n_a r_a + n_b r_b + n_c r_c \geq \frac{m_a h_b h_c}{g_a} + \frac{m_b h_c h_a}{g_b} + \frac{m_c h_a h_b}{g_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c)$$

$$\begin{aligned} \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= a n_a^2 + a(as - s^2) \end{aligned}$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc)$$

$$= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}$$

$$= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s - a} \right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(1)}{\cong} s^2 - 2h_a r_a$$

$$\text{Again, Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b)$$

$$= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c)$$

$$= a g_a^2 + a(s - b)(s - c)$$

$$\therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2$$

$$\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s$$

$$- b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and c

$$= x + y$$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2$$



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$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$\Rightarrow (\text{a}) \text{ is true} \Rightarrow n_a g_a \geq s(s-a)$

$$\Rightarrow n_a g_a r_a \geq s(s-a) \left(\frac{rs}{s-a} \right) = rs^2 \Rightarrow n_a g_a r_a \geq rs^2 \Rightarrow \frac{n_a g_a r_a}{h_b h_c} \geq \frac{bc \cdot rs^2}{4r^2 s^2} = \frac{bc}{4r}$$

$$= \left(\frac{R}{2r} \right) h_a \stackrel{\text{Panaitopol}}{\geq} m_a \Rightarrow n_a r_a \geq \frac{m_a h_b h_c}{g_a} \text{ and analogs}$$

$$\Rightarrow n_a r_a + n_b r_b + n_c r_c \geq \frac{m_a h_b h_c}{g_a} + \frac{m_b h_c h_a}{g_b} + \frac{m_c h_a h_b}{g_c} \text{ (Proved)}$$

1718. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\min \left(\left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1}, \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \right) \geq \frac{R}{2r}$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) \\
 & = an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 & = ag_a^2 + a(s-b)(s-c) \\
 & \quad \therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s-a)^2 \\
 & \Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2
 \end{aligned}$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c=x+y$

Using these substitutions, (a)

$$\begin{aligned}
 & \Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\} \{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2 \\
 & \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true} \\
 & \Rightarrow (\text{a}) \text{ is true} \Rightarrow n_a g_a \geq s(s-a) \text{ and analogs}
 \end{aligned}$$



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$$\Rightarrow \sum n_a g_a \geq s \sum (s - a) = s^2 \Rightarrow \left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1} \geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1}$$

$$= s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\} = \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r}$$

$$\Rightarrow \left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1} \stackrel{(m)}{\geq} \frac{R}{2r}$$

$$\text{Again, } \sum m_a w_a \stackrel{\text{loscu}}{\geq} \sum \left\{ \left(\frac{b+c}{2} \right) \cos \frac{A}{2} \left(\frac{2bc}{b+c} \right) \cos \frac{A}{2} \right\} = \sum \left[bc \left\{ \frac{s(s-a)}{bc} \right\} \right]$$

$$= s \sum (s - a) = s^2$$

$$\Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1} = s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\}$$

$$= \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r} \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \stackrel{(n)}{\geq} \frac{R}{2r}$$

$$(m), (n) \Rightarrow \min \left(\left(\sum n_a g_a \right) \left(\sum h_a h_b \right)^{-1}, \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \right) \geq \frac{R}{2r} \text{ (Proved)}$$

1719. In any ΔABC holds:

$$\frac{s}{2R} \left(5 - \frac{2r}{R} \right) \stackrel{(1)}{\leq} \sum \frac{h_a}{s-a} \stackrel{(2)}{\leq} \frac{s}{2R} \left(2 + \frac{R}{r} \right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{s-a} = \frac{2rs}{a(s-a)} = \frac{2 \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2} \cdot 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\left(2 \cos^2 \frac{B}{2} \right) \left(2 \cos^2 \frac{C}{2} \right)}{2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \frac{2R}{s} (1 + \cos B)(1 + \cos C) \text{ and analogs}$$

$$\Rightarrow \sum \frac{h_a}{s-a} = \frac{2R}{s} \sum (1 + \cos B)(1 + \cos C) = \frac{2R}{s} \sum (1 + (\cos B + \cos C) + \cos B \cos C)$$



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$$\begin{aligned}
 &= \frac{2R}{s} \left(3 + \frac{2(R+r)}{R} + \frac{\frac{(R+r)^2}{R^2} - 3 + \frac{s^2 - 4Rr - r^2}{2R^2}}{2} \right) \\
 &= \frac{2R}{s} \left(\frac{12R^2 + 8R(R+r) + 2(R+r)^2 - 6R^2 + s^2 - 4Rr - r^2}{4R^2} \right) \\
 \therefore \sum \frac{h_a}{s-a} &\stackrel{(m)}{\cong} \frac{s^2 + (4R+r)^2}{2Rs} \text{ and so, (2)} \Leftrightarrow \frac{s^2 + (4R+r)^2}{2Rs} \leq \frac{s(R+2r)}{2Rr} \Leftrightarrow (R+2r)s^2 \\
 &\geq rs^2 + r(4R+r)^2 \\
 &\stackrel{(i)}{\Leftrightarrow} (R+r)s^2 \stackrel{?}{\geq} r(4R+r)^2
 \end{aligned}$$

Now, LHS of (i) $\stackrel{\text{Gerretsen}}{\cong} (R+r)(16Rr - 5r^2) \stackrel{?}{\cong} r(4R+r)^2 \Leftrightarrow 3Rr \stackrel{?}{\geq} 6r^2$

\rightarrow true (Euler) \Rightarrow (i) \Rightarrow (2) is true

$$\text{Again, (m)} \Rightarrow (1) \Leftrightarrow \frac{s^2 + (4R+r)^2}{2Rs} \geq \frac{s(5R - 2r)}{2R^2} \stackrel{(ii)}{\Leftrightarrow} R(4R+r)^2 \stackrel{?}{\geq} (4R - 2r)s^2$$

$$\begin{aligned}
 &\text{Now, RHS of (ii)} \stackrel{\text{Rouche}}{\leq} (4R - 2r) \\
 &- 2r \left(2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \right) \stackrel{?}{\leq} R(4R+r)^2 \\
 &\Leftrightarrow R(4R+r)^2 - (2R^2 + 10Rr - r^2)(4R - 2r) \stackrel{?}{\geq} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr} \\
 &\Leftrightarrow (R - 2r)(8R^2 - 12Rr + r^2) \stackrel{(iii)}{\cong} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr}
 \end{aligned}$$

$\because R - 2r \stackrel{\text{Euler}}{\cong} \therefore$ in order to prove (iii), it suffices to prove : $8R^2 - 12Rr + r^2$

$$> 2(4R - 2r)\sqrt{R^2 - 2Rr}$$

$$\begin{aligned}
 &\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 4(R^2 - 2Rr)(4R - 2r)^2 > 0 \Leftrightarrow r^2(4R+r)^2 > 0 \rightarrow \text{true} \\
 &\Rightarrow (iii) \Rightarrow (ii) \Rightarrow (1) \text{ is true (Proved)}
 \end{aligned}$$



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1720. In ΔABC the following relationship holds:

$$\prod_{cyc} (m_a^5 - h_a^5 + w_a^5) \leq \left(\prod_{cyc} (m_a - h_a + w_a) \right)^5$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned}
 & (x - y + z)^5 = \\
 &= x^5 - y^5 + z^5 + 5xy(y^3 - x^3) + 5xy(x^3 + z^3) + 10x^2y^2(x - y) + 10x^2z^2(x + z) \\
 &\quad + 10y^2z^2(z - y) - 20xyz(x^2 + y^2 + z^2) + 30xyz(xy + yz + zx) \Rightarrow \\
 & (x - y + z)^5 - (x^5 - y^5 + z^5) = \\
 &= 5(x - y)(x + z)(z - y)[x^2 + y^2 + z^2 - (xy + yz + zx)] \geq 0
 \end{aligned}$$

Because $x \geq z \geq y > 0$; $x^2 + y^2 + z^2 - (xy + yz + zx) \geq 0$

Choose: $x = m_a$; $y = h_a$; $z = w_a$; ($x \geq z \geq y > 0$) \Rightarrow

$$(m_a - h_a + w_a)^5 \geq m_a^5 - h_a^5 + w_a^5 \Rightarrow$$

$$\prod_{cyc} (m_a^5 - h_a^5 + w_a^5) \leq \left(\prod_{cyc} (m_a - h_a + w_a) \right)^5$$

Proved.

1721. In ΔABC the following relationship holds:

$$\frac{9(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \leq 1 + 2 \left(\frac{R}{r} \right)^2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned}
 & \text{Let } x = m_a; y = m_b; z = m_c \Rightarrow x + y + z \leq 4R + r; \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{r} \Rightarrow \\
 & \frac{(x + y)(y + z)(z + x)}{xyz} = \frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{y}{z} + \frac{z}{y} + 2 = (x + y + z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 1 \Rightarrow
 \end{aligned}$$



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$$\begin{aligned}
 LHS &= \frac{9}{8} \left[(x+y+z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) - 1 \right] \leq \frac{9}{8} \left[(4R+r) \cdot \frac{1}{r} - 1 \right] = \frac{9}{8} \cdot \frac{4R}{r} \\
 &= \frac{9R}{2r} \stackrel{(*)}{\leq} 1 + 2 \left(\frac{R}{r} \right)^2
 \end{aligned}$$

$$\begin{aligned}
 (*) \Leftrightarrow 2t^2 + 1 - \frac{9}{2}t \geq 0 &\Leftrightarrow 4t^2 - 9t + 2 \geq 0 \Leftrightarrow (t-2)(4t-1) \geq 0 \text{ true by } t \geq 2 \Rightarrow \\
 t-2 \geq 0 &\Rightarrow 4t-1 \geq 8-1 = 7 > 0. \text{ Proved.}
 \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum xy &\leq \sum x^2 \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \leq \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2} \\
 &= \frac{\left(\frac{9}{16}\right) \sum a^2 b^2}{9S^2} \stackrel{\text{Goldstone}}{\geq} \frac{\left(\frac{9}{16}\right) 4R^2 s^2}{9r^2 s^2} = \left(\frac{R}{2r}\right)^2 \\
 \therefore \left(\frac{R}{r}\right)^2 &\geq \frac{4m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \Rightarrow \text{RHS} \stackrel{(1)}{\geq} 1 + \frac{8m_a m_b m_c (m_a + m_b + m_c)}{9S^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } 1 + \frac{abc(a+b+c)}{2S^2} &\geq \frac{9(a+b)(b+c)(c+a)}{8abc} \Leftrightarrow 1 + \frac{8Rrs^2}{2r^2 s^2} \\
 &\geq \frac{18s(s^2 + 2Rr + r^2)}{32Rrs} \stackrel{(i)}{\geq} 16R(4R+r) \stackrel{(i)}{\geq} 9s^2 + 9(2Rr + r^2)
 \end{aligned}$$

$$\text{Now, RHS of (i)} \stackrel{\text{Gerretsen}}{\leq} 9(4R^2 + 4Rr + 3r^2) + 9(2Rr + r^2) \stackrel{?}{\leq} 16R(4R+r)$$

$$\Leftrightarrow 14R^2 - 19Rr - 18r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R-2r)(14R+9r) \stackrel{?}{\geq} 0$$

$$\rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (\text{i}) \text{ is true} \therefore 1 + \frac{abc(a+b+c)}{2S^2}$$

$$\geq \frac{9(a+b)(b+c)(c+a)}{8abc} \text{ applying which on a triangle with sides}$$

$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course $= \frac{S}{3}$, we get :

$$1 + \frac{\frac{8}{27} m_a m_b m_c \left(\frac{2}{3}\right) (m_a + m_b + m_c)}{\frac{2S^2}{9}} \geq \frac{\frac{72}{27} (m_a + m_b)(m_b + m_c)(m_c + m_a)}{\frac{64}{27} m_a m_b m_c}$$



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$$\Rightarrow 1 + \frac{8m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \stackrel{(2)}{\geq} \frac{9(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \therefore (1), (2)$$

$$\Rightarrow \frac{9(m_a + m_b)(m_b + m_c)(m_c + m_a)}{8m_a m_b m_c} \leq 1 + 2 \left(\frac{R}{r} \right)^2$$

1722. In ΔABC the following relationship holds:

$$\frac{a^4}{b^2} + \frac{b^4}{c^2} + \frac{c^4}{a^2} \geq \frac{3R(a^2 + b^2 + c^2)}{2(R + r)}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

Lemma: For $x, y, z > 0$, $\frac{x^2}{y} + \frac{y^2}{z} + \frac{z^2}{x} \geq \frac{3(x^2 + y^2 + z^2)}{x+y+z}$; (1)

Chose: $x = a^2$; $y = b^2$; $z = c^2$

$$(1) \Rightarrow \frac{a^4}{b^2} + \frac{b^4}{c^2} + \frac{c^4}{a^2} \geq \frac{3(a^4 + b^4 + c^4)}{a^2 + b^2 + c^2}$$

We need to prove:

$$\frac{3(a^4 + b^4 + c^4)}{a^2 + b^2 + c^2} \geq \frac{3R(a^2 + b^2 + c^2)}{2(R + r)} \Leftrightarrow$$

$$2(R + r)(a^4 + b^4 + c^4) \geq R(a^2 + b^2 + c^2)^2$$

$$4(R + r)(s^4 - (8Rr + 6r^2)s^2 + r^2(4R + r)^2) \geq 4R(s^2 - 4Rr - r^2)^2$$

$$(R + r)(s^4 - (8Rr + 6r^2)s^2 + r^2(4R + r)^2) \geq$$

$$\geq R(s^4 + 16R^2r^2 + r^4 - 8Rrs^2 - 2s^2r^2 + 8Rr^3)$$

$$(R + r)[s^4 - (8Rr + 6r^2)s^2 + r^2(4R + r)^2] \geq$$

$$\geq R[s^4 - (8Rr + 2r^2)s^2 + r^2(4R + r)^2]$$

$$rs^4 + [R(8Rr + 2r^2) - (R + r)(8Rr + 6r^2)]s^2 + [(R + r) - R]r^2(4R + r)^2 \geq 0$$

$$rs^4 + 2r[R(4R + r) - (R + r)(4R + 3r)]s^2 + r^3(4R + r)^2 \geq 0$$

$$s^4 - 2(6Rr + 3r^2)s^2 + r^2(4R + r)^2 \geq 0$$

$$s^2(s^2 - 12Rr - 6r^2) + r^2(4R + r)^2 \stackrel{(*)}{\geq} 0$$

$$\text{But: } s^2 \geq 16Rr - 5r^2 \Rightarrow LHS_{(*)} \geq s^2(4Rr - 11r^2) + r^2(4R + r)^2 \stackrel{(**)}{\geq} 0$$



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Now, we prove that: $(**)$ is true.

$$(**) \Leftrightarrow s^2(4R - 11r) + r(4R + r)^2 \geq 0$$

If $4R - 11r \geq 0$ then: $s^2(4R - 11r) + r(4R + r)^2 > 0$

If $\begin{cases} 4R - 11r < 0 \\ 2r \leq R \text{ (Euler)} \end{cases} \Rightarrow 2 \leq \frac{R}{r} < \frac{11}{4}$ then from: $s^2 \leq 4R^2 + 4Rr + 3r^2 \Rightarrow$

$$(4R^2 + 4Rr + 3r^2)(4R - 11r) + r(4R + r)^2 \geq 0 \Leftrightarrow$$

$$(4t^2 + 4t + 3)(4t - 11) + (4t + 1)^2 \geq 0; \left(t = \frac{R}{r}; 2 \leq t < \frac{11}{4} \right) \Leftrightarrow$$

$$16t^3 - 12t^2 - 24t - 32 \geq 0 \Leftrightarrow 4(t-2)(4t^2 + 5t + 4) \geq 0$$

Which is true, because: $2 \leq t < \frac{11}{4} \Rightarrow (**)\text{true} \Rightarrow (*)\text{true}$.

1723. In ΔABC the following relationship holds:

$$5 - \frac{2r}{R} \leq \sec^2 \frac{A}{2} + \sec^2 \frac{B}{2} + \sec^2 \frac{C}{2} \leq 1 + \frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } \Omega = \sum_{cyc} \sec^2 \frac{A}{2} = \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2}} = \frac{s^2 + (4R+r)^2}{s^2}$$

$$\begin{aligned} \Omega \stackrel{(1)}{\geq} 5 - \frac{2r}{R} \Leftrightarrow \frac{s^2 + (4R+r)^2}{s^2} \geq 5 - \frac{2r}{R} \Leftrightarrow \frac{(4R+r)^2}{s^2} \geq 4 - \frac{2r}{R} \Leftrightarrow R(4R+r)^2 \\ \geq s^2(4R-2r) \end{aligned}$$

$$\text{But: } s^2 \leq 2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)}$$

$$\begin{aligned} \Rightarrow s^2(4R-2r) &\leq (4R-2r) \left[2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R(R-2r)} \right] \\ &\stackrel{(*)}{\leq} R(4R+r)^2 \end{aligned}$$

$$(*) \xrightarrow[t=\frac{R}{r}\geq 2]{} (4t-2) \left[2t^2 + 10t - 1 + 2(t-2)\sqrt{t(t-2)} \right] \leq t(4t+1)^2$$

$$\Leftrightarrow 4(2t-1)(t-2)\sqrt{t(t-2)} \leq t(4t+1)^2 - (4t-2)(2t^2 + 10t - 1)$$

$$\Leftrightarrow 4(2t-1)(t-2)\sqrt{t(t-2)} \leq (t-2)(8t^2 - 12t + 1)$$

$$\Leftrightarrow (t-2) \left[8t^2 - 12t + 1 - (2t-1)\sqrt{t(t-2)} \right] \geq 0$$



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Because: $t \geq 2 \Rightarrow t - 2 \geq 0$

We just check: $8t^2 - 12t + 1 \geq (2t - 1)\sqrt{t(t - 2)} \stackrel{t \geq 2}{\Leftrightarrow}$

$$(8t^2 - 12t + 1)^2 \geq (2t - 1)^2 t(t - 2)$$

$$(8t^2 - 12t + 1)^2 - (2t - 1)^2 t(t - 2) \geq 0$$

$$60t^4 - 180t^3 + 151t^2 - 22t + 1 > 0$$

Which is clearly true, because:

$$f(t) = 60t^4 - 180t^3 + 151t^2 - 22t + 1; (t \geq 2)$$

$$f'(t) = 240t^3 - 540t^2 + 302t - 22$$

$$f''(t) = 720t^2 - 1080t + 302 = 360t(2t - 3) + 302 \stackrel{t \geq 2}{\geq} 1022 > 0 \Rightarrow$$

$$f' \uparrow [2; \infty) \Rightarrow f'(t) \geq f'(2) = 342 > 0 \Rightarrow$$

$$f \uparrow [2; \infty) \Rightarrow f(t) \geq f(2) = 81 > 0 \Rightarrow (*) \text{true} \Rightarrow (1) \text{true.}$$

$$\begin{aligned} \Omega &\stackrel{(2)}{\leq} 1 + \frac{3R}{2r} \Leftrightarrow \frac{s^2 + (4R + r)^2}{s^2} \leq 1 + \frac{3R}{2r} \Leftrightarrow 1 + \frac{(4R + r)^2}{s^2} 1 + \frac{3R}{2r} \\ &\Leftrightarrow 2r(4R + r)^2 \leq 3Rs^2 \end{aligned}$$

$$\text{But: } s^2 \geq 16Rr - 5r^2 \Rightarrow 3R(16Rr - 5r^2) \stackrel{(**)}{\geq} 2r(4R + r)^2$$

$$(**) \Leftrightarrow 48R^2r - 15Rr^2 \geq 2r(16R^2 + 8Rr + r^2)$$

$$\Leftrightarrow 48R^2 - 15Rr \geq 32R^2 + 16Rr^2 + 2r^2$$

$$\Leftrightarrow 16R^2 - 31Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(16R + r) \geq 0 \text{ true by } R \geq 2r \text{ (Euler)} \Rightarrow (**) \text{ is}$$

true \Rightarrow (2) is true. Proved.

1724. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) + \prod_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) > \frac{9}{2\pi} + \frac{1}{\prod_{cyc} (\pi - \mu(A))}$$

Proposed by Radu Diaconu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

Because: $0 < A, B, C < \frac{\pi}{2} \Rightarrow$

$$\sin^3 A, \sin^3 B, \sin^3 C > \sin^4 A, \sin^4 B, \sin^4 C > 0$$



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$$\cos^3 A, \cos^3 B, \cos^3 C > \cos^4 A, \cos^4 B, \cos^4 C > 0$$

$$\begin{aligned} \Omega &= \sum_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) > \sum_{cyc} \left(\frac{\sin^4 A}{\mu(B)} + \frac{\cos^4 A}{\mu(C)} \right) = \\ &= \sum_{cyc} \left(\frac{(\sin^2 A)^2}{\mu(B)} + \frac{(\cos^2 A)^2}{\mu(C)} \right) \stackrel{\text{Bergström}}{\geq} \sum_{cyc} \frac{(\sin^2 A + \cos^2 A)^2}{\mu(B) + \mu(C)} = \\ &= \sum_{cyc} \frac{1^2}{\mu(B) + \mu(C)} \stackrel{\text{Bergström}}{\geq} \frac{(1+1+1)^2}{2(\mu(A) + \mu(B) + \mu(C))} = \frac{9}{2\pi} \\ \Phi &= \prod_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) > \prod_{cyc} \left(\frac{\sin^4 A}{\mu(B)} + \frac{\cos^4 A}{\mu(C)} \right) = \\ &= \prod_{cyc} \left(\frac{(\sin^2 A)^2}{\mu(B)} + \frac{(\cos^2 A)^2}{\mu(C)} \right) \stackrel{\text{Bergström}}{\geq} \prod_{cyc} \left(\frac{(\sin^2 A + \cos^2 A)^2}{\mu(B) + \mu(C)} \right) = \\ &= \prod_{cyc} \left(\frac{1}{\pi - \mu(A)} \right) \end{aligned}$$

$$LHS = \sum_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) + \prod_{cyc} \left(\frac{\sin^3 A}{\mu(B)} + \frac{\cos^3 A}{\mu(C)} \right) > \frac{9}{2\pi} + \frac{1}{\prod_{cyc} (\pi - \mu(A))}$$

1725. In any ΔABC , n_a – Nagel's cevian, holds:

$$3 \sum a r_a \geq \sqrt{2}(2R - r) \sum (n_a + \sqrt{2h_a r_a})$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s - a) = a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &\quad = a n_a^2 + a(as - s^2) \\ &\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\ &\quad = as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \end{aligned}$$



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$$\begin{aligned}
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \\
 &\Rightarrow n_a + \sqrt{2h_a r_a} \stackrel{\text{CBS}}{\leq} \sqrt{2}\sqrt{s^2 - 2h_a r_a + 2h_a r_a} = \sqrt{2}s \text{ and analogs}
 \end{aligned}$$

$$\Rightarrow \sum(n_a + \sqrt{2h_a r_a}) \leq 3\sqrt{2}s \Rightarrow \sqrt{2}(2R - r) \sum(n_a + \sqrt{2h_a r_a}) \stackrel{(1)}{\geq} 6s(2R - r)$$

$$\begin{aligned}
 \text{Again, } 3 \sum ar_a &= 3 \sum 4R \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2} = 6Rs \sum 2 \sin^2 \frac{A}{2} = 6Rs \sum (1 - \cos A) \\
 &= 6Rs \left(3 - 1 - \frac{r}{R}\right) = 6s(2R - r)
 \end{aligned}$$

$$\stackrel{\text{by (1)}}{\geq} \sqrt{2}(2R - r) \sum(n_a + \sqrt{2h_a r_a}) \text{ (Proved)}$$

1726. In ΔABC , K –Lemoine's point, the following relationship holds:

$$\frac{48r^4}{R^3} \leq AK + BK + CK \leq \frac{3R^2}{2r}$$

Proposed by Marian Ursărescu-Romania

Solution 1 by Bogdan Fuștei-Romania

$$AK = \frac{2bc}{a^2 + b^2 + c^2} \cdot m_a \text{ (and analogs); } bc = 2Rh_a$$

$$\begin{aligned}
 AK &= \frac{4R \cdot h_a m_a}{a^2 + b^2 + c^2} \geq \frac{4R}{a^2 + b^2 + c^2} \cdot h_a^2 \text{ (and analogs); } m_a \geq h_a \\
 a^2 + b^2 + c^2 &\leq 9R^2 \text{ (Leibniz)}
 \end{aligned}$$

$$AK \geq \frac{4R}{a^2 + b^2 + c^2} \cdot h_a^2 \geq \frac{4R}{9R^2} \cdot h_a^2 = \frac{4h_a^2}{9R} \text{ (and analogs)}$$

$$h_a + h_b + h_c \geq 9r \Rightarrow (h_a + h_b + h_c)^2 \geq 81r^2$$

$$3(h_a^2 + h_b^2 + h_c^2) \stackrel{\text{CBS}}{\geq} (h_a + h_b + h_c)^2 \Rightarrow h_a^2 + h_b^2 + h_c^2 \geq 27r^2 \Rightarrow$$

$$\frac{4(h_a^2 + h_b^2 + h_c^2)}{9R} \geq \frac{4 \cdot 27r^2}{9R} = \frac{12r^2}{R} \Rightarrow AK + BK + CK \geq \frac{12r^2}{R}; (1)$$

We must show that: $\frac{12r^2}{R} \geq \frac{48r^4}{R^3} \Leftrightarrow 1 \geq \frac{4r^2}{R^2} \Leftrightarrow R^2 \geq 4r^2 \Leftrightarrow R \geq 2r$ (Euler)



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$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2);$$

$$AK \leq \frac{4R}{a^2 + b^2 + c^2} \cdot m_a^2 (\text{and analogs}) \Rightarrow$$

$$AK + BK + CK \leq \frac{4R \cdot \frac{3}{4}(a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} = 3R; (2)$$

$$\text{We must show: } 3R \leq \frac{3R^2}{2r} \Leftrightarrow 1 \leq \frac{R}{2r} \Leftrightarrow R \geq 2r (\text{Euler})$$

From (1),(2) we have:

$$\frac{48r^4}{R^3} \leq AK + BK + CK \leq \frac{3R^2}{2r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} AK &= \frac{2bcm_a}{\sum a^2} \text{ and analogs and } \frac{m_a}{\sum a^2} \leq \frac{1}{2\sqrt{3}a} \text{ and analogs } \therefore \sum AK \leq \sum \frac{2bc}{2\sqrt{3}a} \\ &= \frac{\sum b^2c^2}{4\sqrt{3}Rrs} \stackrel{\text{Goldstone}}{\leq} \frac{4R^2s^2}{4\sqrt{3}Rrs} = \frac{Rs}{\sqrt{3}r} \\ &\stackrel{\text{Mitrinovic}}{\leq} \frac{R^2 \left(\frac{3\sqrt{3}}{2} \right)}{\sqrt{3}r} = \frac{3R^2}{2r} \therefore AK + BK + CK \stackrel{(1)}{\leq} \frac{3R^2}{2r} \end{aligned}$$

Now, WLOG we may assume $a \geq b \geq c \therefore m_a \leq m_b \leq m_c$ and $bc \leq ca \leq ab \therefore \sum AK$

$$= \left(\frac{2}{\sum a^2} \right) \sum bcm_a$$

$$\begin{aligned} &\stackrel{\text{Chebyshev}}{\leq} \left(\frac{2}{3 \sum a^2} \right) (\sum bc) (\sum m_a) \stackrel{\text{Tereshin}}{\leq} \left(\frac{2}{3 \sum a^2} \right) 2R (\sum h_a) \left(\sum \frac{b^2 + c^2}{4R} \right) \\ &\geq \left(\frac{2}{6R \sum a^2} \right) 2R (9r) \sum a^2 = 6r \stackrel{?}{\geq} \frac{48r^4}{R^3} \Leftrightarrow R^3 \stackrel{?}{\geq} 8r^3 \\ &\Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler)} \therefore \frac{48r^4}{R^3} \stackrel{(2)}{\leq} AK + BK + CK \therefore (1), (2) \Rightarrow \frac{48r^4}{R^3} \\ &\leq AK + BK + CK \leq \frac{3R^2}{2r} \text{ (Proved)} \end{aligned}$$



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1727. In ΔABC the following relationship holds:

$$\sqrt[n]{m_a + r_a} + \sqrt[n]{m_b + r_b} + \sqrt[n]{m_c + r_c} \leq \frac{6n + 18R}{2n}, n \in \mathbb{N}, n \geq 2$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by George Florin Ţerban – Romania

Let: $f: (0, \infty) \rightarrow \mathbb{R}$, $f(x) = \sqrt[n]{x} = x^{\frac{1}{n}}$, $f'(x) = \frac{1}{n}x^{\frac{1}{n}-1}$,

$$f''(x) = \frac{1}{n}\left(\frac{1}{n} - 1\right)x^{\frac{1}{n}-2} < 0 \Rightarrow f \text{ –concave on } (0, \infty)$$

$$f\left(\frac{x+y+z}{3}\right) \geq \frac{f(x)+f(y)+f(z)}{3}; \text{ (Jensen)}$$

$$\sum_{cyc} \sqrt[n]{x} \leq 3 \cdot \sqrt[n]{\frac{1}{3} \sum_{cyc} x}$$

$$\sum_{cyc} \sqrt[n]{m_a + r_a} \leq 3 \cdot \sqrt[n]{\frac{\sum(m_a + r_a)}{3}} = 3 \cdot \sqrt[n]{\frac{\sum m_a + \sum r_a}{3}} =$$

$$= 3 \cdot \sqrt[n]{\frac{\sum m_a + 4R + r}{3}} \stackrel{\text{Euler}}{\leq} 3 \cdot \sqrt[n]{\frac{\frac{9R}{2} + 4R + \frac{R}{2}}{3}} = 3 \cdot \sqrt[n]{3R} \stackrel{(1)}{\leq} \frac{6n + 18R}{2n} =$$

$$(1) \Leftrightarrow 3 \cdot \sqrt[n]{3R} \leq 3 + \frac{9R}{n} \Leftrightarrow \sqrt[n]{3R} \leq 1 + \frac{3R}{n} \Leftrightarrow$$

$$\left(1 + \frac{3R}{n}\right)^n \stackrel{\text{Bernoulli}}{\geq} 1 + \frac{3R}{n} \cdot n = 1 + 3R > 3R \Leftrightarrow 1 + \frac{3R}{n} > \sqrt[n]{3R} \text{ true. Proved.}$$

1728. In any ΔABC , holds:

$$w_a + w_b + w_c \geq s + \left(6 - 3\sqrt{3} + \frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c}\right)r$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Now, } b + c - a &= 4R \cos \frac{A}{2} \cos \frac{B-C}{2} - 4R \cos \frac{A}{2} \sin \frac{A}{2} \\ &= 4R \cos \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) = 8R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &\stackrel{(1)}{\Rightarrow} s - a \triangleq 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{aligned}$$



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$$\text{Again, } AI = \frac{r}{\sin \frac{A}{2}} = \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}} = 4R \sin \frac{B}{2} \sin \frac{C}{2} \stackrel{\text{by (1)}}{\cong} \frac{s-a}{\cos \frac{A}{2}} \Rightarrow \cos \frac{A}{2} \stackrel{(2)}{\cong} \frac{s-a}{AI}$$

$$\text{We have, } \tan \frac{A}{4} = \frac{1 - \cos \frac{A}{2}}{\sin \frac{A}{2}} \stackrel{\text{by (2)}}{\cong} \frac{1 - \frac{s-a}{AI}}{\frac{r}{AI}} = \frac{AI - (s-a)}{r} \stackrel{(a)}{\cong} s - a + rtan \frac{A}{4}$$

$$\text{Similarly, } BI \stackrel{(b)}{\cong} s - b + rtan \frac{B}{4} \text{ and } CI \stackrel{(c)}{\cong} s - c + rtan \frac{C}{4} \therefore (a) + (b) + (c)$$

$$\Rightarrow \sum AI \stackrel{(3)}{\cong} s + r \sum \tan \frac{A}{4}$$

$$\text{Let } f(x) = \tan \left(\frac{x}{4} \right) \forall x \in (0, \pi) \therefore f''(x) = \frac{\tan \left(\frac{x}{4} \right) \sec^2 \left(\frac{x}{4} \right)}{8} > 0 \Rightarrow f(x) \text{ is convex}$$

$$\text{Let } \tan \frac{\pi}{12} = m \therefore \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}} = \frac{2m}{1-m^2} \Rightarrow m^2 + 2\sqrt{3}m - 1 = 0 \Rightarrow m$$

$$= \frac{-2\sqrt{3} \pm \sqrt{12+4}}{2} = 2 - \sqrt{3} \Rightarrow \tan \frac{\pi}{12} \stackrel{(4)}{\cong} 2 - \sqrt{3}$$

$$\text{Now, by (4), } \sum AI = s + r \sum \tan \frac{A}{4} \stackrel{\text{Jensen}}{\geq}$$

$$\geq s + 3rtan \frac{\pi}{12} \left(\text{as } f(x) = \tan \left(\frac{x}{4} \right) \text{ is convex which has been proved earlier} \right)$$

$$= s + 3r(2 - \sqrt{3}) = s - 3\sqrt{3}r + 6r \Rightarrow \sum AI \stackrel{(4)}{\cong} s - 3\sqrt{3}r + 6r$$

Let I be the incenter and let us join BI and AI and let AI produced meet BC at D.

Angle bisector theorem on $\triangle ABC \Rightarrow$

$$\frac{CD}{BD} = \frac{b}{c} \Rightarrow \frac{a}{BD} = \frac{b+c}{c} \stackrel{(i)}{\cong} BD \stackrel{(i)}{\cong} \frac{ac}{b+c} \text{ and angle bisector theorem on } \triangle ABD \Rightarrow \frac{AI}{DI} = \frac{AC}{BC} = \frac{a}{b+c}$$

$$= \frac{c}{BD} \stackrel{\text{by (i)}}{\cong} \frac{b+c}{a} \Rightarrow \frac{w_a}{DI} = \frac{2s}{a} \Rightarrow DI = \frac{aw_a}{2s}$$

$$\Rightarrow w_a = AI + DI = AI + \frac{aw_a}{2s} \text{ and analogs} \Rightarrow \sum w_a$$

$$= \sum AI + \frac{1}{2s} \sum aw_a \stackrel{\text{by (4)}}{\cong} s - 3\sqrt{3}r + 6r + \frac{1}{2s} \sum aw_a$$



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$$\therefore \sum w_a \stackrel{(5)}{\geq} s - 3\sqrt{3}r + 6r + \frac{1}{2s} \sum aw_a$$

$$\begin{aligned} r \sum \frac{w_a}{h_a} &= r \sum \frac{aw_a}{2rs} \stackrel{(6)}{\cong} \frac{1}{2s} \sum aw_a \therefore \text{by (5) and (6), } \sum w_a \geq s - 3\sqrt{3}r + 6r + r \sum \frac{w_a}{h_a} \\ &= s + \left(6 - 3\sqrt{3} + \frac{w_a}{h_a} + \frac{w_b}{h_b} + \frac{w_c}{h_c} \right) r \quad (\text{Proved}) \end{aligned}$$

1729. In any } ABC, holds:

$$\frac{m_a w_a + m_b w_b + m_c w_c}{h_a h_b + h_b h_c + h_c h_a} \geq \left(\frac{r_a + r_b + r_c + s\sqrt{3}}{m_a + m_b + m_c + h_a + h_b + h_c} \right)^2$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum m_a w_a &\stackrel{\text{loscu}}{\geq} \sum \left\{ \left(\frac{b+c}{2} \right) \cos \frac{A}{2} \left(\frac{2bc}{b+c} \right) \cos \frac{A}{2} \right\} = \sum \left[bc \left\{ \frac{s(s-a)}{bc} \right\} \right] = s \sum (s-a) \\ &= s^2 \end{aligned}$$

$$\begin{aligned} \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} &\geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1} = s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\} \\ &= \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r} \Rightarrow \text{LHS} \stackrel{(m)}{\geq} \frac{R}{2r} \end{aligned}$$

$$(m) \Rightarrow \text{it suffices to prove : } \sqrt{\frac{R}{2r}} \geq \frac{r_a + r_b + r_c + s\sqrt{3}}{m_a + m_b + m_c + h_a + h_b + h_c}$$

$$\Leftrightarrow \sqrt{\frac{R}{2r}} (m_a + m_b + m_c) + \sqrt{\frac{R}{2r}} (h_a + h_b + h_c) \stackrel{(1)}{\geq} r_a + r_b + r_c + s\sqrt{3}$$

$$\text{Now, } r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\cong} 4R \cos^2 \frac{A}{2}$$

$$\begin{aligned} \text{Now, } (b+c)^2 &\geq 32R \cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\cong} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\ &= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a) \end{aligned}$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c$$

$$\geq 4\sqrt{2R} \cos \frac{A}{2} \Rightarrow \sum m_a \stackrel{\text{loscu}}{\geq} \sum \left(\frac{b+c}{2} \cos \frac{A}{2} \right)$$



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$$\begin{aligned}
 & \geq \sqrt{2Rr} \sum 2\cos^2 \frac{A}{2} = \sqrt{2Rr} \sum (1 + \cos A) = \sqrt{2Rr} \left(4 + \frac{r}{R} \right) = \sqrt{\frac{2r}{R}} \left(\sum r_a \right) \\
 & \Rightarrow \sqrt{\frac{R}{2r}} (m_a + m_b + m_c) \stackrel{(a)}{\geq} r_a + r_b + r_c \\
 \text{Now, } & \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 \geq 3s^2 \Leftrightarrow \frac{(s^2 + 4Rr + r^2)^2}{8Rr} \stackrel{(ii)}{\geq} 3s^2 \Leftrightarrow (s^2 + 4Rr + r^2)^2 \geq 24Rrs^2 \\
 & \Leftrightarrow s^4 - s^2(16Rr - 2r^2) + r^2(4R + r)^2 \stackrel{\text{Gerretsen}}{\geq} 0 \\
 \text{LHS of (ii)} & \stackrel{\text{Trucht}}{\geq} s^2(16Rr - 5r^2) - s^2(16Rr - 2r^2) + r^2(4R + r)^2 \\
 & = r^2\{(4R + r)^2 - 3s^2\} \stackrel{(b)}{\geq} 0 \Rightarrow (\text{ii}) \text{ is true} \\
 \therefore & \sqrt{\frac{R}{2r}} (h_a + h_b + h_c) \stackrel{(b)}{\geq} s\sqrt{3} \therefore (\text{a}) + (\text{b}) \Rightarrow (\text{1}) \text{ is true} \therefore \frac{m_a w_a + m_b w_b + m_c w_c}{h_a h_b + h_b h_c + h_c h_a} \\
 & \geq \left(\frac{r_a + r_b + r_c + s\sqrt{3}}{m_a + m_b + m_c + h_a + h_b + h_c} \right)^2 \text{ (Proved)}
 \end{aligned}$$

1730. In ΔABC the following relationship holds:

$$\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a} \leq \frac{3R}{2r}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Marian Ursărescu-Romania

From Cauchy Inequality, we have:

$$\left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a} \right)^2 \leq (m_a^2 + m_b^2 + m_c^2) \left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \right)$$

We must show that:

$$(m_a^2 + m_b^2 + m_c^2) \left(\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \right) \leq \frac{9R^2}{4r^2}; (1)$$

But: $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2)$ and $a^2 + b^2 + c^2 \leq 9R^2$ (Neuberg) \Rightarrow

$$m_a^2 + m_b^2 + m_c^2 \leq \frac{27R^2}{4}; (2)$$

From (1),(2) we must show:



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$$\frac{1}{m_a^2} + \frac{1}{m_b^2} + \frac{1}{m_c^2} \leq \frac{1}{3r^2}; \quad (3)$$

$$\text{But: } m_a \geq \sqrt{s(s-a)} \Rightarrow \frac{1}{m_a^2} \leq \frac{1}{s(s-a)}; \quad (4)$$

From (3),(4) we must show:

$$\frac{1}{s} \left(\frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} \right) \leq \frac{1}{3r^2} \text{ but } \frac{1}{s-a} + \frac{1}{s-b} + \frac{1}{s-c} = \frac{4R+r}{sr} \Rightarrow$$

We must show that:

$$\frac{4R+r}{s^2r} \leq \frac{1}{3r^2} \Leftrightarrow 3r(4R+r) \leq s^2 \quad (\text{Doucet})$$

1731. In ΔABC the following relationship holds:

$$\sum_{\text{cyc}} \frac{h_a}{s-a} \leq \sum_{\text{cyc}} \frac{r_a}{s-a}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\sum_{\text{cyc}} \frac{h_a}{s-a} = \sum_{\text{cyc}} \frac{2S}{a(s-a)} = 2S \sum_{\text{cyc}} \frac{1}{a(s-a)}$$

$$\sum_{\text{cyc}} \frac{r_a}{s-a} = \sum_{\text{cyc}} \frac{S}{(s-a)^2} = S \sum_{\text{cyc}} \frac{1}{(s-a)^2}$$

Let: $x = s-a$; $y = s-b$; $z = s-c$, ($x, y, z > 0$) $\Rightarrow x+y+z = s$; $a = x+y$,

$b = x+z$, $c = x+y$ then

$$\sum_{\text{cyc}} \frac{h_a}{s-a} \leq \sum_{\text{cyc}} \frac{r_a}{s-a} \Leftrightarrow$$

$$2 \left[\frac{1}{x(y+z)} + \frac{1}{y(z+x)} + \frac{1}{z(x+y)} \right] \stackrel{(*)}{\leq} \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \text{ which is true, because}$$

$$\frac{1}{x(y+z)} = \frac{1}{xy+zx} \stackrel{BCS}{\leq} \frac{1}{4} \left(\frac{1}{xy} + \frac{1}{xz} \right); \quad (1)$$

$$\frac{1}{y(z+x)} = \frac{1}{yz+yx} \stackrel{BCS}{\leq} \frac{1}{4} \left(\frac{1}{yz} + \frac{1}{yx} \right); \quad (2)$$

$$\frac{1}{z(x+y)} = \frac{1}{zx+zy} \stackrel{BCS}{\leq} \frac{1}{4} \left(\frac{1}{zx} + \frac{1}{zy} \right); \quad (3)$$



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From (1),(2),(3) we get:

$$2 \left[\frac{1}{x(y+z)} + \frac{1}{y(z+x)} + \frac{1}{z(x+y)} \right] \leq \frac{1}{x} \cdot \frac{1}{y} + \frac{1}{y} \cdot \frac{1}{z} + \frac{1}{z} \cdot \frac{1}{x} \leq \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \Rightarrow (*) \text{ is true.}$$

Solution Soumava Chakraborty-Kolkata-India

$$\frac{h_a}{s-a} = \frac{2rs}{a(s-a)} = \frac{2 \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \cdot 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2} \cdot 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\left(2 \cos^2 \frac{B}{2}\right) \left(2 \cos^2 \frac{C}{2}\right)}{2 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}$$

$$= \frac{2R}{s} (1 + \cos B)(1 + \cos C) \text{ and analogs}$$

$$\begin{aligned} \Rightarrow \sum \frac{h_a}{s-a} &= \frac{2R}{s} \sum (1 + \cos B)(1 + \cos C) = \frac{2R}{s} \sum (1 + (\cos B + \cos C) + \cos B \cos C) \\ &= \frac{2R}{s} \left(3 + \frac{2(R+r)}{R} + \frac{\frac{(R+r)^2}{R^2} - 3 + \frac{s^2 - 4Rr - r^2}{2R^2}}{2} \right) \\ &= \frac{2R}{s} \left(\frac{12R^2 + 8R(R+r) + 2(R+r)^2 - 6R^2 + s^2 - 4Rr - r^2}{4R^2} \right) \end{aligned}$$

$$= \frac{s^2 + (4R+r)^2}{2Rs} \stackrel{\text{Euler}}{\geq} \frac{s^2 + (4R+r)^2}{4rs} \therefore \sum \frac{h_a}{s-a} \stackrel{(m)}{\leq} \frac{s^2 + (4R+r)^2}{4rs}$$

$$\begin{aligned} \text{Again, } \frac{r_a}{s-a} &= \frac{\tan \frac{A}{2}}{4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{\tan \frac{A}{2} \cdot 4R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}{4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} = \frac{s \cdot \tan \frac{A}{2}}{\tan \frac{B}{2} \cdot \tan \frac{C}{2}} \\ &= \frac{sr_a}{r_b r_b} = \frac{sr_a^2}{rs^2} \Rightarrow \frac{r_a}{s-a} = \frac{r_a^2}{rs} \text{ and analogs} \end{aligned}$$

$$\begin{aligned} \Rightarrow \sum \frac{r_a}{s-a} &\stackrel{(n)}{\cong} \frac{(4R+r)^2 - 2s^2}{rs} \therefore (m), (n) \Rightarrow \text{it suffices to prove : } s^2 + (4R+r)^2 \\ &\leq 4(4R+r)^2 - 8s^2 \Leftrightarrow (4R+r)^2 \geq 3s^2 \end{aligned}$$

$$\rightarrow \text{true (Trucht)} \therefore \sum \frac{h_a}{s-a} \leq \sum \frac{r_a}{s-a} \text{ (Proved)}$$

1732. In ΔABC the following relationship holds:

$$\frac{a}{\mu(B)} + \frac{b}{\mu(C)} + \frac{c}{\mu(A)} + \frac{abc}{\mu(A)\mu(B)\mu(C)} > \frac{4s}{\pi} \left(\frac{s}{3R} + \frac{54r^2}{\pi^2} \right)$$

Proposed by Radu Diaconu-Romania



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Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \mu(A)\mu(B)\mu(C) &\stackrel{AM-GM}{\leq} \frac{(\mu(A) + \mu(B) + \mu(C))^3}{27} = \frac{\pi^3}{27} \Rightarrow \frac{1}{\mu(A)\mu(B)\mu(C)} \geq \frac{27}{\pi^3} \\ &\Rightarrow \frac{abc}{\mu(A)\mu(B)\mu(C)} \geq \frac{27abc}{\pi^3} = \frac{27}{\pi^3} \cdot 4Rrs \stackrel{R \geq 2r}{\geq} \frac{54 \cdot 4}{\pi^3} \cdot sr^2; (1) \end{aligned}$$

In any ΔABC : $\sqrt{ab} + \sqrt{bc} + \sqrt{ca} > \frac{1}{2}(a + b + c) \Leftrightarrow$

$$4(ab + bc + ca) + 2 \sum_{cyc} \sqrt{abc}(\sqrt{a} + \sqrt{b} + \sqrt{c}) > (a + b + c)^2$$

Which is clearly true, because:

$$\begin{aligned} 4(ab + bc + ca) &> (a + b + c)^2 \Leftrightarrow 4(s^2 + 4Rr + r^2) > 4s^2 \Leftrightarrow \\ &4(4Rr + r^2) > 0 \text{ (true)} \end{aligned}$$

$$\begin{aligned} \frac{a}{\mu(B)} + \frac{b}{\mu(C)} + \frac{c}{\mu(A)} &= \frac{\sqrt{a^2}}{\mu(B)} + \frac{\sqrt{b^2}}{\mu(C)} + \frac{\sqrt{c^2}}{\mu(A)} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{\mu(A) + \mu(B) + \mu(C)} = \\ &= \frac{(\sqrt{a} + \sqrt{b} + \sqrt{c})^2}{\pi} = \frac{a + b + c + 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca})}{\pi} > \\ &> \frac{a + b + c + 2 \cdot \frac{1}{2}(a + b + c)}{\pi} = \frac{4s}{\pi} \stackrel{(2)}{>} \frac{4s^2}{\pi \cdot 3R} \\ (2) &\Leftrightarrow 3R > s \Leftrightarrow 9R^2 > s^2 \end{aligned}$$

Which is true, because: $s^2 \leq 4R^2 + 4Rr + 3r^2$ (Gerretsen).

So, we need to prove that: $4R^2 + 4Rr + 3r^2 < 9R^2 \Leftrightarrow 5R^2 - 4Rr - 3r^2 > 0 \Leftrightarrow$

$$5t^2 - 4t + 3 > 0 \left(t = \frac{R}{r} \geq 2 \text{ (Euler)} \right) \Leftrightarrow t(5t - 4) - 3 > 0 \text{ (true)}$$

because $t(5t - 4) - 3 \geq 2(5 \cdot 2 - 4) - 3 = 9 > 0$. Proved.

1733. In any ΔABC , holds:

$$\frac{w_a^2}{h_a^2} + \frac{w_b^2}{h_b^2} + \frac{w_c^2}{h_c^2} \geq \frac{9w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sum \frac{w_a^2}{h_a^2} &\stackrel{A-G}{\geq} 3\sqrt[3]{x^2} \left(\text{where } x = \frac{w_a w_b w_c}{h_a h_b h_c} \right) \geq \frac{9w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c} = \frac{9x}{x+2} \Leftrightarrow x^2 \\
 &\geq \frac{27x^3}{(x+2)^3} \Leftrightarrow (x+2)^3 - 27x \geq 0 \\
 \Leftrightarrow (x+8)(x-1)^2 &\geq 0 \rightarrow \text{true}
 \end{aligned}$$

$$\therefore \frac{w_a^2}{h_a^2} + \frac{w_b^2}{h_b^2} + \frac{w_c^2}{h_c^2} \geq \frac{9w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c} \quad (\text{Proved})$$

1734. In any ΔABC , holds:

$$2 \left(\sin^3 \frac{A}{2} + \sin^3 \frac{B}{2} + \sin^3 \frac{C}{2} \right) - \left(\sin^2 \frac{A}{2} \sin \frac{B}{2} + \sin^2 \frac{B}{2} \sin \frac{C}{2} + \sin^2 \frac{C}{2} \sin \frac{A}{2} \right) < 3$$

Proposed by Nguyen Van Canh - Vietnam

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \sin \frac{A}{2} < 1 \Rightarrow \sin^3 \frac{A}{2} < \sin^2 \frac{A}{2} \text{ and analogs} \Rightarrow 2 \sum \sin^3 \frac{A}{2} &< 2 \sum \sin^2 \frac{A}{2} \\
 &= \sum (1 - \cos A) = 3 - 1 - \frac{r}{R} < 3 \\
 \Rightarrow 2 \sum \sin^3 \frac{A}{2} - \sum_{\text{cyclic}} \sin^2 \frac{A}{2} \sin \frac{B}{2} &< 3 - \sum_{\text{cyclic}} \sin^2 \frac{A}{2} \sin \frac{B}{2} < 3 \quad (\text{Proved})
 \end{aligned}$$

1735. In any ΔABC , holds:

$$\sqrt{\frac{2h_b h_c}{h_b^2 + h_c^2}} + \sqrt{\frac{2h_c h_a}{h_c^2 + h_a^2}} + \sqrt{\frac{2h_a h_b}{h_a^2 + h_b^2}} \geq \frac{3h_a h_b h_c}{m_a m_b m_c}$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India



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$$\sum \sqrt{\frac{2h_b h_c}{h_b^2 + h_c^2}} = \sum \sqrt{\frac{2ca \cdot ab}{c^2 a^2 + a^2 b^2}} = \sum \sqrt{\frac{2bc}{b^2 + c^2}}$$

$$= \sum \sqrt{x} \left(\text{where } x = \frac{2bc}{b^2 + c^2}, y = \frac{2ca}{c^2 + a^2} \text{ and } z = \frac{2ab}{a^2 + b^2} \right)$$

$$\stackrel{\text{A-G}}{\geq} 3\sqrt[6]{xyz} \Rightarrow \sum \sqrt{\frac{2h_b h_c}{h_b^2 + h_c^2}} \stackrel{(1)}{\geq} 3\sqrt[6]{xyz}$$

$$\text{Now, } \frac{3h_a h_b h_c}{m_a m_b m_c} \leq \frac{3s_a s_b s_c}{m_a m_b m_c} = 3 \prod \left\{ \frac{\left(\frac{2bc}{b^2 + c^2}\right) m_a}{m_a} \right\} = 3xyz \Rightarrow \frac{3h_a h_b h_c}{m_a m_b m_c} \stackrel{(2)}{\geq} 3xyz$$

(1), (2) \Rightarrow it suffices to prove : $3\sqrt[6]{xyz} \geq 3xyz \Leftrightarrow xyz \leq (xyz)^6 \Leftrightarrow (xyz)^5 \leq 1$

$$\Leftrightarrow xyz \leq 1 \Leftrightarrow \prod \left(\frac{2bc}{b^2 + c^2} \right) \leq 1 \rightarrow \text{true}$$

$\because \frac{2bc}{b^2 + c^2} \leq 1$ and analogs as $(b - c)^2 \geq 0$ and analogs

$$\therefore \sqrt{\frac{2h_b h_c}{h_b^2 + h_c^2}} + \sqrt{\frac{2h_c h_a}{h_c^2 + h_a^2}} + \sqrt{\frac{2h_a h_b}{h_a^2 + h_b^2}} \geq \frac{3h_a h_b h_c}{m_a m_b m_c} \text{ (Proved)}$$

1736. In any ΔABC , holds:

$$\sum \frac{\sqrt{r_a h_a}}{w_a} \geq \left(1 + \frac{2}{3} \sum \frac{m_a}{h_a} \right) \sqrt{\frac{2r}{R}}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{\sqrt{r_a h_a}}{w_a} &= \sum \left(\sqrt{\frac{2rs \cdot \tan \frac{A}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}}} \left(\frac{b+c}{2bc \cos \frac{A}{2}} \right) \right) = s \sqrt{\frac{r}{2R}} \sum \frac{b+c}{2bc \left(\frac{s(s-a)}{bc} \right)} \\ &= \frac{1}{2} \sqrt{\frac{r}{2R}} \sum \frac{s+s-a}{s-a} \end{aligned}$$



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$$= \frac{1}{2} \sqrt{\frac{r}{2R}} \left(3 + \frac{s \sum(s-b)(s-c)}{sr^2} \right) = \frac{1}{2} \sqrt{\frac{r}{2R}} \left(3 + \frac{4R+r}{r} \right) = 2 \sqrt{\frac{r}{2R}} \left(\frac{R+r}{r} \right)$$

$$\therefore \sum \frac{\sqrt{r_a h_a}}{w_a} \stackrel{(1)}{\cong} \sqrt{\frac{2r}{R}} \left(\frac{R+r}{r} \right)$$

$$\text{Again, } \left(1 + \frac{2}{3} \sum \frac{m_a}{h_a} \right) \sqrt{\frac{2r}{R}} \stackrel{\text{Panaitopol}}{\geq} \left(1 + \left(\frac{2}{3} \right) \frac{3R}{2r} \right) \sqrt{\frac{2r}{R}}$$

$$= \sqrt{\frac{2r}{R}} \left(\frac{R+r}{r} \right) \stackrel{\text{by (1)}}{\cong} \sum \frac{\sqrt{r_a h_a}}{w_a} \quad (\text{Proved})$$

1737. In any acute ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \sqrt{2(r_a^2 + r_b^2)} \geq \sum (n_a + g_a)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\cong} 4R \cos^2 \frac{A}{2}$$

$$\begin{aligned} & \text{Now, Stewart's theorem } \Rightarrow b^2(s-c) + c^2(s-b) \\ &= a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\ &= a g_a^2 + a(s-b)(s-c) \end{aligned}$$

Adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$\begin{aligned} &= a n_a^2 + a g_a^2 + 2a(s-b)(s-c) \\ &\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2) \\ &= 2(n_a^2 + g_a^2) + a^2 - (b-c)^2 \\ &\Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \\ &\Rightarrow 4m_a^2 + (b-c)^2 + 4r_b r_c = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b-c)^2 + 4s(s-a) \\ &= 2(n_a^2 + g_a^2) + 4r_b r_c \end{aligned}$$



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$$\Rightarrow 4m_a^2 + 4m_a^2 = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow n_a^2 + g_a^2 = 4m_a^2 - 2r_b r_c$$

$$\therefore n_a + g_a \stackrel{?}{\leq} \sqrt{2}\sqrt{n_a^2 + g_a^2} = \sqrt{2}\sqrt{4m_a^2 - 2r_b r_c} \stackrel{?}{\leq} \sqrt{2(r_b^2 + r_c^2)}$$

$$\Leftrightarrow 4m_a^2 - 2r_b r_c \stackrel{?}{\leq} r_b^2 + r_c^2 \Leftrightarrow 4m_a^2 \stackrel{?}{\leq} (r_b + r_c)^2 \Leftrightarrow 2m_a \stackrel{?}{\leq} r_b + r_c \stackrel{\text{by (i)}}{\equiv} 4R \cos^2 \frac{A}{2}$$

$$\Leftrightarrow m_a \stackrel{?}{\leq} R(1 + \cos A) \rightarrow \text{true for acute angle } A$$

$$\therefore \sqrt{2(r_b^2 + r_c^2)} \geq n_a + g_a \text{ and analogs} \Rightarrow \sum \sqrt{2(r_b^2 + r_c^2)} \geq \sum (n_a + g_a)$$

$$\Rightarrow \sum \sqrt{2(r_a^2 + r_b^2)} \geq \sum (n_a + g_a) \text{ (Proved)}$$

1738. In any ΔABC , g_a – Gergonne's cevian holds:

$$\frac{9r_a r_b r_c}{g_a g_b g_c} + \frac{2r}{R} \geq 10$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\frac{9r_a r_b r_c}{w_a w_b w_c} = \frac{9rs^2}{\prod \left(\frac{2bc \cos \frac{A}{2}}{b+c} \right)} = \frac{9rs^2 \prod (b+c)}{8a^2 b^2 c^2 \left(\prod \cos \frac{A}{2} \right)} = \frac{9rs^2 \cdot 2s(s^2 + 2Rr + r^2)}{128R^2 r^2 s^2 \left(\frac{s}{4R} \right)}$$

$$= \frac{9(s^2 + 2Rr + r^2)}{16Rr}$$

$$\Rightarrow \frac{9r_a r_b r_c}{w_a w_b w_c} + \frac{2r}{R} = \frac{9(s^2 + 2Rr + r^2)}{16Rr} + \frac{2r}{R} = \frac{9(s^2 + 2Rr + r^2) + 32r^2}{16Rr} \geq 10$$

$$\Leftrightarrow 9s^2 + 18Rr + 41r^2 \geq 160Rr$$

$$\stackrel{(1)}{\Leftrightarrow} 9s^2 \stackrel{?}{\geq} 142Rr - 41r^2$$

Now, $9s^2 \geq 144Rr - 45r^2 \stackrel{?}{\geq} 142Rr - 41r^2 \Leftrightarrow 2Rr \stackrel{?}{\geq} 4r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true (Euler)}$

$$\Rightarrow (1) \text{ is true} \therefore \frac{9r_a r_b r_c}{w_a w_b w_c} + \frac{2r}{R} \stackrel{(a)}{\stackrel{?}{\geq}} 10$$



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$$\begin{aligned}
 \text{Now, triangle inequality } \Rightarrow g_a \leq AI + r \stackrel{?}{\leq} w_a &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abc \cos \frac{A}{2}}{a(b+c)} \\
 &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rrs \cos \frac{A}{2}}{4R(b+c) \sin \frac{A}{2} \cos \frac{A}{2}} \\
 &\Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a+b+c}{(b+c) \sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} + \frac{a}{(b+c) \sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c) \sin \frac{A}{2} \stackrel{?}{\leq} a \\
 &\Leftrightarrow 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leq} 4R \sin \frac{A}{2} \cos \frac{A}{2} \\
 &\Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true } \therefore w_a \geq g_a \text{ and analogs} \Rightarrow \frac{9r_a r_b r_c}{g_a g_b g_c} + \frac{2r}{R} \\
 &\geq \frac{9r_a r_b r_c}{w_a w_b w_c} + \frac{2r}{R} \stackrel{\text{by (a)}}{\stackrel{?}{\leq}} 10 \text{ (Proved)}
 \end{aligned}$$

1739. In any ΔABC , holds:

$$\frac{R}{2r} \geq 1 + \frac{s^2(r_a - r_b)^2}{(s^2 + r_a r_b)^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \frac{s^2(r_a - r_b)^2}{(s^2 + r_a r_b)^2} &= \frac{s^2 \left(\frac{rs}{s-a} - \frac{rs}{s-b} \right)^2}{\left(s^2 + \frac{rs}{s-a} \cdot \frac{rs}{s-b} \right)^2} = \frac{s^2 \left(\frac{r^2 s^2}{r^4 s^2} \right) (a-b)^2 (s-c)^2}{\left(s^2 + \frac{s(s-a)(s-b)(s-c)}{(s-a)(s-b)} \right)^2} \\
 &= \frac{s^2 (a-b)^2 (s-c)^2}{s^2 r^2 (s+s-c)^2} = \frac{(a-b)^2 (s-c)^2}{r^2 (a+b)^2} \\
 &= \frac{\left(16R^2 \sin^2 \frac{A-B}{2} \sin^2 \frac{C}{2} \right) \left(16R^2 \cos^2 \frac{C}{2} \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \right)}{\left(16R^2 \sin^2 \frac{A}{2} \sin^2 \frac{B}{2} \sin^2 \frac{C}{2} \right) \left(16R^2 \cos^2 \frac{A-B}{2} \cos^2 \frac{C}{2} \right)} = \frac{1 - \cos^2 \frac{A-B}{2}}{\cos^2 \frac{A-B}{2}} \\
 &= \frac{1}{\cos^2 \frac{A-B}{2}} - 1 \Rightarrow 1 + \frac{s^2(r_a - r_b)^2}{(s^2 + r_a r_b)^2} \stackrel{(1)}{\stackrel{?}{\leq}} \frac{1}{\cos^2 \frac{A-B}{2}}
 \end{aligned}$$



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$$\text{Now, } r_a + r_b = s \left(\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \right) = \frac{s \sin \left(\frac{A+B}{2} \right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{C}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{C}{2}$$

$$\therefore r_a + r_b \stackrel{(i)}{\cong} 4R \cos^2 \frac{C}{2}$$

$$\begin{aligned} \text{Now, } (a+b)^2 &\geq 32Rr \cos^2 \frac{C}{2} \stackrel{\text{by (i)}}{\cong} 8r(r_a + r_b) = 8r^2 s \left(\frac{1}{s-a} + \frac{1}{s-b} \right) \\ &= 8(s-a)(s-b)(s-c) \frac{c}{(s-a)(s-b)} = 4c(a+b-c) \end{aligned}$$

$$\begin{aligned} \Leftrightarrow (a+b)^2 + 4c^2 - 4c(a+b) &\geq 0 \Leftrightarrow (a+b-2c)^2 \geq 0 \rightarrow \text{true} \therefore a+b \\ &\geq 4\sqrt{2Rr} \cos \frac{C}{2} \Rightarrow 4R \cos \frac{C}{2} \cos \frac{A-B}{2} \geq 4\sqrt{2Rr} \cos \frac{C}{2} \\ \Rightarrow \cos \frac{A-B}{2} &\geq \sqrt{\frac{2r}{R}} \Rightarrow \frac{1}{\cos^2 \frac{A-B}{2}} \leq \frac{R}{2r} \stackrel{\text{by (1)}}{\cong} 1 + \frac{s^2(r_a - r_b)^2}{(s^2 + r_a r_b)^2} \leq \frac{R}{2r} \text{ (Proved)} \end{aligned}$$

1740. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\left(\sum (n_a - g_a) \right) \left(\sum \frac{n_a^2 |b - c|}{h_a^2} \right) \geq s^2 \sum \frac{(n_b - g_b)(n_c - g_c)}{h^2}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Triangle inequality } \Rightarrow g_a &\leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abc \cos \frac{A}{2}}{a(b+c)} \\ \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r &\stackrel{?}{\leq} \frac{8Rr \cos \frac{A}{2}}{4R(b+c) \sin \frac{A}{2} \cos \frac{A}{2}} \\ \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 &\stackrel{?}{\leq} \frac{a+b+c}{(b+c) \sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a}{(b+c) \sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c) \sin \frac{A}{2} \stackrel{?}{\leq} a \\ \Leftrightarrow 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} &\stackrel{?}{\leq} 4R \sin \frac{A}{2} \cos \frac{A}{2} \end{aligned}$$



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$$\Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\geq} 1 \rightarrow \text{true} \therefore g_a \leq w_a \stackrel{(1)}{\geq} m_a$$

$$\begin{aligned}
 \text{Now, Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) = a n_a^2 + a(s-b)(s-c) \\
 &\Rightarrow 4an_a^2 - 4am_a^2 \\
 &= 4b^2(s-c) + 4c^2(s-b) - 4a(s-b)(s-c) - a(2b^2 + 2c^2 - a^2) \\
 &= 2b^2(a+b-c) + 2c^2(c+a-b) - a(c+a-b)(a+b-c) - a(2b^2 + 2c^2 - a^2) \\
 &= a(b-c)^2 + 2b^3 + 2c^3 - 2b^2c - 2bc^2 \\
 &= a(b-c)^2 + 2(b+c)(b^2 + c^2 - bc) - 2bc(b+c) = a(b-c)^2 + 2(b+c)(b-c)^2 \\
 &= (a+2b+2c)(b-c)^2 \geq 0 \Rightarrow 4an_a^2 \geq 4am_a^2 \\
 &\Rightarrow n_a \stackrel{(2)}{\geq} m_a \therefore (1), (2) \Rightarrow n_a \stackrel{(3)}{\geq} g_a
 \end{aligned}$$

$$\begin{aligned}
 \text{Again, Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) \\
 &= an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 &= ag_a^2 + a(s-b)(s-c) \\
 &\quad \therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s-a)^2 \\
 &\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \stackrel{(a)}{\leq} a^2 s^2 (s-a)^2
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } s-a = x, s-b = y \text{ and } s-c = z \therefore s = x+y+z \Rightarrow a = y+z, b = z+x \text{ and } c \\
 = x+y
 \end{aligned}$$

Using these substitutions, (a)

$$\begin{aligned}
 &\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\} \{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2 \\
 &\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true} \\
 &\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \stackrel{(4)}{\geq} s(s-a)
 \end{aligned}$$

$$\begin{aligned}
 \text{Also, Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) \\
 &= an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 &= ag_a^2 + a(s-b)(s-c)
 \end{aligned}$$



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$$\begin{aligned}
 & \text{Adding the above two, we get : } (b^2 + c^2)(2s - b - c) \\
 &= an_a^2 + ag_a^2 + 2a(s - b)(s - c) \\
 \Rightarrow 2a(b^2 + c^2) &= 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\
 &= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\
 \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 &= 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 \Rightarrow 4m_a^2 + (b - c)^2 + 4r_b r_c &= 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b - c)^2 + 4s(s - a) \\
 &= 2(n_a^2 + g_a^2) + 4s(s - a) \\
 \Rightarrow 4m_a^2 + 4m_a^2 &= 2(n_a^2 + g_a^2) + 4s(s - a) \stackrel{(5)}{\Rightarrow} n_a^2 + g_a^2 \stackrel{(5)}{\cong} 4m_a^2 - 2s(s - a) \\
 &\Rightarrow b^2 + c^2 - 2bc - (n_a^2 + g_a^2 - 2n_a g_a) \\
 \stackrel{\text{by (5)}}{\cong} b^2 + c^2 - 2bc + 2n_a g_a - (4m_a^2 - 2s(s - a)) &\stackrel{\text{by (4)}}{\geq} b^2 + c^2 - 2bc + 2s(s - a) \\
 &- 4m_a^2 + 2s(s - a) \\
 &= b^2 + c^2 - 2bc + (b + c + a)(b + c - a) - (2b^2 + 2c^2 - a^2) \\
 &= b^2 + c^2 - 2bc + (b + c)^2 - a^2 - (2b^2 + 2c^2 - a^2) = 0 \\
 \Rightarrow (b - c)^2 \geq (n_a - g_a)^2 &\stackrel{\text{by (3)}}{\Leftrightarrow} |b - c| \stackrel{(6)}{\geq} n_a - g_a \text{ and analogs}
 \end{aligned}$$

Let AX, BY and CZ be the nagel cevians intersecting at N_a. Now, AZ = s - b, BZ

$$= s - a, BX = s - c, CX = s - b, CY = s - a$$

and AY = s - c

$$\begin{aligned}
 & \text{By Van Aubel's theorem, } \frac{AN_a}{XN_a} = \frac{AZ}{BZ} + \frac{AY}{CY} = \frac{s - b}{s - a} + \frac{s - c}{s - a} = \frac{2s - b - c}{s - a} = \frac{a}{s - a} \\
 \Rightarrow \frac{XN_a}{AN_a} &= \frac{s - a}{a} \Rightarrow \frac{XN_a + AN_a}{AN_a} = \frac{s}{a} \Rightarrow \frac{AN_a}{n_a} = \frac{a}{s}
 \end{aligned}$$

$\Rightarrow AN_a = \frac{an_a}{s}$ and analogs and choosing M \equiv N_a and x = n_a - g_a, y = n_b - g_b and z = n_c - g_c in Klamkin's polar moment of

inertia inequality, we get

$$\begin{aligned}
 & : \left(\sum (n_a - g_a) \right) \left(\frac{(n_a - g_a)a^2 n_a^2}{s^2} + \frac{(n_b - g_b)b^2 n_b^2}{s^2} + \frac{(n_c - g_c)c^2 n_c^2}{s^2} \right) \\
 & \geq \sum a^2 (n_b - g_b)(n_c - g_c)
 \end{aligned}$$



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$$\begin{aligned}
 & \Rightarrow \left(\sum (\mathbf{n}_a - \mathbf{g}_a) \right) \left(\frac{(\mathbf{n}_a - \mathbf{g}_a) \left(\frac{\mathbf{a}^2}{4r^2s^2} \right) n_a^2}{s^2} + \frac{(\mathbf{n}_b - \mathbf{g}_b) \left(\frac{\mathbf{b}^2}{4r^2s^2} \right) n_b^2}{s^2} \right. \\
 & \quad \left. + \frac{(\mathbf{n}_c - \mathbf{g}_c) \left(\frac{\mathbf{c}^2}{4r^2s^2} \right) n_c^2}{s^2} \right) - \sum \left(\frac{\mathbf{a}^2}{4r^2s^2} \right) (\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c) \geq 0 \\
 & \Rightarrow \left(\sum (\mathbf{n}_a - \mathbf{g}_a) \right) \left(\sum \frac{(\mathbf{n}_a - \mathbf{g}_a) n_a^2}{h_a^2} \right) - s^2 \sum \frac{(\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c)}{h_a^2} \stackrel{(m)}{\geq} 0 \\
 \text{Now, (6) and analogs} & \Rightarrow \sum \frac{n_a^2 |\mathbf{b} - \mathbf{c}|}{h_a^2} \geq \sum \frac{(\mathbf{n}_a - \mathbf{g}_a) n_a^2}{h_a^2} \\
 & \Rightarrow \left(\sum (\mathbf{n}_a - \mathbf{g}_a) \right) \left(\sum \frac{n_a^2 |\mathbf{b} - \mathbf{c}|}{h_a^2} \right) - s^2 \sum \frac{(\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c)}{h_a^2} \\
 & \geq \left(\sum (\mathbf{n}_a - \mathbf{g}_a) \right) \left(\sum \frac{(\mathbf{n}_a - \mathbf{g}_a) n_a^2}{h_a^2} \right) - s^2 \sum \frac{(\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c)}{h_a^2} \stackrel{\text{by (m)}}{\geq} 0 \\
 & \Rightarrow \left(\sum (\mathbf{n}_a - \mathbf{g}_a) \right) \left(\sum \frac{n_a^2 |\mathbf{b} - \mathbf{c}|}{h_a^2} \right) \geq s^2 \sum \frac{(\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c)}{h_a^2} \text{ (Proved)}
 \end{aligned}$$

1741. In ΔABC the following relationship holds:

$$\frac{s}{5R^2 - 2r^2} \leq \frac{a}{m_b^2 + m_c^2} + \frac{b}{m_c^2 + m_a^2} + \frac{c}{m_a^2 + m_b^2} \leq \frac{3}{s}$$

Proposed by Marin Chirciu-Romania

Solution by Marian Ursărescu-Romania

In any ΔABC we have: $m_a \geq \sqrt{s(s-a)} \Rightarrow m_b^2 + m_c^2 \geq s(s-b) + s(s-c) = as \Rightarrow \frac{a}{m_b^2 + m_c^2} \leq \frac{1}{s}$ and similarly, then: $\sum_{\text{cyc}} \frac{a}{m_b^2 + m_c^2} \leq \frac{3}{s}$ (1)

Let: $a \leq b \leq c \Rightarrow m_a \geq m_b \geq m_c \Rightarrow \frac{1}{m_b^2 + m_c^2} \leq \frac{1}{m_c^2 + m_a^2} \leq \frac{1}{m_a^2 + m_b^2} \xrightarrow{\text{Chebyshev's}}$

$$\frac{a}{m_b^2 + m_c^2} + \frac{b}{m_c^2 + m_a^2} + \frac{c}{m_a^2 + m_b^2} \geq \frac{a+b+c}{3} \cdot \left(\frac{1}{m_b^2 + m_c^2} + \frac{1}{m_c^2 + m_a^2} + \frac{1}{m_a^2 + m_b^2} \right)$$

We must show:



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$$\frac{2s}{3} \sum_{cyc} \frac{1}{m_b^2 + m_c^2} \geq \frac{2s}{5R^2 - 2r^2} \Leftrightarrow \sum_{cyc} \frac{1}{m_b^2 + m_c^2} \geq \frac{3}{5R^2 - 2r^2} \quad (1)$$

But:

$$\left(\sum_{cyc} \frac{1}{m_b^2 + m_c^2} \right) \left(\sum_{cyc} (m_a^2 + m_b^2 + m_c^2) \right) \geq 9 \Rightarrow \sum_{cyc} \frac{1}{m_b^2 + m_c^2} \geq \frac{9}{2(m_a^2 + m_b^2 + m_c^2)} \quad (2)$$

From (1)+(2) we must show:

$$\begin{aligned} \frac{9}{2(m_a^2 + m_b^2 + m_c^2)} &\geq \frac{3}{5R^2 - 2r^2} \Leftrightarrow \frac{2}{3}(m_a^2 + m_b^2 + m_c^2) \leq 5R^2 - 2r^2 \\ \Leftrightarrow \frac{2}{3} \cdot \frac{3}{4}(a^2 + b^2 + c^2) &\leq 5R^2 - 2r^2 \Leftrightarrow a^2 + b^2 + c^2 \leq 10R^2 - 4r^2 \quad (3) \end{aligned}$$

From Gerretsen we have: $a^2 + b^2 + c^2 \leq 8R^2 + 4r^2 \leq a^2 + b^2 + c^2 \leq 10R^2 - 4r^2 \Leftrightarrow$

$$R \geq 2r$$

1742. In ΔABC the following relationship holds:

$$\frac{8m_a m_b m_c (m_a + m_b + m_c)}{9F^2} + 1 \geq \frac{\sqrt{3}(m_a + m_b + m_c)^2}{3F}, F = [ABC]$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have: $MA = MB, NA = NC, PA = PB$

$BMXN$ – parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A, D \equiv M, E \equiv X \Rightarrow [DEF] = \frac{3}{4}[ABC]$

$$R_m = \frac{m_a m_b m_c}{3[ABC]}, r_m = \frac{3[ABC]}{2(m_a + m_b + m_c)} \Rightarrow \frac{R_m}{2r_m} = \frac{1}{9} \cdot \frac{m_a m_b m_c (m_a + m_b + m_c)}{F^2}$$

So, the inequality becomes as:

$$\begin{aligned} 8 \cdot \frac{R_m}{2r_m} + 1 &\geq \frac{\sqrt{3}(m_a + m_b + m_c)^2}{3 \cdot \frac{4[DEF]}{3}} = \frac{\sqrt{3}(m_a + m_b + m_c)^2}{4[DEF]} \Leftrightarrow \\ 4 \cdot \frac{R_m}{r_m} + 1 &\stackrel{(1)}{\geq} \frac{\sqrt{3}(m_a + m_b + m_c)^2}{4[DEF]} \end{aligned}$$

Hence, for any ΔABC we need to prove that:



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$$4 \cdot \frac{R}{r} + 1 \stackrel{(2)}{\geq} \frac{\sqrt{3}(a+b+c)^2}{4[ABC]}$$

$$\Leftrightarrow 4 \cdot \frac{R}{r} + 1 \geq \frac{\sqrt{3} \cdot 4s^2}{4sr} \Leftrightarrow 4 \cdot \frac{R}{r} + 1 \geq \frac{\sqrt{3}s}{r} \Leftrightarrow$$

$$4R + r \geq \sqrt{3}s \text{ (true)} \Rightarrow (2) \text{ true} \Rightarrow (1) \text{ true}.$$

Solution 2 by Bogdan Fuștei-Romania

$$m_a, m_b, m_c - \text{it can be the sides of triangle with } S_m - \text{area}, S = [ABC]; \frac{3}{4}S = S_m$$

$$m_a m_b m_c = 4S_m R_m = 3SR_m \Rightarrow R_m = \frac{m_a m_b m_c}{3S}; S_m = s_m r_m; s_m = \frac{m_a + m_b + m_c}{2}$$

$$\frac{3}{4}S = \frac{r_m(m_a + m_b + m_c)}{2} \Rightarrow 2r_m = \frac{3S}{m_a + m_b + m_c}$$

$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

Then, the inequality becomes:

$$8 \cdot \frac{R_m}{2r_m} + 1 \geq \frac{\sqrt{3}}{3} \cdot \frac{(m_a + m_b + m_c)^2}{S}; 3S = 4S_m \Rightarrow$$

$$1 + \frac{4R_m}{r_m} \geq \frac{\sqrt{3}(m_a + m_b + m_c)^2}{4S_m} \Rightarrow$$

$$1 + \frac{4R}{r} \geq \frac{\sqrt{3} \cdot (a+b+c)^2}{4S} = \frac{\sqrt{3}}{4} \cdot \frac{4s^2}{S} = \frac{\sqrt{3}}{4} \cdot \frac{4s^2}{sr} \Leftrightarrow$$

$$1 + \frac{4R}{r} \geq \frac{s\sqrt{3}}{r} \Leftrightarrow \frac{4R + r}{r} \geq \frac{s\sqrt{3}}{r}$$

But: $r_a + r_b + r_c = 4R + r; s^2 = r_a r_b + r_b r_c + r_c r_a$

$$r_a^2 + r_b^2 + r_c^2 + 2(r_a r_b + r_b r_c + r_c r_a) \geq 3(r_a r_b + r_b r_c + r_c r_a)$$

$$r_a^2 + r_b^2 + r_c^2 \geq r_a r_b + r_b r_c + r_c r_a$$

$$2r_a^2 + 2r_b^2 + 2r_c^2 \geq 2r_a r_b + 2r_b r_c + 2r_c r_a$$

$$(r_a - r_b)^2 + (r_b - r_c)^2 + (r_c - r_a)^2 \geq 0 \text{ (true)}$$

1743. In ΔABC denote $DE = m_a, EF = m_b, FD = m_c, R_m, r_m$ – circumradii and inradii in ΔDEF . Prove that:

$$\frac{R_m}{2r_m} \geq \frac{m_a}{h_a}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam

We have $MB = MC, NA = NC, PA = PB$



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$BMXN$ – parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A, D \equiv M, E \equiv X \Rightarrow [DEF] = \frac{3}{4} [ABC]$

$$\begin{aligned}
 R_m &= \frac{m_a m_b m_c}{3[ABC]}; r_m = \frac{3[ABC]}{2(m_a + m_b + m_c)} \Rightarrow \\
 \frac{R_m}{2r_m} &= \frac{1}{9} \cdot \frac{m_a m_b m_c (m_a + m_b + m_c)}{S^2} = \\
 &= \frac{1}{9} \cdot \frac{m_a m_b m_c (m_a + m_b + m_c)}{\frac{16}{9} \cdot \frac{1}{16} (m_a + m_b + m_c)(m_b - m_a + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)} \\
 &= \frac{m_a m_b m_c}{(m_b - m_a + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)} \stackrel{(*)}{\geq} \frac{m_a}{h_a}
 \end{aligned}$$

More, $m'_a = \frac{3}{4}a \Rightarrow a = \frac{4}{3}m'_a = \sqrt{\frac{2(m_b^2 + m_c^2) - m_a^2}{4}} = \frac{2}{3}\sqrt{2(m_b^2 + m_c^2) - m_a^2}$

$$[DEF] = \frac{3}{4}S \Rightarrow$$

$$S = \frac{4}{3}[DEF] = \frac{1}{3}\sqrt{(m_a + m_b + m_c)(m_b - m_a + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)}$$

Then,

$$\begin{aligned}
 (*) \Leftrightarrow 2S \cdot m_b m_c &\geq a(m_b - m_a + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c) \Leftrightarrow \\
 m_b m_c \sqrt{m_a + m_b + m_c} &\geq \\
 \sqrt{2(m_b^2 + m_c^2) - m_a^2} \cdot \sqrt{(m_b - m_a + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)} & \\
 yz\sqrt{x+y+z} &\geq \sqrt{2(y^2 + z^2) - x^2} \cdot \sqrt{(y+z-x)(z+x-y)(x+y-z)}
 \end{aligned}$$

Where ($m_a = x; m_b = y; m_c = z$).

$$\begin{aligned}
 \frac{xyz}{(x+y-z)(y+z-x)(z+x-y)} &\geq \\
 \sqrt{\frac{2(y^2 + z^2) - x^2}{4}} & \\
 \geq \frac{2}{x} \sqrt{\left(\frac{x+y+z}{2}\right)\left(\frac{x+y-z}{2}\right)\left(\frac{y+z-x}{2}\right)\left(\frac{x+z-y}{2}\right)} &
 \end{aligned}$$

$\Leftrightarrow \frac{R_x}{2r_x} \geq \frac{m_x}{h_x}$. Which is clearly true for any ΔXYZ . Proved.

Solution 2 by Bogdan Fuștei-Romania

m_a, m_b, m_c – it can be the sides of triangle with S_m – area, $S = [ABC]; \frac{3}{4}S = S_m$

$$m_a m_b m_c = 4S_m R_m = 3S R_m \Rightarrow R_m = \frac{m_a m_b m_c}{3S}; S_m = s_m r_m; s_m = \frac{m_a + m_b + m_c}{2}$$

$$\frac{3}{4}S = \frac{r_m(m_a + m_b + m_c)}{2} \Rightarrow 2r_m = \frac{3S}{m_a + m_b + m_c}$$



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$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

Let denote h_{m_a} –the altitudes in ΔDEF for the side m_a .

$$S_m = \frac{h_{m_a} \cdot m_a}{2} \Rightarrow \frac{3}{4}S = \frac{h_{m_a} \cdot m_a}{2} \Rightarrow \frac{3}{2}S = h_{m_a} \cdot m_a \Rightarrow h_{m_a} = \frac{3s}{2m_a} \text{ (and analogs)}$$

$$m_a^2 = \frac{2(b^2 + c^2) - a^2}{4}; m_b^2 = \frac{2(a^2 + c^2) - b^2}{4}; m_c^2 = \frac{2(a^2 + b^2) - c^2}{4}$$

$$m_b^2 + m_c^2 = \frac{2(a^2 + c^2) - b^2 + 2(a^2 + b^2) - c^2}{4} = \frac{4a^2 + b^2 + c^2}{4}$$

$$2(m_b^2 + m_c^2) = \frac{8a^2 + 2b^2 + 2c^2}{4}$$

$$m_{m_a}^2 = \frac{2(m_b^2 + m_c^2) - m_a^2}{4 \cdot 4} = \frac{8a^2 + 2b^2 + 2c^2 - 2b^2 - 2c^2 + a^2}{4 \cdot 4}$$

$$m_{m_a}^2 = \frac{9a^2}{4 \cdot 4} \Rightarrow m_{m_a} = \frac{3}{4}a$$

$$\frac{m_{m_a}}{h_{m_a}} = \frac{3}{4}a \cdot \frac{2m_a}{3S} = \frac{a \cdot m_a}{2S} = \frac{a \cdot m_a}{a \cdot h_a} = \frac{m_a}{h_a}$$

So, we have $\frac{m_{m_a}}{h_{m_a}} = \frac{m_a}{h_a}$ (and analogs)

$$\frac{R_m}{2r_m} \geq \frac{m_{m_a}}{h_{m_a}} = \frac{m_a}{h_a} \text{ (Panaitopol's Ineq. for } \Delta m_a m_b m_c)$$

$$\text{Or } \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{m_a}{h_a}$$

1744. In any } ABC, n_a – Nagel's cevian, holds:

$$\left(\sum \frac{n_a}{h_a} \right) \left(\sum \frac{h_a}{n_a} \right) \geq s^2 \sum \frac{a^2}{bcn_b n_c}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let AX, BY and CZ be the nagel cevians intersecting at N_a. Now, AZ = s – b, BZ

$$= s - a, BX = s - c, CX = s - b, CY = s - a$$

$$\text{and } AY = s - c$$



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By Van Aubel's theorem, $\frac{AN_a}{XN_a} = \frac{AZ}{BZ} + \frac{AY}{CY} = \frac{s-b}{s-a} + \frac{s-c}{s-a} = \frac{2s-b-c}{s-a} = \frac{a}{s-a}$

$$\Rightarrow \frac{XN_a}{AN_a} = \frac{s-a}{a} \Rightarrow \frac{XN_a + AN_a}{AN_a} = \frac{s}{a} \Rightarrow \frac{AN_a}{n_a} = \frac{a}{s}$$

$\Rightarrow AN_a = \frac{an_a}{s}$ and analogs and choosing $M \equiv N_a$ and $x = \frac{1}{an_a}, y = \frac{1}{bn_b}$ and $z = \frac{1}{cn_c}$ in Klamkin's polar moment of inertia

$$\begin{aligned} \text{inequality, we get : } & \left(\sum \frac{1}{an_a} \right) \left[\sum \left\{ \left(\frac{1}{an_a} \right) \left(\frac{a^2 n_a^2}{s^2} \right) \right\} \right] \geq \sum \left\{ a^2 \left(\frac{1}{bn_b} \right) \left(\frac{1}{cn_c} \right) \right\} \\ & \Rightarrow \left(\sum \frac{1}{an_a} \right) \left(\sum an_a \right) \geq s^2 \sum \frac{a^2}{bcn_b n_c} \\ & \Rightarrow \left[\sum \left\{ \frac{1}{\left(\frac{2rs}{h_a} \right) n_a} \right\} \right] \left[\sum \left\{ \left(\frac{2rs}{h_a} \right) n_a \right\} \right] \geq s^2 \sum \frac{a^2}{bcn_b n_c} \Rightarrow \left(\sum \frac{n_a}{h_a} \right) \left(\sum \frac{h_a}{n_a} \right) \\ & \geq s^2 \sum \frac{a^2}{bcn_b n_c} \text{ (Proved)} \end{aligned}$$

1745. In any } ABC, n_a – Nagel's cevian holds:

$$s\sqrt{2} \sum \frac{1}{w_a} \geq 2 \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right) + \sum \frac{n_a}{w_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Proof : } r_b + r_c &= s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \\ &\stackrel{(i)}{\therefore} r_b + r_c \cong 4R \cos^2 \frac{A}{2} \end{aligned}$$

$$\begin{aligned} \text{Now, Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &\quad = an_a^2 + a(as - s^2) \\ &\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &\quad = as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \end{aligned}$$



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$$\begin{aligned}
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \stackrel{(ii)}{\geq} s^2 - 2h_a r_a \\
 &\quad 1 - \sqrt{\frac{2r}{r_b + r_c}} \geq \frac{n_a}{s\sqrt{2}} \Leftrightarrow 1 + \frac{2r}{r_b + r_c} - 2\sqrt{\frac{2r}{r_b + r_c}} \\
 &\geq \frac{n_a^2}{2s^2} \stackrel{\text{by (i),(ii)}}{\Leftrightarrow} 1 + \frac{2r}{4R\cos^2 \frac{A}{2}} - 2\sqrt{\frac{2r}{4R\cos^2 \frac{A}{2}}} \geq \frac{s^2 - 2h_a r_a}{2s^2} \\
 &\Leftrightarrow 2s^2 + \frac{rs^2}{R\cos^2 \frac{A}{2}} - 4s^2\sqrt{\frac{2r}{4R\cos^2 \frac{A}{2}}} \geq s^2 - 2\left(\frac{2rs}{a}\right)\left(\frac{rs}{s-a}\right) \\
 &\Leftrightarrow s^2 + \frac{4Rrs \cdot rs^2}{Rsa(s-a)} + \frac{4r^2s^2}{a(s-a)} - 2s^2\sqrt{\frac{8r}{4R}}\sec \frac{A}{2} \geq 0 \\
 &\Leftrightarrow s^2 \left(1 + \frac{8r^2}{a(s-a)} - 2\sqrt{\frac{2r}{R}}\sec \frac{A}{2}\right) \geq 0 \Leftrightarrow s^2 \left(1 + \frac{8sbc r^2}{abcs(s-a)} - 2\sqrt{\frac{2r}{R}}\sec \frac{A}{2}\right) \geq 0 \\
 &\Leftrightarrow s^2 \left(1 + \left(\frac{8sr^2}{4Rrs}\right)\left(\frac{bc}{s(s-a)}\right) - 2\sqrt{\frac{2r}{R}}\sec \frac{A}{2}\right) \geq 0 \\
 &\Leftrightarrow s^2 \left(1 + \frac{2r}{R}\sec^2 \frac{A}{2} - 2\sqrt{\frac{2r}{R}}\sec \frac{A}{2}\right) \geq 0 \Leftrightarrow s^2 \left(1 - \sqrt{\frac{2r}{R}}\sec \frac{A}{2}\right)^2 \geq 0 \\
 &\rightarrow \text{true} \\
 &\Rightarrow 1 - \sqrt{\frac{2r}{r_b + r_c}} \geq \frac{n_a}{s\sqrt{2}} \Rightarrow 1 - \frac{n_a}{s\sqrt{2}} \geq \sqrt{\frac{2r}{r_b + r_c}} \Rightarrow \frac{s\sqrt{2} - n_a}{s\sqrt{2}} \geq \sqrt{\frac{2r}{r_b + r_c}} \Rightarrow s\sqrt{2} - n_a \\
 &\geq 2s\sqrt{\frac{r}{4R\cos^2 \frac{A}{2}}} = s\sqrt{\frac{r}{R}}\left(\frac{1}{\cos \frac{A}{2}}\right) \\
 &\Rightarrow s\sqrt{2} - n_a \geq s\sqrt{\frac{r}{R}}\left(\frac{1}{\cos \frac{A}{2}}\right) \Rightarrow \frac{s\sqrt{2} - n_a}{w_a} \geq s\sqrt{\frac{r}{R}}\left(\frac{1}{\cos \frac{A}{2}}\right)\left(\frac{b+c}{2bccos \frac{A}{2}}\right) \\
 &= s\sqrt{\frac{r}{R}}\left(\frac{b+c}{2s(s-a)}\right) = \frac{1}{2}\sqrt{\frac{r}{R}}\left(\frac{b+c}{s-a}\right) \\
 &\Rightarrow \frac{s\sqrt{2} - n_a}{w_a} \geq \frac{1}{2}\sqrt{\frac{r}{R}}\left(\frac{b+c}{s-a}\right) \text{ and analogs} \Rightarrow \sum \frac{s\sqrt{2} - n_a}{w_a} \geq \frac{1}{2}\sqrt{\frac{r}{R}}\sum \frac{s+s-a}{s-a} \\
 &= \frac{1}{2}\sqrt{\frac{r}{R}}\left\{3 + \frac{s}{sr^2}\sum (s-b)(s-c)\right\}
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{1}{2} \sqrt{\frac{r}{R}} \left(3 + \frac{4R+r}{r} \right) = \sqrt{\frac{r}{R}} \left\{ \frac{2(R+r)}{r} \right\} = \frac{2(R+r)}{\sqrt{Rr}} = 2 \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right) \Rightarrow s\sqrt{2} \sum \frac{1}{w_a} \\
 &\geq 2 \left(\sqrt{\frac{R}{r}} + \sqrt{\frac{r}{R}} \right) + \sum \frac{n_a}{w_a} \quad (\text{Proved})
 \end{aligned}$$

1746. In ΔABC , N – nine point center, the following relationship holds:

$$\left(\frac{a^2 + R^2}{NB} \right)^2 + \left(\frac{b^2 + R^2}{NC} \right)^2 + \left(\frac{c^2 + R^2}{NA} \right)^2 \geq 192r^2$$

Proposed by Daniel Sitaru-Romania

Solution by Tran Hong-Dong Thap-Vietnam

The nine-point center N satisfies:

$$AN^2 + BN^2 + CN^2 = 3R^2 - ON^2$$

$$\text{Other, } ON = \frac{OH}{2} \quad (N - \text{midpoint } OH) \Rightarrow$$

$$\begin{aligned}
 AN^2 + BN^2 + CN^2 &= 3R^2 - \frac{1}{4} OH^2 = 3R^2 - \frac{1}{4} (9R^2 - (a^2 + b^2 + c^2)) = \\
 &= \frac{a^2 + b^2 + c^2 + 3R^2}{4}. \text{ Now,}
 \end{aligned}$$

$$\begin{aligned}
 Lhs &= \sum_{cyc} \left(\frac{a^2 + R^2}{NB} \right)^2 = \sum_{cyc} \frac{(a^2 + R^2)^2}{NB^2} \stackrel{\text{Bergstrom}}{\geq} \frac{(a^2 + b^2 + c^2 + 3R^2)^2}{AN^2 + BN^2 + CN^2} = \\
 &= \frac{(a^2 + b^2 + c^2 + 3R^2)^2}{a^2 + b^2 + c^2 + 3R^2} = 4(a^2 + b^2 + c^2 + 3R^2) \geq 4(4\sqrt{3}s + 3R^2) = \\
 &= 16\sqrt{3} \cdot sr + 12R^2 \stackrel{s \geq 3\sqrt{3}r, R \geq 2r}{\geq} 16 \cdot 9r^2 + 12 \cdot 4r^2 = 192r^2. \text{ Proved.}
 \end{aligned}$$

1747. In ΔABC the following relationship holds:

$$5 + \frac{(m_a + m_b + m_c)^4}{9F^2} \geq \frac{32m_a m_b m_c (m_a + m_b + m_c)}{9F^2}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution by Tran Hong-Dong Thap-Vietnam

We have: $MA = MC, NA = NC, PA = PB$

$BMXN$ – parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A, D \equiv M, E \equiv X \Rightarrow [DEF] = \frac{3}{4}[ABC] = \frac{3}{4}F^2$

$$R_m = \frac{m_a m_b m_c}{3F^2}; r_m = \frac{3F^2}{2(m_a + m_b + m_c)}$$

$$\frac{R_m}{2r_m} = \frac{1}{9} \cdot \frac{m_a m_b m_c (m_a + m_b + m_c)}{F^2}$$

$$[DEF] = S_{\Delta DEF} = \frac{3}{4}F \Rightarrow F^2 = \left(\frac{4}{3}\right)^2 F^2 =$$

$$= \frac{16}{9} \cdot \frac{1}{16} (m_a + m_b + m_c)(-m_a + m_b + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)$$

$$= \frac{1}{9} (m_a + m_b + m_c)(-m_a + m_b + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)$$

So, inequality becomes as:

$$5 + \frac{(m_a + m_b + m_c)^4}{9 \cdot \frac{1}{9} (m_a + m_b + m_c)(-m_a + m_b + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)} \geq 32 \cdot \frac{R_m}{2r_m}$$

$$5 + \frac{(m_a + m_b + m_c)^3}{(-m_a + m_b + m_c)(m_a - m_b + m_c)(m_a + m_b - m_c)} \geq 16 \cdot \frac{R_m}{2r_m}; (1)$$

Now, for any triangle ABC we need to prove:

$$5 + \frac{(a + b + c)^3}{(a + b - c)(b + c - a)(c + a - b)} \geq 16 \cdot \frac{R}{r}; (2)$$

$$\Leftrightarrow 5 + \frac{(a + b + c)^4}{16 \left(\frac{a + b + c}{2}\right) \left(\frac{a + b - c}{2}\right) \left(\frac{b + c - a}{2}\right) \left(\frac{c + a - b}{2}\right)} \geq 16 \cdot \frac{R}{r} \Leftrightarrow$$

$$5 + \frac{(2s)^4}{16(sr)^2} \geq 16 \cdot \frac{R}{r} \Leftrightarrow 5r^2 + s^2 \geq 16Rr \Leftrightarrow s^2 \geq 16Rr - 5r^2 \text{ (Gerretsen)}$$

$\Rightarrow (2)$ is true $\Rightarrow (1)$ is true. Proved.

1748. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:



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$$\sqrt{\frac{R}{2r} \sum (n_a + g_a)} \geq 2 \sum r_a$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \text{Proof : Stewart's theorem } \Rightarrow b^2(s - c) + c^2(s - b) \\
 & = a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\
 & = a g_a^2 + a(s - b)(s - c) \\
 & \quad \text{Adding the above two, we get : } (b^2 + c^2)(2s - b - c) \\
 & = a n_a^2 + a g_a^2 + 2a(s - b)(s - c) \\
 & \Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\
 & = 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\
 & \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 & \Rightarrow 4m_a^2 + (b - c)^2 + 4r_b r_c = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b - c)^2 + 4s(s - a) \\
 & = 2(n_a^2 + g_a^2) + 4s(s - a) \\
 & \Rightarrow 4m_a^2 + 4m_a^2 = 2(n_a^2 + g_a^2) + 4s(s - a) \Rightarrow n_a^2 + g_a^2 \stackrel{(1)}{\geq} 4m_a^2 - 2s(s - a) \\
 & \quad a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2 \\
 & \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) \\
 & \quad - a(s - b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2
 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\begin{aligned}
 & \Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 \\
 & \quad - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2 \\
 & \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true} \\
 & \Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \stackrel{(2)}{\geq} s(s - a) \\
 & \quad \text{by (1) and (2)}
 \end{aligned}$$

$$\text{Now, } n_a^2 + g_a^2 + 2n_a g_a \stackrel{(3)}{\geq} 4m_a^2 - 2s(s - a) + 2s(s - a) = 4m_a^2$$

$$\Rightarrow (n_a + g_a)^2 \geq 4m_a^2 \Rightarrow n_a + g_a \stackrel{(3)}{\geq} 2m_a$$

$$\begin{aligned}
 \text{Also, } r_b + r_c &= s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \\
 &\stackrel{(i)}{\therefore} r_b + r_c \stackrel{(3)}{\geq} 4R \cos^2 \frac{A}{2}
 \end{aligned}$$



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$$\begin{aligned}
 \text{Now, } (b+c)^2 &\geq 32Rr\cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{=} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right) \\
 &= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a) \\
 \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) &\geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true}
 \end{aligned}$$

$$\therefore b+c \stackrel{(4)}{\geq} 4\sqrt{2Rr}\cos \frac{A}{2}$$

$$\begin{aligned}
 \text{Now, } \sum (n_a + g_a) &\stackrel{\text{by (3)}}{\geq} 2 \sum m_a \stackrel{\text{loscu}}{\geq} 2 \sum \left(\frac{b+c}{2} \cos \frac{A}{2} \right) \stackrel{\text{by (4)}}{\geq} 2\sqrt{2Rr} \sum 2 \cos^2 \frac{A}{2} \\
 &= 2\sqrt{2Rr} \sum (1 + \cos A) = 2\sqrt{2Rr} \left(3 + 1 + \frac{r}{R} \right) \\
 &= 2\sqrt{2Rr} \left(\frac{4R+r}{R} \right) = 2\sqrt{\frac{2r}{R}} \left(\sum r_a \right) \Rightarrow \sqrt{\frac{R}{2r}} \sum (n_a + g_a) \geq 2 \sum r_a \quad (\text{Proved})
 \end{aligned}$$

1749. In any } ABC, holds:

$$5 + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq \frac{8r_a r_b r_c}{w_a w_b w_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof : } \left(\sum_{\text{cyc}} \frac{x}{y} \right)^2 \geq \frac{3xyz + \sum\{xy(x+y)\}}{xyz} = 3 + \sum \frac{x+y}{z} = 3 + \sum_{\text{cyc}} \frac{y}{x} + \sum_{\text{cyc}} \frac{x}{y}$$

$$\Leftrightarrow \sum_{\text{cyc}} \frac{x^2}{y^2} + \sum_{\text{cyc}} \frac{y}{x} \stackrel{(1)}{\geq} 3 + \sum_{\text{cyc}} \frac{x}{y}$$

$$\text{Now, } \sum_{\text{cyc}} \frac{x^2}{y^2} \geq \frac{1}{3} \left(\sum_{\text{cyc}} \frac{x}{y} \right)^2 \stackrel{\text{A-G}}{\geq} \frac{1}{3} \cdot 3 \cdot \sum_{\text{cyc}} \frac{x}{y} \Rightarrow \sum_{\text{cyc}} \frac{x^2}{y^2} \geq \sum_{\text{cyc}} \frac{x}{y} \text{ and } \because \sum_{\text{cyc}} \frac{y}{x} \stackrel{\text{A-G}}{\geq} 3$$

$$\therefore \sum_{\text{cyc}} \frac{x^2}{y^2} + \sum_{\text{cyc}} \frac{y}{x} \geq 3 + \sum_{\text{cyc}} \frac{x}{y} \Rightarrow (1) \text{ is true}$$

$$\begin{aligned}
 \therefore \left(\sum_{\text{cyc}} \frac{x}{y} \right)^2 &\geq \frac{3xyz + \sum\{xy(x+y)\}}{xyz} \Rightarrow \left(\sum_{\text{cyc}} \frac{r_a^2}{r_b^2} \right)^2 \geq \frac{3r_a^2 r_b^2 r_c^2 + \sum\{r_a^2 r_b^2 (r_a^2 + r_b^2)\}}{r_a^2 r_b^2 r_c^2} \\
 &= \frac{3r_a^2 r_b^2 r_c^2 + \sum\{r_a^2 r_b^2 (\sum r_a^2 - r_c^2)\}}{r^2 s^4}
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{3r_a^2 r_b^2 r_c^2 + (\sum r_a^2)(\sum r_a^2 r_b^2) - 3r_a^2 r_b^2 r_c^2}{r^2 s^4} = \frac{\{(4R+r)^2 - 2s^2\}(\sum r_a r_b)^2 - 2r_a r_b r_c (\sum r_a)\}}{r^2 s^4} \\
 &= \frac{\{(4R+r)^2 - 2s^2\}\{s^4 - 2rs^2(4R+r)\}}{r^2 s^4} \\
 &\stackrel{\text{cyc}}{\Rightarrow} \sum \frac{r_a^2}{r_b^2} \stackrel{\text{(i)}}{\leq} \frac{\sqrt{\{(4R+r)^2 - 2s^2\}(s^2 - 8Rr - 2r^2)}}{rs}
 \end{aligned}$$

$$\begin{aligned}
 \text{Again, } \frac{8r_a r_b r_c}{w_a w_b w_c} - 5 &= \frac{8rs^2(s^2 + 2Rr + r^2)}{16Rr^2 s^2} - 5 = \frac{s^2 + 2Rr + r^2}{2Rr} - 5 \\
 &\stackrel{\text{(ii)}}{\Rightarrow} \frac{8r_a r_b r_c}{w_a w_b w_c} \stackrel{\text{(ii)}}{\geq} \frac{s^2 - 8Rr + r^2}{2Rr}
 \end{aligned}$$

$$\begin{aligned}
 \text{(i), (ii)} \Rightarrow \text{it suffices to prove : } & \frac{\sqrt{\{(4R+r)^2 - 2s^2\}(s^2 - 8Rr - 2r^2)}}{rs} \\
 &\geq \frac{s^2 - 8Rr + r^2}{2Rr} \\
 &\Leftrightarrow \frac{\{(4R+r)^2 - 2s^2\}(s^2 - 8Rr - 2r^2)}{s^2} \geq \frac{(s^2 - 8Rr + r^2)^2}{4R^2} \\
 &\Leftrightarrow 4R^2\{(4R+r)^2 - 2s^2\}(s^2 - 8Rr - 2r^2) \geq s^2(s^2 - 8Rr + r^2)^2 \\
 &\Leftrightarrow s^6 + s^4(8R^2 - 16Rr + 2r^2) - s^2(64R^4 + 96R^3r - 44R^2r^2 + 16Rr^3 - r^4) \\
 &\quad \stackrel{\text{(a)}}{+} 8R^2r(4R+r)^3 \stackrel{\text{(a)}}{\leq} 0
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, Rouche} \Rightarrow s^2 - (m - n) &\geq 0 \text{ and } s^2 - (m + n) \leq 0, \text{ where } m \\
 &= 2R^2 + 10Rr - r^2 \text{ and } n = 2(R - 2r)\sqrt{R^2 - 2Rr} \\
 \therefore (s^2 - (m + n))(s^2 - (m - n)) &\leq 0 \Rightarrow s^4 - s^2(2m) + m^2 - n^2 \leq 0 \\
 &\Rightarrow s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3 \stackrel{\text{(iii)}}{\leq} 0 \\
 \text{and } \because \text{ by Gerretsen, } s^2 - 16Rr + 5r^2 &\geq 0
 \end{aligned}$$

$$\therefore \{s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R+r)^3\}(s^2 - 16Rr + 5r^2) \stackrel{\text{(iv)}}{\leq} 0$$

(iv) \Rightarrow in order to prove (a), it suffices to prove :

$$\begin{aligned}
 s^6 + s^4(8R^2 - 16Rr + 2r^2) - s^2(64R^4 + 96R^3r - 44R^2r^2 + 16Rr^3 - r^4) \\
 + 8R^2r(4R+r)^3
 \end{aligned}$$



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$$\begin{aligned}
 & \leq \{s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3\}(s^2 - 16Rr + 5r^2) \\
 & \Leftrightarrow s^4(12R^2 + 20Rr - 5r^2) \\
 & - s^2(64R^4 + 224R^3r + 304R^2r^2 - 104Rr^3 + 10r^4) + r(512R^5 + 1408R^4r \\
 & \quad + 544R^3r^2 - 40R^2r^3 - 44Rr^4 - 5r^5) \stackrel{(b)}{\geq} 0
 \end{aligned}$$

Now, (iii) $\Rightarrow (12R^2 + 20Rr - 5r^2)\{s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3\} \stackrel{(v)}{\geq} 0$

(v) \Rightarrow in order to prove (b), it suffices to prove

$$\begin{aligned}
 & : s^4(12R^2 + 20Rr - 5r^2) - s^2(64R^4 + 224R^3r + 304R^2r^2 - 104Rr^3 \\
 & \quad + 10r^4) \\
 & + r(512R^5 + 1408R^4r + 544R^3r^2 - 40R^2r^3 - 44Rr^4 - 5r^5) \\
 & \leq (12R^2 + 20Rr - 5r^2)\{s^4 - s^2(4R^2 + 20Rr - 2r^2) + r(4R + r)^3\} \\
 & \Leftrightarrow s^2(4R^3 - 24R^2r - 13Rr^2 + 9r^3) + r(64R^4 + 112R^3r + 60R^2r^2 + 13Rr^3 + r^4) \geq 0 \\
 & \Leftrightarrow s^2(R - 2r)(4R^2 - 8Rr) + r(64R^4 + 112R^3r + 60R^2r^2 + 13Rr^3 \\
 & \quad + r^4) \stackrel{(c)}{\geq} s^2(8R^2r + 29Rr^2 - 9r^3)
 \end{aligned}$$

Now, LHS of (c) $\stackrel{\text{Gerretsen}}{\underset{(m)}{\geq}} (16Rr - 5r^2)(R - 2r)(4R^2 - 8Rr) + r(64R^4 + 112R^3r + 60R^2r^2 + 13Rr^3 + r^4)$ and

RHS of (c) $\stackrel{\text{Gerretsen}}{\underset{(n)}{\leq}} (4R^2 + 4Rr + 3r^2)(8R^2r + 29Rr^2 - 9r^3) \therefore (m), (n)$

\Rightarrow in order to prove (c), it suffices to prove :

$$\begin{aligned}
 & (16Rr - 5r^2)(R - 2r)(4R^2 - 8Rr) + r(64R^4 + 112R^3r + 60R^2r^2 + 13Rr^3 + r^4) \\
 & \geq (4R^2 + 4Rr + 3r^2)(8R^2r + 29Rr^2 - 9r^3) \\
 & \Leftrightarrow 48t^4 - 156t^3 + 146t^2 - 59t + 14 \stackrel{?}{\geq} 0 \left(\text{where } t = \frac{R}{r} \right) \\
 & \Leftrightarrow (t - 2)\{(t - 2)(48t^2 + 36t + 98) + 189\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \\
 & \Rightarrow (c) \Rightarrow (b) \Rightarrow (a) \text{ is true} \therefore 5 + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq \frac{8r_a r_b r_c}{w_a w_b w_c} \text{ (Proved)}
 \end{aligned}$$



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1750. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \sqrt{r_a}(n_a + g_a) \geq s \sqrt{\frac{2}{R}} \sum \left(\sqrt{\frac{r_b}{r_c}} + \sqrt{\frac{r_c}{r_b}} \right)$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) \\ &= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= a g_a^2 + a(s - b)(s - c) \end{aligned}$$

$$\begin{aligned} \text{Adding the above two, we get : } &(b^2 + c^2)(2s - b - c) \\ &= a n_a^2 + a g_a^2 + 2a(s - b)(s - c) \\ \Rightarrow 2a(b^2 + c^2) &= 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\ &= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\ \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 &= 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\ \Rightarrow 4m_a^2 + (b - c)^2 + 4r_b r_c &= 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b - c)^2 + 4s(s - a) \\ &= 2(n_a^2 + g_a^2) + 4s(s - a) \\ \Rightarrow 4m_a^2 + 4m_a^2 &= 2(n_a^2 + g_a^2) + 4s(s - a) \Rightarrow n_a^2 + g_a^2 \stackrel{(1)}{\cong} 4m_a^2 - 2s(s - a) \\ a n_a^2 \cdot a g_a^2 &\geq a^2 s^2 (s - a)^2 \\ \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) \\ &- a(s - b)(s - c)\} \stackrel{(a)}{\cong} a^2 s^2 (s - a)^2 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\begin{aligned} \Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 \\ - yz(y + z)\} &\geq x^2(y + z)^2(x + y + z)^2 \\ \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 &\geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true} \\ \Rightarrow (a) \text{ is true} &\Rightarrow n_a g_a \stackrel{(2)}{\cong} s(s - a) \end{aligned}$$



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by (1)and (2)

$$\text{Now, } \mathbf{n}_a^2 + \mathbf{g}_a^2 + 2\mathbf{n}_a\mathbf{g}_a \stackrel{\text{by (1)and (2)}}{\leq} 4\mathbf{m}_a^2 - 2s(s-a) + 2s(s-a) = 4\mathbf{m}_a^2$$

$$\Rightarrow (\mathbf{n}_a + \mathbf{g}_a)^2 \geq 4\mathbf{m}_a^2 \Rightarrow \mathbf{n}_a + \mathbf{g}_a \stackrel{(3)}{\geq} 2\mathbf{m}_a$$

$$\text{Also, } r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\geq} 4R \cos^2 \frac{A}{2}$$

$$\text{Now, } (\mathbf{b} + \mathbf{c})^2 \geq 32Rr \cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\geq} 8r(r_b + r_c) = 8r^2s \left(\frac{1}{s-b} + \frac{1}{s-c} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore b+c \stackrel{(4)}{\geq} 4\sqrt{2Rr} \cos \frac{A}{2}$$

Now, $\sqrt{r_a r_b r_c}(\mathbf{n}_a$

$$+ \mathbf{g}_a) \stackrel{\text{by (3)}}{\geq} 2(s\sqrt{r})\mathbf{m}_a \stackrel{\text{Ioscu}}{\geq} 2 \cdot \frac{\mathbf{b} + \mathbf{c}}{2} \cdot (s\sqrt{r}) \cos \frac{A}{2} \stackrel{\text{by (4)}}{\geq} sr(\sqrt{2R}) \left(4 \cos^2 \frac{A}{2} \right) \stackrel{\text{by (i)}}{\geq} S(\sqrt{2R}) \left(\frac{r_b + r_c}{R} \right)$$

$$= S \sqrt{\frac{2}{R}} (r_b + r_c)$$

$$\Rightarrow \sqrt{r_a}(\mathbf{n}_a + \mathbf{g}_a) \geq S \sqrt{\frac{2}{R}} \left(\sqrt{\frac{r_b}{r_c}} + \sqrt{\frac{r_c}{r_b}} \right) \text{ and analogs} \Rightarrow \sum \sqrt{r_a}(\mathbf{n}_a + \mathbf{g}_a)$$

$$\geq S \sqrt{\frac{2}{R}} \sum \left(\sqrt{\frac{r_b}{r_c}} + \sqrt{\frac{r_c}{r_b}} \right) \text{ (Proved)}$$

1751. In any ΔABC , holds:

$$\frac{3w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \frac{w_a w_b w_c}{h_a h_b h_c} &\leq \frac{\sqrt{s(s-a)}\sqrt{s(s-b)}\sqrt{s(s-c)}}{\frac{16R^2 r^2 s^2}{8R^3}} = \frac{R \cdot s \cdot rs}{2r^2 s^2} = \frac{R}{2r} \Rightarrow \frac{R}{2r} \geq \frac{w_a w_b w_c}{h_a h_b h_c} \\
 &\geq \frac{3w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c} \\
 \Leftrightarrow x \geq \frac{3x}{x+2} \left(\text{where } x = \frac{w_a w_b w_c}{h_a h_b h_c} \right) &\Leftrightarrow x^2 + 2x \geq 3x \Leftrightarrow x(x-1) \geq 0 \rightarrow \text{true} \therefore x \\
 &= \frac{w_a w_b w_c}{h_a h_b h_c} \geq 1 \therefore \frac{3w_a w_b w_c}{w_a w_b w_c + 2h_a h_b h_c} \leq \frac{R}{2r} \text{ (QED)}
 \end{aligned}$$

1752. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sqrt{\frac{R}{2r}} \geq \frac{1}{3} \sum \frac{r_b + r_c}{n_a + g_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem} &\Rightarrow b^2(s-c) + c^2(s-b) \\
 &= a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 &= a g_a^2 + a(s-b)(s-c)
 \end{aligned}$$

Adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$= a n_a^2 + a g_a^2 + 2a(s-b)(s-c)$$

$$\begin{aligned}
 \Rightarrow 2a(b^2 + c^2) &= 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2) \\
 &= 2(n_a^2 + g_a^2) + a^2 - (b-c)^2
 \end{aligned}$$

$$\Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2 + g_a^2)$$

$$\begin{aligned}
 \Rightarrow 4m_a^2 + (b-c)^2 + 4r_b r_c &= 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b-c)^2 + 4s(s-a) \\
 &= 2(n_a^2 + g_a^2) + 4s(s-a)
 \end{aligned}$$

$$\Rightarrow 4m_a^2 + 4m_a^2 = 2(n_a^2 + g_a^2) + 4s(s-a) \stackrel{(1)}{\Rightarrow} n_a^2 + g_a^2 \cong 4m_a^2 - 2s(s-a)$$



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$$an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s - a)^2$$

$$\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s - b)(s - c)\} \stackrel{(a)}{\leq} a^2 s^2 (s - a)^2$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \stackrel{(2)}{\leq} s(s - a)$$

by (1) and (2)

$$\text{Now, } n_a^2 + g_a^2 + 2n_a g_a \stackrel{(2)}{\leq} 4m_a^2 - 2s(s - a) + 2s(s - a) = 4m_a^2$$

$$\Rightarrow (n_a + g_a)^2 \geq 4m_a^2 \Rightarrow n_a + g_a \stackrel{(3)}{\leq} 2m_a$$

$$\text{Also, } r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\cong} 4R \cos^2 \frac{A}{2}$$

$$\text{Now, } (b + c)^2 \geq 32Rr \cos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\cong} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s - b} + \frac{1}{s - c} \right)$$

$$= 8(s - a)(s - b)(s - c) \frac{a}{(s - b)(s - c)} = 4a(b + c - a)$$

$$\Leftrightarrow (b + c)^2 + 4a^2 - 4a(b + c) \geq 0 \Leftrightarrow (b + c - 2a)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore b + c \stackrel{(4)}{\geq} 4\sqrt{2Rr} \cos \frac{A}{2}$$

$$\text{Now, } n_a + g_a \stackrel{\text{by (3)}}{\leq} 2m_a \stackrel{\text{loscu}}{\leq} 2 \cdot \frac{b + c}{2} \cos \frac{A}{2} \stackrel{\text{by (4)}}{\leq} \sqrt{2Rr} \left(4 \cos^2 \frac{A}{2} \right) \stackrel{\text{by (i)}}{\cong} \sqrt{2Rr} \left(\frac{r_b + r_c}{R} \right)$$

$$= \sqrt{\frac{2r}{R}} (r_b + r_c) \Rightarrow \sqrt{\frac{R}{2r}} \geq \frac{r_b + r_c}{n_a + g_a} \text{ and analogs}$$



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$$\Rightarrow 3 \sqrt{\frac{R}{2r}} \geq \sum \frac{r_b + r_c}{n_a + g_a} \Rightarrow \sqrt{\frac{R}{2r}} \geq \frac{1}{3} \sum \frac{r_b + r_c}{n_a + g_a} \text{ (Proved)}$$

1753. In any ΔABC , holds:

$$\frac{m_a}{m_b} + \frac{m_b}{m_a} \leq \frac{R}{r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Tereshin} \Rightarrow m_c \geq \frac{a^2 + b^2}{4R} \Rightarrow \frac{4RSm_c}{S} \geq a^2 + b^2 \Rightarrow \frac{abcm_c}{S} \geq a^2 + b^2 \Rightarrow \frac{cm_c}{S} \geq \frac{a}{b} + \frac{b}{a}$$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course

$= \frac{S}{3}$ and medians of course $= \frac{a}{2}, \frac{b}{2}, \frac{c}{2}$ we get :

$$\frac{\left(\frac{2m_c}{3}\right)\left(\frac{c}{2}\right)}{\frac{S}{3}} \geq \frac{\frac{2m_a}{3}}{\frac{2m_b}{3}} + \frac{\frac{2m_b}{3}}{\frac{2m_a}{3}} \Rightarrow \frac{2m_c}{\left(\frac{2S}{c}\right)} \geq \frac{m_a}{m_b} + \frac{m_b}{m_a} \Rightarrow \frac{m_a}{m_b} + \frac{m_b}{m_a}$$

$$\leq \frac{2m_c}{h_c} \stackrel{\text{Panaitopol}}{\leq} \frac{R}{r} \text{ (Proved)}$$

1754. In any ΔABC , holds:

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{3\mu}{2} \stackrel{(1)}{\geq} 1 + \mu \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \right) \forall \mu \leq \frac{10}{9}$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(1) \Leftrightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \stackrel{(2)}{\geq} \mu \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right)$$

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1$$

$$= \frac{2s^2 - 8Rr - 2r^2 - (s^2 + 4Rr + r^2)}{s^2 + 4Rr + r^2} \stackrel{(i)}{\cong} \frac{s^2 - 12Rr - 3r^2}{s^2 + 4Rr + r^2} \text{ and } \frac{10}{9} \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right)$$

$$+ \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2}$$



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$$\begin{aligned}
 &= \frac{10}{9} \left\{ \frac{\sum a(c+a)(a+b)}{\prod(b+c)} - \frac{3}{2} \right\} = \frac{10}{9} \left[\frac{\sum \{a(\sum ab + a^2)\}}{2s(s^2 + 2Rr + r^2)} - \frac{3}{2} \right] \\
 &= \frac{10}{9} \left\{ \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2) - 3s(s^2 + 2Rr + r^2)}{2s(s^2 + 2Rr + r^2)} \right\} \\
 &\stackrel{(ii)}{\equiv} \frac{5}{9} \left(\frac{s^2 - 10Rr - 7r^2}{s^2 + 2Rr + r^2} \right) \therefore (i), (ii) \Rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \\
 &\geq \frac{10}{9} \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right) \\
 \Leftrightarrow &\frac{s^2 - 12Rr - 3r^2}{s^2 + 4Rr + r^2} \geq \frac{5}{9} \left(\frac{s^2 - 10Rr - 7r^2}{s^2 + 2Rr + r^2} \right) \Leftrightarrow 9(s^2 + 2Rr + r^2)(s^2 - 12Rr - 3r^2) \\
 &\geq 5(s^2 - 10Rr - 7r^2)(s^2 + 4Rr + r^2) \\
 &\Leftrightarrow s^4 - s^2(15Rr - 3r^2) \stackrel{(3)}{\geq} r^2(4R^2 - 7Rr - 2r^2) \\
 \text{Now, LHS of (3)} &\stackrel{\text{Gerretsen}}{\geq} s^2(Rr - 2r^2) \stackrel{\text{Gerretsen}}{\geq} (Rr - 2r^2)(16Rr - 5r^2) \stackrel{?}{\geq} r^2(4R^2 - 7Rr - 2r^2) \\
 &\Leftrightarrow 2R^2 - 5Rr + 2r^2 \stackrel{?}{\geq} 0 \\
 \Leftrightarrow &(R - 2r)(2R - r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r \Rightarrow (3) \text{ is true} \therefore \frac{a^2 + b^2 + c^2}{ab + bc + ca} - 1 \\
 &\geq \frac{10}{9} \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right) \\
 &\stackrel{\frac{10}{9} \geq \mu}{\geq} \mu \left(\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \right) \left(\because \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq 0 \text{ by Nesbitt} \right) \\
 &\Rightarrow (2) \Rightarrow (1) \text{ is true (Proved)}
 \end{aligned}$$

It is to be noted that $(s^2 - 12Rr - 3r^2)$ and $(s^2 - 10Rr - 7r^2)$

$$\geq 0 \text{ using } s^2 - 16Rr + 5r^2 \stackrel{\text{Gerretsen}}{\geq} 0 \text{ and } R \stackrel{\text{Euler}}{\geq} 2r$$

1755. In ΔABC the following relationship holds:

$$\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} - \frac{1}{2} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan



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Solution 1 by Bogdan Fuștei-Romania

We known that: $\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc}$

$$\frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \frac{x^2 + y^2 + z^2}{xy + yz + zx}; \forall x, y, z > 0 \Rightarrow \sqrt{\frac{R}{2r}} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

We show that: $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{1}{2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca}$

$$\sum_{cyc} \frac{a}{b+c} \leq \frac{ab + bc + ca + a^2 + b^2 + c^2}{2(ab + bc + ca)}$$

$$\text{But: } (ab + bc + ca) \cdot \frac{a}{b+c} = [a(b+c) + bc] \cdot \frac{a}{b+c} = a^2 + \frac{abc}{b+c}$$

So, we have: $(ab + bc + ca) \sum_{cyc} \frac{a}{b+c} = a^2 + b^2 + c^2 + abc \sum_{cyc} \frac{1}{b+c}$ and then

$$a^2 + b^2 + c^2 + abc \sum_{cyc} \frac{1}{b+c} \leq a^2 + b^2 + c^2 + \frac{1}{2}(ab + bc + ca) \Leftrightarrow$$

$$abc \sum_{cyc} \frac{1}{b+c} \leq \frac{1}{2}(ab + bc + ca) \Leftrightarrow \sum_{cyc} \frac{2}{b+c} \leq \frac{ab + bc + ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Leftrightarrow$$

$$\sum_{cyc} \frac{4}{b+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

$$\frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c} \Leftrightarrow \frac{b+c}{b} + \frac{b+c}{c} \geq 4 \Leftrightarrow \frac{b}{c} + \frac{c}{b} \geq 2 \Leftrightarrow b^2 + c^2 \geq 2bc \Leftrightarrow$$

$(b - c)^2 \geq 0$; (and analogs). Summing, we get:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sum_{cyc} \frac{4}{b+c}$$

So, we have prove that:

$$\sum_{cyc} \frac{a}{b+c} < \frac{1}{2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \text{ and from } \sqrt{\frac{R}{2r}} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \Rightarrow$$

$$\sum_{cyc} \frac{a}{b+c} \leq \frac{1}{2} + \sqrt{\frac{R}{2r}}$$

m_a, m_b, m_c – can be the sides of on triangle, then we can write:



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$$\sum_{cyc} \frac{m_a}{m_b + m_c} \leq \frac{1}{2} + \sqrt{\frac{R_m}{2r_m}}$$

$$S_m = \frac{3}{4} \cdot S; \quad S_m = [m_a m_b m_c]; \quad m_a m_b m_c = 4 S_m R_m = 3 S R_m \Rightarrow R_m = \frac{m_a m_b m_c}{3S}$$

$$\begin{cases} S_m = s_m \cdot r_m \\ s_m = \frac{m_a + m_b + m_c}{2} \Rightarrow r_m = \frac{3S}{2(m_a + m_b + m_c)} \\ \frac{R_m}{2r_m} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \end{cases}$$

$m_a + m_b + m_c \leq 4R + r$ and $m_a m_b m_c \leq \frac{1}{2} \cdot R s^2$ then

$$\frac{R_m}{2r_m} \leq \frac{1}{2} \cdot \frac{R(4R+r)s^2}{9S^2} = \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2}$$

We must show that: $\left(\frac{R}{2r}\right)^2 \geq \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2} \Leftrightarrow$

$$\frac{R^2}{4r^2} \geq \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2} \Leftrightarrow \frac{R}{2} \geq \frac{4R+r}{9} \Leftrightarrow 9R \geq 8R + 2r \Leftrightarrow R \geq 2r \text{ (Euler)}$$

Finally, $\left(\frac{R}{2r}\right)^2 \geq \frac{R_m}{2r_m} \Rightarrow \frac{R}{2r} \geq \sqrt{\frac{R_m}{2r_m}}; (1)$

$$\sum_{cyc} \frac{m_a}{m_b + m_c} \leq \frac{1}{2} + \sqrt{\frac{R_m}{2r_m}}; (2)$$

From (1), (2) we have:

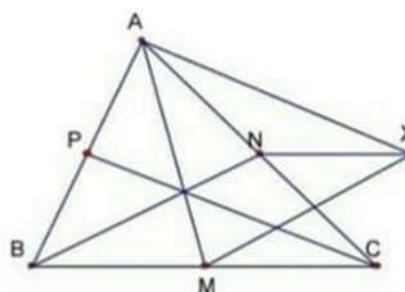
$$\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} - \frac{1}{2} \leq \frac{R}{2r}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $MB = MC; NA = NC; PA = PB$

$BMXN$ – parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A; D \equiv M; E \equiv X \Rightarrow S_{\Delta DEF} = \frac{3}{4} S_{\Delta ABC}$





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$$\begin{aligned}
 \text{Let: } \Omega &= \frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \stackrel{BCS}{\leq} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{(m_a + m_b)^2}} \leq \\
 &\stackrel{(m_a+m_b)^2 \geq 4m_a m_b}{\leq} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{4m_a m_b}} = \frac{1}{2} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{m_a m_b}} \leq \\
 &\stackrel{\sum_{cyc} \frac{1}{m_a m_b} \leq \frac{\sqrt{3}}{S}}{\leq} \frac{1}{2} \sqrt{\frac{3}{4}(a^2 + b^2 + c^2)} \cdot \sqrt{\frac{\sqrt{3}}{S}} \stackrel{(1)}{\leq} \frac{1}{2} + \frac{R}{2r} \\
 (1) \Leftrightarrow \frac{3}{4}(a^2 + b^2 + c^2) \cdot \frac{\sqrt{3}}{sr} &\leq \left(1 + \frac{R}{r}\right)^2 \Leftrightarrow \frac{3\sqrt{3}(s^2 - 4Rr - r^2)}{2s} \leq \frac{(R+r)^2}{r} \Leftrightarrow \\
 2s(R+r)^2 &\geq 3\sqrt{3}r(s^2 - 4Rr - r^2) \Leftrightarrow 4s^2(R+r)^4 \geq 27r^2(s^2 - 4Rr - r^2)^2
 \end{aligned}$$

But: $16Rr - 5r^2 \leq s^2 \leq 4R^2 - 4Rr + 3r^2$ (Gerretsen)

$$4s^2(R+r)^4 \geq 4(16Rr - 5r^2)(R+r)^4; (2)$$

$$\begin{aligned}
 27r^2(s^2 - 4Rr - r^2)^2 &\leq 27r^2(4R^2 - 4Rr + 3r^2 - 4Rr - r^2)^2 = \\
 &= 27r^2(4R^2 + 2r^2)^2 = 27 \cdot 4r^2(2R^2 + r^2)^2; (3)
 \end{aligned}$$

From (2), (3) we need to prove:

$$4(16Rr - 5r^2)(R+r)^4 \geq 27 \cdot 4r^2(2R^2 + r^2)^2 \Leftrightarrow$$

$$(16Rr - 5r)(R+r)^4 \geq 27r(2R^2 + r^2)^2; \left(t = \frac{R}{r} \geq 2\right) \Leftrightarrow$$

$$\begin{aligned}
 (16t - 5)(t+1)^4 &\geq 27(2t^2 + 1)^2 \Leftrightarrow (16t - 5)(t+1)^4 - 27(2t^2 + 1)^2 \geq 0 \Leftrightarrow \\
 16t^5 - 49t^4 + 76t^3 - 64t^2 - 4t - 32 &\geq 0 \Leftrightarrow \\
 (t-2)(16t^4 - 17t^3 + 42t^2 + 10t + 16) &\geq 0
 \end{aligned}$$

Which is true because $t \geq 2 \Rightarrow t-2 \geq 0$

$$\begin{aligned}
 16t^4 - 17t^3 + 42t^2 + 10t + 16 &= t^3(16t - 17) + 42t^2 + 10t + 16 \stackrel{t \geq 2}{\geq} \\
 &\geq 2^3(16 \cdot 2 - 17) + 52 \cdot 2^2 + 10 \cdot 2 + 16 = 324 > 0. \text{ Proved.}
 \end{aligned}$$

Lastly, we prove that:

$$\sum_{cyc} \frac{1}{m_a m_b} \leq \frac{\sqrt{3}}{S} = \frac{\sqrt{3}}{\frac{4}{3} S_{\Delta DEF}} = \frac{3\sqrt{3}}{4S_{\Delta DEF}}$$

In fact, for any $\triangle ABC$ we have:



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$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{a+b+c}{abc} = \frac{2s}{4Rsr} \stackrel{R \geq \frac{2}{3\sqrt{3}}s}{\leq} \frac{1}{2} \cdot \frac{3\sqrt{3}}{2sr} = \frac{3\sqrt{3}}{4s}$$

Apply to $\Delta DEF \Rightarrow$ proved.

1756. In any ΔABC , n_a – Nagel's cevian, holds:

$$\left(\frac{n_a}{r_a}\right)^2 + \left(\frac{n_b}{r_b}\right)^2 + \left(\frac{n_c}{r_c}\right)^2 \geq 13 + \frac{4R - 2(h_a + h_b + h_c)}{r}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= a n_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\
 &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left(\frac{2\Delta}{a}\right) \left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 = s^2 - 2h_a r_a \Rightarrow \left(\frac{n_a}{r_a}\right)^2 \\
 &= \frac{s^2}{s^2 \tan^2 \frac{A}{2}} - \frac{2h_a}{r_a} = \operatorname{cosec}^2 \frac{A}{2} - 1 - \frac{\frac{4rs}{a}}{\frac{rs}{s - a}} \\
 &= \frac{bc(s - a)}{(s - a)(s - b)(s - c)} - 1 - 4 \left(\frac{s - a}{a}\right) = \frac{bc(s - a)}{sr^2} + 3 - \frac{4sbc}{4Rrs} \\
 &= \frac{2R}{r^2} h_a - \frac{4Rrs}{sr^2} + 3 - \frac{2}{r} h_a \Rightarrow \left(\frac{n_a}{r_a}\right)^2 = \frac{2R}{r^2} h_a - \frac{4R}{r} + 3 - \frac{2}{r} h_a \\
 \text{and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \quad &\left(\frac{n_a}{r_a}\right)^2 + \left(\frac{n_b}{r_b}\right)^2 + \left(\frac{n_c}{r_c}\right)^2 \\
 &= 9 + \frac{2R}{r^2} \sum h_a - \frac{12R}{r} - \frac{2}{r} \sum h_a \stackrel{\sum h_a \geq 9r}{\geq} 9 + \left(\frac{2R}{r^2}\right) 9r - \frac{12R}{r} - \frac{2}{r} \sum h_a
 \end{aligned}$$



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$$\begin{aligned}
 &= 9 + \frac{2R}{r} + \frac{4R}{r} - \frac{2}{r} \sum h_a \stackrel{\text{Euler}}{\geq} 13 + \frac{4R}{r} - \frac{2}{r} \sum h_a \\
 &= 13 + \frac{4R - 2(h_a + h_b + h_c)}{r} \quad (\text{Proved})
 \end{aligned}$$

1757. In any } ABC, holds:

$$\min\left(\frac{R}{r}, \frac{r_a}{r_b} + \frac{r_b}{r_a}\right) \geq \frac{a}{b} + \frac{b}{a}$$

Proposed by Alex Szoros – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Let } a = y + z, b = z + x \text{ and } c = x + y \therefore 2s = a + b + c = 2(x + y + z) \Rightarrow s \\
 = x + y + z \therefore s - a = x, s - b = y \\
 \therefore \frac{r_a}{r_b} + \frac{r_b}{r_a} \geq \frac{a}{b} + \frac{b}{a} \Leftrightarrow \frac{y}{x} + \frac{x}{y} \geq \frac{y+z}{z+x} + \frac{z+x}{y+z} \Leftrightarrow (x^2 + y^2)(z+x)(y+z) \\
 \geq xy((z+x)^2 + (y+z)^2) \\
 \Leftrightarrow z(x^3 + y^3) - xyz(x+y) + z^2(x-y)^2 \geq 0 \Leftrightarrow z(x+y)(x-y)^2 + z^2(x-y)^2 \geq 0 \\
 \rightarrow \text{true} \therefore \frac{r_a}{r_b} + \frac{r_b}{r_a} \geq \frac{a}{b} + \frac{b}{a} \text{ and } \because \frac{R}{r} \stackrel{\text{Bandila}}{\geq} \frac{a}{b} + \frac{b}{a} \\
 \therefore \frac{R}{r}, \frac{r_a}{r_b} + \frac{r_b}{r_a} \geq \frac{a}{b} + \frac{b}{a} \Rightarrow \min\left(\frac{R}{r}, \frac{r_a}{r_b} + \frac{r_b}{r_a}\right) \geq \frac{a}{b} + \frac{b}{a} \quad (\text{Proved})
 \end{aligned}$$

1758. In any } ABC, holds:

$$\sqrt{\frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{m_a m_b m_c (m_a + m_b + m_c)}} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Now, } abc(a+b+c) = 4Rrs(2s) \stackrel{\text{Euler}}{\geq} 16r^2s^2 \Rightarrow abc(a+b+c) \\
 \geq 16s^2 \text{ and applying this last inequality on a triangle with }
 \end{aligned}$$



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sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course $= \frac{S}{3}$, we get

$$\therefore \frac{8}{27} m_a m_b m_c \left\{ \frac{2}{3} (m_a + m_b + m_c) \right\} \geq \frac{16S^2}{9}$$

$$\Rightarrow m_a m_b m_c (m_a + m_b + m_c) \geq 9S^2 \Rightarrow \frac{1}{\sqrt{m_a m_b m_c (m_a + m_b + m_c)}} \stackrel{(1)}{\leq} \frac{1}{3rs}$$

$$\text{Also, } m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 = \frac{9}{16} \sum a^2 b^2 \stackrel{\text{Goldstone}}{\leq} \left(\frac{9}{16} \right) 4R^2 s^2$$

$$\Rightarrow \sqrt{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2} \stackrel{(2)}{\leq} \frac{3Rs}{2} \therefore (1), (2) \Rightarrow$$

$$\sqrt{\frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{m_a m_b m_c (m_a + m_b + m_c)}} \leq \frac{\left(\frac{3Rs}{2}\right)}{3rs} = \frac{R}{2r} \text{ (Proved)}$$

1759. In ΔABC the following relationship holds:

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} + \frac{s_a s_b s_c}{2m_a m_b m_c} \geq 2$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Tran Hong-Dong Thap-Vietnam

$$s_a = \frac{2bc}{b^2 + c^2} \cdot m_a \Rightarrow \frac{s_a s_b s_c}{m_a m_b m_c} = \frac{8a^2 b^2 c^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \Rightarrow$$

$$\frac{s_a s_b s_c}{2m_a m_b m_c} = \frac{4a^2 b^2 c^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}$$

So, we need to prove:

$$\frac{a^2}{b^2 + c^2} + \frac{b^2}{c^2 + a^2} + \frac{c^2}{a^2 + b^2} + \frac{4a^2 b^2 c^2}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)} \geq 2 \quad (*)$$

Let: $x = a^2; y = b^2; z = c^2$

$$(*) \Leftrightarrow \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{4xyz}{(x+y)(y+z)(z+x)} \geq 2 \Leftrightarrow$$

$$x^3 + y^3 + z^3 + 3xyz \geq xy(x+y) + yz(y+z) + zx(z+x)$$



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Which is true by Schur's inequality. Proved.

1760. In any ΔABC , holds:

$$\left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \geq \left(s\sqrt{3} + \sum r_a \right)^2 \left(\sum (m_a + h_a) \right)^{-2}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum m_a w_a &\stackrel{\text{loscu}}{\leq} \sum \left\{ \left(\frac{b+c}{2} \right) \cos \frac{A}{2} \left(\frac{2bc}{b+c} \right) \cos \frac{A}{2} \right\} = \sum \left[bc \left\{ \frac{s(s-a)}{bc} \right\} \right] = s \sum (s-a) \\ &= s^2 \end{aligned}$$

$$\Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \geq s^2 \left\{ \sum \left(\frac{bc}{2R} \right) \left(\frac{ca}{2R} \right) \right\}^{-1} = s^2 \left\{ \left(\frac{4Rrs}{4R^2} \right)^{-1} \right\} \left\{ \left(\sum a \right)^{-1} \right\}$$

$$= \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r} \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \stackrel{(m)}{\geq} \frac{R}{2r}$$

$$\text{Now, } r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\equiv} 4R \cos^2 \frac{A}{2}$$

$$\text{Now, } (b+c)^2 \geq 32Rrcos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\equiv} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c$$

$$\geq 4\sqrt{2Rr} \cos \frac{A}{2} \Rightarrow \sum m_a \stackrel{\text{loscu}}{\leq} \sum \left(\frac{b+c}{2} \cos \frac{A}{2} \right)$$

$$\geq \sqrt{2Rr} \sum 2 \cos^2 \frac{A}{2} = \sqrt{2Rr} \sum (1 + \cos A) = \sqrt{2Rr} \left(4 + \frac{r}{R} \right) = \sqrt{\frac{2r}{R}} \left(\sum r_a \right)$$

$$\Rightarrow \left(\sum (m_a + h_a) \right)^2 \geq \left(\sqrt{\frac{2r}{R}} \left(\sum r_a \right) + \sum h_a \right)^2$$



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$$\begin{aligned}
 &= \frac{2r}{R} \left(\sum r_a \right)^2 + 2 \sqrt{\frac{2r}{R}} \left(\sum r_a \right) \left(\sum h_a \right) + \left(\sum h_a \right)^2 \\
 &\Rightarrow \left(\sum (m_a + h_a) \right)^2 \stackrel{(n)}{\geq} \frac{2r}{R} \left(\sum r_a \right)^2 + 2 \sqrt{\frac{2r}{R}} \left(\sum r_a \right) \left(\sum h_a \right) + \left(\sum h_a \right)^2 \\
 &\quad (m), (n) \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \cdot \left(\sum (m_a + h_a) \right)^2 \\
 &\geq \left(\sum r_a \right)^2 + 2 \sqrt{\frac{R}{2r}} \left(\sum r_a \right) \left(\sum h_a \right) + \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 \\
 &\Rightarrow \boxed{\left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \cdot \left(\sum (m_a + h_a) \right)^2 - \left(s\sqrt{3} + \sum r_a \right)^2} \stackrel{(u)}{\geq} \left(\sum r_a \right)^2 \\
 &\quad + 2 \sqrt{\frac{R}{2r}} \left(\sum r_a \right) \left(\sum h_a \right) + \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 \\
 &\quad - \left(3s^2 + 2s\sqrt{3} \left(\sum r_a \right) + \left(\sum r_a \right)^2 \right)
 \end{aligned}$$

$$= \boxed{2 \left(\sum r_a \right) \left\{ \sqrt{\frac{R}{2r}} \sum h_a - s\sqrt{3} \right\} + \left\{ \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 - 3s^2 \right\}}$$

$$\text{Now, } \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 \geq 3s^2 \Leftrightarrow \frac{(s^2 + 4Rr + r^2)^2}{8Rr} \geq 3s^2 \Leftrightarrow (s^2 + 4Rr + r^2)^2 \geq 24Rrs^2$$

$$\Leftrightarrow s^4 - s^2(16Rr - 2r^2) + r^2(4R + r)^2 \stackrel{(1)}{\geq} 0$$

$$\text{LHS of (1)} \stackrel{\text{Gerretsen}}{\geq} s^2(16Rr - 5r^2) - s^2(16Rr - 2r^2) + r^2(4R + r)^2$$

$$= r^2 \{ (4R + r)^2 - 3s^2 \} \stackrel{\text{Trucht}}{\geq} 0 \Rightarrow (1) \text{ is true}$$

$$\therefore \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 \geq 3s^2 \Rightarrow \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 - 3s^2 \stackrel{(ii)}{\geq} 0 \text{ and } \sqrt{\frac{R}{2r}} \sum h_a - s\sqrt{3} \stackrel{(iii)}{\geq} 0$$

$$\therefore (ii), (iii)$$



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$$\begin{aligned}
 & \Rightarrow 2 \left(\sum r_a \right) \left\{ \sqrt{\frac{R}{2r}} \sum h_a - s\sqrt{3} \right\} + \left\{ \left(\frac{R}{2r} \right) \left(\sum h_a \right)^2 - 3s^2 \right\} \stackrel{(v)}{\geq} 0 \therefore (u), (v) \\
 & \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \cdot \left(\sum (m_a + h_a) \right)^2 \geq (s\sqrt{3} + \sum r_a)^2 \\
 & \Rightarrow \left(\sum m_a w_a \right) \left(\sum h_a h_b \right)^{-1} \geq (s\sqrt{3} + \sum r_a)^2 \left(\sum (m_a + h_a) \right)^{-2}
 \end{aligned}$$

1761. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\frac{2(a^2 + b^2)}{(a+b)^2} + \frac{2(b^2 + c^2)}{(b+c)^2} + \frac{2(c^2 + a^2)}{(c+a)^2} \leq \frac{3n_a g_a r_a}{h_a h_b h_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : } R \left(1 - \frac{2r}{R} \right) & \geq \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2 \sin B \sin C}{2R} \Leftrightarrow 1 - \frac{8R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \\
 & \geq \sin^2 B + \sin^2 C - 2 \sin B \sin C = (\sin B - \sin C)^2 \\
 & \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(2 \sin \frac{B}{2} \sin \frac{C}{2} \right) \geq \left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2} \right)^2 \\
 & \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) \geq 4 \sin^2 \frac{A}{2} \left(1 - \cos^2 \frac{B-C}{2} \right) \\
 & \Leftrightarrow 1 - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 4 \sin^2 \frac{A}{2} \geq 4 \sin^2 \frac{A}{2} - 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} \\
 & \Leftrightarrow 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 1 \geq 0 \\
 & \Leftrightarrow \left(2 \sin \frac{A}{2} \cos \frac{B-C}{2} - 1 \right)^2 \stackrel{(1)}{\geq} 0 \rightarrow \text{true} \Rightarrow R - 2r \geq \frac{b^2 + c^2}{4R} - \frac{bc}{2R}
 \end{aligned}$$

$$\Rightarrow (b - c)^2 \stackrel{(1)}{\leq} 4R(R - 2r) \text{ and analogs}$$

$$\begin{aligned}
 & \text{Again, Stewart's theorem } \Rightarrow b^2(s-c) + c^2(s-b) \\
 & = a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 & = a g_a^2 + a(s-b)(s-c) \\
 & \quad \therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s-a)^2 \\
 & \Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2
 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$



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Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \geq s(s-a)$$

$$\Rightarrow \frac{3n_a g_a r_a}{h_a h_b h_c} \stackrel{(2)}{\leq} \frac{3s(s-a) \left(\frac{rs}{s-a}\right) \cdot 8R^3}{16R^2 r^2 s^2} = \frac{3R}{2r}$$

$$\text{Now, } \sum \frac{2(a^2 + b^2)}{(a+b)^2} = \sum \frac{(b+c)^2 + (b-c)^2}{(b+c)^2}$$

$$= 3 + \sum \frac{(b-c)^2}{(b+c)^2} \stackrel{\text{by (1) and analogs}}{\leq} 3 + \sum \frac{4R(R-2r)}{(b+c)^2} \stackrel{A-G}{\leq} 3$$

$$+ \sum \frac{4R(R-2r)}{4bc}$$

$$= 3 + R(R-2r) \left(\frac{2s}{4Rrs} \right) = 3 + \frac{R-2r}{2r} \Rightarrow \sum \frac{2(a^2 + b^2)}{(a+b)^2} \stackrel{(3)}{\leq} 3 + \frac{R-2r}{2r} \therefore (2), (3)$$

$$\Rightarrow \text{it suffices to prove : } \frac{3R}{2r} \geq 3 + \frac{R-2r}{2r}$$

$$\Leftrightarrow 3 \left(\frac{R-2r}{2r} \right) \geq \frac{R-2r}{2r} \Leftrightarrow \frac{R-2r}{r} \geq 0 \rightarrow \text{true (Euler)}$$

$$\therefore \frac{2(a^2 + b^2)}{(a+b)^2} + \frac{2(b^2 + c^2)}{(b+c)^2} + \frac{2(c^2 + a^2)}{(c+a)^2} \leq \frac{3n_a g_a r_a}{h_a h_b h_c} \text{ (Proved)}$$

1762. In any ΔABC , holds:

$$\sum \left(\frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right) \leq 4 \left(1 + \frac{R}{r} \right)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum \frac{m_a}{w_a} &\stackrel{\text{Tsintsifas}}{\leq} \sum \left(\frac{b^2 + c^2}{2bc} \right) = \frac{1}{2} \sum \left(\frac{b}{c} + \frac{c}{b} \right) = \frac{1}{2} \sum \left(\frac{c}{a} + \frac{b}{a} \right) = \frac{1}{2} \sum \frac{b+c}{a} \\ &= \frac{1}{2} \sum \frac{4R \cos \frac{A}{2} \cos \frac{B-C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}} \leq \frac{1}{2} \sum \frac{1}{\sin \frac{A}{2}} \\ \left(\because 0 \leq \cos \frac{B-C}{2} \leq 1 \text{ and analogs} \right) &= \frac{1}{2r} \sum AI \Rightarrow \sum \frac{m_a}{w_a} \stackrel{(i)}{\leq} \frac{1}{2r} \sum AI \end{aligned}$$



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$$\begin{aligned}
 \sum \sqrt{\frac{m_a}{r_a}} &\stackrel{\text{Panaitopol}}{\leq} \sum \sqrt{\frac{Rh_a}{2rr_a}} = \sum \sqrt{\frac{2Rrs(s-a)}{2r^2as}} \\
 &= \sum \sqrt{\frac{R(s-a)}{ra}} \stackrel{\text{by (1)}}{\cong} \sum \sqrt{\frac{4R^2 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4R \cos \frac{A}{2} \sin \frac{A}{2} \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}} = \frac{1}{2} \sum \frac{1}{\sin \frac{A}{2}} \\
 &= \frac{1}{2r} \sum AI \\
 &\Rightarrow \sum \sqrt{\frac{m_a}{r_a}} \stackrel{\text{(ii)}}{\leq} \frac{1}{2r} \sum AI
 \end{aligned}$$

Now, $x^4 + 1 \geq \frac{1}{2}(x^2 + 1)^2 \stackrel{A-G}{\geq} \frac{1}{2}(2x)(x^2 + 1) \Rightarrow x^4 + 1 \geq x(x^2 + 1) \Rightarrow x^2 + \frac{1}{x^2} \stackrel{(1)}{\geq} x + \frac{1}{x}$

Choosing $x = \sqrt{\frac{a}{b}}$ **in (1), we get :** $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \leq \frac{a}{b} + \frac{b}{a}$ **and analogs** $\Rightarrow \sum \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right)$

$$\begin{aligned}
 &\leq \sum \left(\frac{a}{b} + \frac{b}{a} \right) = \sum \left(\frac{c}{a} + \frac{b}{a} \right) = \sum \frac{b+c}{a} \\
 &= \sum \frac{4R \cos \frac{A}{2} \cos \frac{B-C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}} \leq \sum \frac{1}{\sin \frac{A}{2}} \left(\because 0 \leq \cos \frac{B-C}{2} \leq 1 \text{ and analogs} \right) = \frac{1}{r} \sum AI \\
 &\Rightarrow \sum \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right) \stackrel{\text{(iii)}}{\leq} \frac{1}{r} \sum AI
 \end{aligned}$$

$$\begin{aligned}
 \therefore \sum \frac{m_a}{w_a} + \sum \sqrt{\frac{m_a}{r_a}} + \sum \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right) &\leq \frac{1}{2r} \sum AI + \frac{1}{2r} \sum AI + \frac{1}{r} \sum AI = \frac{2}{r} \sum AI \\
 &= 2 \sum \frac{1}{\sin \frac{A}{2}} = 2 \sum \sqrt{\frac{bc(s-a)}{(s-a)(s-b)(s-c)}}
 \end{aligned}$$



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$$\stackrel{\text{CBS}}{\leq} \frac{2}{r\sqrt{s}} \sqrt{\sum ab} \sqrt{\sum (s-a)} = \frac{2\sqrt{s}}{r\sqrt{s}} \sqrt{s^2 + 4Rr + r^2} = \frac{2\sqrt{s^2 + 4Rr + r^2}}{r}$$

$$\therefore \sum \frac{m_a}{w_a} + \sum \sqrt{\frac{m_a}{r_a}} + \sum \left(\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right) \leq \frac{2\sqrt{s^2 + 4Rr + r^2}}{r}$$

$$\stackrel{\text{Gerretsen}}{\leq} \frac{2\sqrt{4R^2 + 8Rr + 4r^2}}{r} = \frac{4(R+r)}{r} = 4 \left(1 + \frac{R}{r} \right) \text{ (Proved)}$$

1763. Let ABC be an arbitrary triangle ; denote by A the set $A = \{r_a, r_b, r_c\}$.

With the usual notations in triangle , prove that :

a) $R - 2r \leq \frac{1}{2}(\max A - \min A)$

b) Let $t \in \mathbb{R}$ be such that in any triangle $R - 2r \leq t(\max A - \min A)$. Then $t \geq \frac{1}{2}$.

Proposed by Cristian Miu – Romania

Solution by proposer

a) Suppose that the sides of the given triangle verify : $c \geq b \geq a$. Denote by $x = p-a, y = p-b, z = p-c$. It results that $x \geq y \geq z$. Applying the formula

$$r_a = \frac{S}{p-a} = \frac{\sqrt{xyz(x+y+z)}}{x} = \sqrt{\frac{yz(x+y+z)}{x}}, \text{ etc., it results that } r_c \geq r_b \geq r_a . \text{ We have also}$$

$$R = \frac{abc}{4S} = \frac{\prod(x+y)}{4\sqrt{xyz(x+y+z)}}, r = \frac{S}{p} = \sqrt{\frac{xyz}{x+y+z}}.$$

From $r_c \geq r_b \geq r_a$ it results that $\max A = r_c, \min A = r_a$. We have :

$$R - 2r \leq \frac{1}{2}(r_c - r_a) \Leftrightarrow \frac{\prod(x+y)}{4\sqrt{xyz(x+y+z)}} - 2\sqrt{\frac{xyz}{x+y+z}} \leq \frac{1}{2} \left(\frac{\sqrt{xyz(x+y+z)}}{z} - \frac{\sqrt{xyz(x+y+z)}}{x} \right) \Leftrightarrow \\ \prod(x+y) - 8xyz \leq 2y(x+y+z)(x-z) \Leftrightarrow x^2y + xy^2 + 6xyz \geq x^2z + xz^2 + 3y^2z + 3yz^2$$

which results by adding the inequalities : $x^2y \geq x^2z, 3xyz \geq 3y^2z, 3xyz \geq 3yz^2, xy^2 \geq xz^2$, which are true according to the hypothesis $x \geq y \geq z$.

b) As above , the inequality from enunciation is equivalent to the following : for any $x, y, z \in R$ satisfying $x \geq y \geq z$, we have :

$$\frac{\prod(x+y)}{4\sqrt{xyz(x+y+z)}} - 2\sqrt{\frac{xyz}{x+y+z}} \leq t \left(\frac{\sqrt{xyz(x+y+z)}}{z} - \frac{\sqrt{xyz(x+y+z)}}{x} \right) \Leftrightarrow \\ \Leftrightarrow \prod(x+y) - 8xyz \leq 4ty(x+y+z)(x-z)$$



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Taking $y = z$ in the above inequality it results that for any $x, z \in R$ for which $x \geq z$ the following inequality is verified :

$$2z(x+z)^2 - 8xz^2 \leq 4tz(x+2z)(x-z) \Leftrightarrow (x+z)^2 - 4xz \leq 2t(x+2z)(x-z) \Leftrightarrow x-z \leq 2t(x+2z)$$

Since $x \geq z$, it results that there is an $u \geq 0$ such that $x = z + u$. Doing this substitution, the latest inequality becomes equivalent to : $u \leq 2t(3z+u) \Leftrightarrow u(1-2t) \leq 6zt$, which is

satisfied for any u, z positive real numbers only when $1-2t \leq 0 \Leftrightarrow t \geq \frac{1}{2}$, q.e.d.

1764. In ΔABC the following relationship holds:

$$\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} - \frac{1}{2} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Bogdan Fuștei-Romania

We known that: $\frac{R}{r} \geq \frac{abc+a^3+b^3+c^3}{2abc}$

$$\frac{x^3+y^3+z^3}{4xyz} + \frac{1}{4} \geq \frac{x^2+y^2+z^2}{xy+yz+zx}; \forall x, y, z > 0 \Rightarrow \sqrt{\frac{R}{2r}} \geq \frac{a^2+b^2+c^2}{ab+bc+ca}$$

We show that: $\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq \frac{1}{2} + \frac{a^2+b^2+c^2}{ab+bc+ca}$

$$\sum_{cyc} \frac{a}{b+c} \leq \frac{ab+bc+ca+a^2+b^2+c^2}{2(ab+bc+ca)}$$

But: $(ab+bc+ca) \cdot \frac{a}{b+c} = [a(b+c) + bc] \cdot \frac{a}{b+c} = a^2 + \frac{abc}{b+c}$

So, we have: $(ab+bc+ca) \sum_{cyc} \frac{a}{b+c} = a^2 + b^2 + c^2 + abc \sum_{cyc} \frac{1}{b+c}$ and then

$$a^2 + b^2 + c^2 + abc \sum_{cyc} \frac{1}{b+c} \leq a^2 + b^2 + c^2 + \frac{1}{2}(ab+bc+ca) \Leftrightarrow$$

$$abc \sum_{cyc} \frac{1}{b+c} \leq \frac{1}{2}(ab+bc+ca) \Leftrightarrow \sum_{cyc} \frac{2}{b+c} \leq \frac{ab+bc+ca}{abc} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \Leftrightarrow$$

$$\sum_{cyc} \frac{4}{b+c} \leq \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

$$\frac{1}{b} + \frac{1}{c} \geq \frac{4}{b+c} \Leftrightarrow \frac{b+c}{b} + \frac{b+c}{c} \geq 4 \Leftrightarrow \frac{b}{c} + \frac{c}{b} \geq 2 \Leftrightarrow b^2 + c^2 \geq 2bc \Leftrightarrow$$



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$(b - c)^2 \geq 0$; (and analogs). Summing, we get:

$$\sum_{cyc} \frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sum_{cyc} \frac{4}{b+c}$$

So, we have prove that:

$$\sum_{cyc} \frac{a}{b+c} < \frac{1}{2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \text{ and from } \sqrt{\frac{R}{2r}} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \Rightarrow$$

$$\sum_{cyc} \frac{a}{b+c} \leq \frac{1}{2} + \sqrt{\frac{R}{2r}}$$

m_a, m_b, m_c – can be the sides of on triangle, then we can write:

$$\sum_{cyc} \frac{m_a}{m_b + m_c} \leq \frac{1}{2} + \sqrt{\frac{R_m}{2r_m}}$$

$$S_m = \frac{3}{4} \cdot S; S_m = [m_a m_b m_c]; m_a m_b m_c = 4 S_m R_m = 3 S R_m \Rightarrow R_m = \frac{m_a m_b m_c}{3S}$$

$$\begin{cases} S_m = s_m \cdot r_m \\ S_m = \frac{m_a + m_b + m_c}{2} \Rightarrow r_m = \frac{3S}{2(m_a + m_b + m_c)} \end{cases}$$

$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

$$m_a + m_b + m_c \leq 4R + r \text{ and } m_a m_b m_c \leq \frac{1}{2} \cdot R s^2 \text{ then}$$

$$\frac{R_m}{2r_m} \leq \frac{1}{2} \cdot \frac{R(4R+r)s^2}{9S^2} = \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2}$$

$$\text{We must show that: } \left(\frac{R}{2r}\right)^2 \geq \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2} \Leftrightarrow$$

$$\frac{R^2}{4r^2} \geq \frac{1}{2} \cdot \frac{R(4R+r)}{9r^2} \Leftrightarrow \frac{R}{2} \geq \frac{4R+r}{9} \Leftrightarrow 9R \geq 8R + 2r \Leftrightarrow R \geq 2r \text{ (Euler)}$$

$$\text{Finally, } \left(\frac{R}{2r}\right)^2 \geq \frac{R_m}{2r_m} \Rightarrow \frac{R}{2r} \geq \sqrt{\frac{R_m}{2r_m}}; (1)$$

$$\sum_{cyc} \frac{m_a}{m_b + m_c} \leq \frac{1}{2} + \sqrt{\frac{R_m}{2r_m}}; (2)$$

From (1),(2) we have:

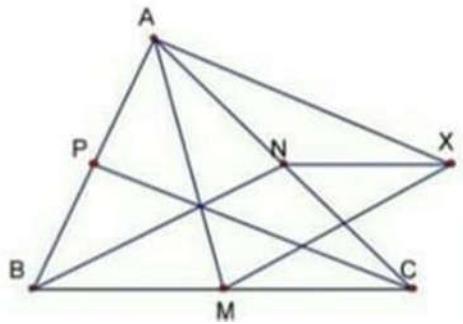
$$\frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} - \frac{1}{2} \leq \frac{R}{2r}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $MB = MC; NA = NC; PA = PB$

$BMXN$ — parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A; D \equiv M; E \equiv X \Rightarrow S_{\Delta DEF} = \frac{3}{4} S_{\Delta ABC}$



$$\begin{aligned} \text{Let: } \Omega &= \frac{m_a}{m_b + m_c} + \frac{m_b}{m_c + m_a} + \frac{m_c}{m_a + m_b} \stackrel{BCS}{\leq} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{(m_a + m_b)^2}} \leq \\ &\leq \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{4m_a m_b}} = \frac{1}{2} \sqrt{\sum_{cyc} m_a^2} \cdot \sqrt{\sum_{cyc} \frac{1}{m_a m_b}} \leq \\ &\leq \sum_{cyc} \frac{1}{m_a m_b} \stackrel{\Sigma_{cyc} \frac{1}{m_a m_b} \leq \frac{\sqrt{3}}{S}}{\leq} \frac{1}{2} \sqrt{\frac{3}{4}(a^2 + b^2 + c^2)} \cdot \sqrt{\frac{\sqrt{3}}{S}} \stackrel{(1)}{\leq} \frac{1}{2} + \frac{R}{2r} \end{aligned}$$

$$(1) \Leftrightarrow \frac{3}{4}(a^2 + b^2 + c^2) \cdot \frac{\sqrt{3}}{sr} \leq \left(1 + \frac{R}{r}\right)^2 \Leftrightarrow \frac{3\sqrt{3}(s^2 - 4Rr - r^2)}{2s} \leq \frac{(R+r)^2}{r} \Leftrightarrow$$

$$2s(R+r)^2 \geq 3\sqrt{3}r(s^2 - 4Rr - r^2) \Leftrightarrow 4s^2(R+r)^4 \geq 27r^2(s^2 - 4Rr - r^2)^2$$

But: $16Rr - 5r^2 \leq s^2 \leq 4R^2 - 4Rr + 3r^2$ (Gerretsen)

$$4s^2(R+r)^4 \geq 4(16Rr - 5r^2)(R+r)^4; (2)$$

$$\begin{aligned} 27r^2(s^2 - 4Rr - r^2)^2 &\leq 27r^2(4R^2 - 4Rr + 3r^2 - 4Rr - r^2)^2 = \\ &= 27r^2(4R^2 + 2r^2)^2 = 27 \cdot 4r^2(2R^2 + r^2)^2; (3) \end{aligned}$$

From (2), (3) we need to prove:



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$$4(16Rr - 5r^2)(R + r)^4 \geq 27 \cdot 4r^2(2R^2 + r^2)^2 \Leftrightarrow$$

$$(16Rr - 5r)(R + r)^4 \geq 27r(2R^2 + r^2)^2; \left(t = \frac{R}{r} \geq 2\right) \Leftrightarrow$$

$$(16t - 5)(t + 1)^4 \geq 27(2t^2 + 1)^2 \Leftrightarrow (16t - 5)(t + 1)^4 - 27(2t^2 + 1)^2 \geq 0 \Leftrightarrow$$

$$16t^5 - 49t^4 + 76t^3 - 64t^2 - 4t - 32 \geq 0 \Leftrightarrow$$

$$(t - 2)(16t^4 - 17t^3 + 42t^2 + 10t + 16) \geq 0$$

Which is true because $t \geq 2 \Rightarrow t - 2 \geq 0$

$$\begin{aligned} 16t^4 - 17t^3 + 42t^2 + 10t + 16 &= t^3(16t - 17) + 42t^2 + 10t + 16 \stackrel{t \geq 2}{\geq} \\ &\geq 2^3(16 \cdot 2 - 17) + 52 \cdot 2^2 + 10 \cdot 2 + 16 = 324 > 0. \text{Proved.} \end{aligned}$$

Lastly, we prove that:

$$\sum_{cyc} \frac{1}{m_a m_b} \leq \frac{\sqrt{3}}{S} = \frac{\sqrt{3}}{\frac{4}{3} S_{\Delta DEF}} = \frac{3\sqrt{3}}{4 S_{\Delta DEF}}$$

In fact, for any ΔABC we have:

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{a+b+c}{abc} = \frac{2s}{4Rs} \stackrel{R \geq \frac{2}{3\sqrt{3}}s}{\leq} \frac{1}{2} \cdot \frac{3\sqrt{3}}{2sr} = \frac{3\sqrt{3}}{4S}$$

Apply to $\Delta DEF \Rightarrow$ proved.

1765. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$|\mathbf{r}_a - \mathbf{r}_b| |\mathbf{r}_b - \mathbf{r}_c| |\mathbf{r}_c - \mathbf{r}_a| \geq (\mathbf{n}_a - \mathbf{g}_a)(\mathbf{n}_b - \mathbf{g}_b)(\mathbf{n}_c - \mathbf{g}_c)$$

Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Triangle inequality } \Rightarrow g_a &\leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abccos \frac{A}{2}}{a(b+c)} \\ &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rrscos \frac{A}{2}}{4R(b+c)\sin \frac{A}{2}\cos \frac{A}{2}} \end{aligned}$$



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$$\begin{aligned}
 & \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a+b+c}{(b+c)\sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a}{(b+c)\sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c)\sin \frac{A}{2} \stackrel{?}{\leq} a \\
 & \Leftrightarrow 4R\cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leq} 4R\sin \frac{A}{2} \cos \frac{A}{2} \\
 & \Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true} \therefore g_a \leq w_a \stackrel{(1)}{\leq} m_a
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, Stewart's theorem } & \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\
 & \Rightarrow 4an_a^2 - 4am_a^2 \\
 & = 4b^2(s-c) + 4c^2(s-b) - 4a(s-b)(s-c) - a(2b^2 + 2c^2 - a^2) \\
 & = 2b^2(a+b-c) + 2c^2(c+a-b) - a(c+a-b)(a+b-c) - a(2b^2 + 2c^2 - a^2) \\
 & = a(b-c)^2 + 2b^3 + 2c^3 - 2b^2c - 2bc^2 \\
 & = a(b-c)^2 + 2(b+c)(b^2 + c^2 - bc) - 2bc(b+c) = a(b-c)^2 + 2(b+c)(b-c)^2 \\
 & = (a+2b+2c)(b-c)^2 \geq 0 \Rightarrow 4an_a^2 \geq 4am_a^2 \\
 & \Rightarrow n_a \stackrel{(2)}{\leq} m_a \therefore (1), (2) \Rightarrow n_a \stackrel{(3)}{\leq} g_a \\
 \text{Again, Stewart's theorem } & \Rightarrow b^2(s-c) + c^2(s-b) \\
 & = an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 & = ag_a^2 + a(s-b)(s-c) \\
 & \therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s-a)^2 \\
 & \Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \stackrel{(a)}{\leq} a^2 s^2 (s-a)^2
 \end{aligned}$$

Let $s-a=x$, $s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z$, $b=z+x$ and $c=x+y$

Using these substitutions, (a)

$$\begin{aligned}
 & \Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\} \{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2 \\
 & \Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \stackrel{(4)}{\geq} 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}
 \end{aligned}$$

$\Rightarrow (a)$ is true $\Rightarrow n_a g_a \stackrel{?}{\leq} s(s-a)$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b)$

$$\begin{aligned}
 & = an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 & = ag_a^2 + a(s-b)(s-c)
 \end{aligned}$$

Adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$\begin{aligned}
 & = an_a^2 + ag_a^2 + 2a(s-b)(s-c) \\
 & \Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2) \\
 & = 2(n_a^2 + g_a^2) + a^2 - (b-c)^2 \\
 & \Rightarrow 2(b^2 + c^2) - a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b-c)^2 = 2(n_a^2 + g_a^2) \\
 & \Rightarrow 4m_a^2 + (b-c)^2 + 4r_b r_c = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b-c)^2 + 4s(s-a) \\
 & = 2(n_a^2 + g_a^2) + 4s(s-a)
 \end{aligned}$$



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$$\begin{aligned}
 & \Rightarrow 4m_a^2 + 4m_a^2 = 2(n_a^2 + g_a^2) + 4s(s-a) \stackrel{(5)}{\Rightarrow} n_a^2 + g_a^2 \triangleq 4m_a^2 - 2s(s-a) \\
 & \Rightarrow b^2 + c^2 - 2bc - (n_a^2 + g_a^2 - 2n_a g_a) \\
 & \stackrel{\text{by (5)}}{\triangleq} b^2 + c^2 - 2bc + 2n_a g_a - (4m_a^2 - 2s(s-a)) \stackrel{\text{by (4)}}{\geq} b^2 + c^2 - 2bc + 2s(s-a) \\
 & - 4m_a^2 + 2s(s-a) \\
 & = b^2 + c^2 - 2bc + (b+c+a)(b+c-a) - (2b^2 + 2c^2 - a^2) \\
 & = b^2 + c^2 - 2bc + (b+c)^2 - a^2 - (2b^2 + 2c^2 - a^2) = 0 \\
 & \stackrel{\text{by (3)}}{\Rightarrow} (b-c)^2 \geq (n_a - g_a)^2 \stackrel{(6)}{\Rightarrow} |b-c| \geq n_a - g_a \text{ and analogs} \\
 & \Rightarrow |a-b||b-c||c-a| \stackrel{(6)}{\geq} (n_a - g_a)(n_b - g_b)(n_c - g_c) \\
 \text{Now, } |r_a - r_b||r_b - r_c||r_c - r_a| &= \left| \frac{rs}{s-a} - \frac{rs}{s-b} \right| \left| \frac{rs}{s-b} - \frac{rs}{s-c} \right| \left| \frac{rs}{s-c} - \frac{rs}{s-a} \right| \\
 &= \frac{r^3 s^3}{\{\prod(s-a)\}^2} |a-b||b-c||c-a| \\
 &= \frac{s}{r} |a-b||b-c||c-a| \stackrel{\text{Mitrinovic}}{\geq} 3\sqrt{3} |a-b||b-c||c-a| \\
 &\geq |a-b||b-c||c-a| \stackrel{(6)}{\geq} (n_a - g_a)(n_b - g_b)(n_c - g_c) \text{ (Proved)}
 \end{aligned}$$

1766. 1) In any ΔABC the following relationship holds:

$$\frac{a^2 b^2 + b^2 c^2 + c^2 a^2}{abc(a+b+c)} \leq \frac{R}{2r}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Marin Chirciu-Romania

Using known identity in triangle: $abc = 4Rrs$; $a+b+c = 2s$

$$\sum a^2 b^2 = s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2$$

The inequality becomes as:

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2}{4Rrs \cdot 2s} \leq \frac{R}{2r} \Leftrightarrow s^2(4R^2 + 8Rr - 2r^2 - s^2) \geq r^2(4R + r)^2$$

which result from $\frac{r(4R+r)^2}{R+r} \leq 16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ (*Gerretsen*) and

$$4R^2 + 8Rr - 2r^2 - s^2 > 0 \text{ see } s^2 \leq 4R^2 + 4Rr + 3r^2$$

It remains to prove that:



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$$\frac{r(4R+r)^2}{R+r}(4R^2 + 8Rr - 2r^2 - 4R^2 - 4Rr - 3r^2) \geq r^2(4R+r)^2 \Leftrightarrow R \geq 2r(\text{Euler})$$

Equality holds if and only if triangle is equilateral.

Remark. The inequality it can be developed.

2) In any ΔABC the following relationship holds:

$$\frac{a^2b^2 + b^2c^2 + c^2a^2}{abc(a+b+c)} + n \leq (n+1)\frac{R}{2r}, n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using known identity in triangle: $abc = 4Rrs; a + b + c = 2s$

$$\sum a^2b^2 = s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2$$

The inequality becomes as:

$$\frac{s^4 + s^2(2r^2 - 8Rr) + r^2(4R + r)^2}{4Rrs \cdot 2s} + n \leq (n+1)\frac{R}{2r} \Leftrightarrow \\ s^2(4(n+1)R^2 + 8(1-n)Rr - 2r^2 - s^2) \geq r^2(4R + r)^2$$

which result from $\frac{r(4R+r)^2}{R+r} \leq 16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ (*Gerretsen*) and
 $4(n+1)R^2 + 8(1-n)Rr - 2r^2 - s^2 > 0$ see $s^2 \leq 4R^2 + 4Rr + 3r^2$ (*Gerretsen*)

It remains to prove that:

$$\frac{r(4R+r)^2}{R+r}(4(n+1)R^2 + 8(1-n)Rr - 2r^2 - 4R^2 - 4Rr - 3r^2) \geq r^2(4R + r)^2 \Leftrightarrow \\ 4nR^2 + (3 - 8n)Rr - 6r^2 \geq 0 \Leftrightarrow (R - 2r)(4nR + 3r) \geq 0 \Leftrightarrow \\ R \geq 2r(\text{Euler})$$

Equality holds if and only if triangle is equilateral.

Note. For $n = 1$ we get the proposed problem by Adil Abdullayev-RMM-2/2020

1767. In acute ΔABC the following relationship holds:

$$1 + \sum_{cyc} \frac{m_a}{w_a} \leq \frac{2R}{r} \cdot \frac{h_a + h_b + h_c}{r_a + r_b + r_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Marin Chirciu-Romania

Lemma. In acute ΔABC the following relationship holds:



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$$\sum_{cyc} \frac{m_a}{w_a} \leq \frac{s^2 + r^2 - 2Rr}{4Rr}$$

Proof. We have: $s_a \leq w_a \Leftrightarrow \frac{2bc}{b^2+c^2} \cdot m_a \leq w_a \Leftrightarrow \frac{m_a}{w_a} \leq \frac{b^2+c^2}{2bc}$ (*Tsintsifas*). We get:

$$\sum_{cyc} \frac{m_a}{w_a} \leq \sum_{cyc} \frac{b^2 + c^2}{2bc}$$

Let's solve the proposed problem.

Using lemma and $\sum h_a = \frac{s^2 + r^2 + 4Rr}{2R}$; $\sum r_a = 4R + r$ it suffices to prove that:

$$1 + \frac{s^2 + r^2 - 2Rr}{4Rr} \leq \frac{2R}{r} \cdot \frac{\frac{s^2 + r^2 + 4Rr}{2R}}{4R+r} \text{ which result from } s^2 \leq 4R^2 + 4Rr + 3r^2 (\text{Gerretsen})$$

Remain to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr + r^2 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0 \Leftrightarrow \\ (R - 2r)(2R + r) \geq 0 \text{ true from } R \geq 2r (\text{Euler}).$$

Equality holds if and only if the triangle is equilateral.

Remark. The inequality it can be developed.

In acute ΔABC the following relationship holds:

$$1 + \sum_{cyc} \frac{m_a}{w_a} \leq \frac{2R}{r} \cdot \frac{r_a + r_b + r_c}{h_a + h_b + h_c}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using lemma and $\sum h_a = \frac{s^2 + r^2 + 4Rr}{2R}$; $\sum r_a = 4R + r$ it suffices to prove that:

$$1 + \frac{s^2 + r^2 - 2Rr}{4Rr} \leq \frac{2R}{r} \cdot \frac{4R + r}{\frac{s^2 + r^2 + 4Rr}{2R}} \Leftrightarrow \frac{s^2 + r^2 + 2Rr}{4Rr} \leq \frac{4R^2(4R + r)}{r(s^2 + r^2 + 4Rr)} \Leftrightarrow$$

$(s^2 + r^2 + 2Rr)(s^2 + r^2 + 4Rr) \leq 16R^3(4R + r) \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2$ which result from $s^2 \leq 4R^2 + 4Rr + 3r^2$ (*Gerretsen*).

Remain to prove that:

$$(4R^2 + 4Rr + 3r^2 + r^2 + 2Rr)(4R^2 + 4Rr + 3r^2 + r^2 + 4Rr) \leq 16R^3(4R + r) \Leftrightarrow$$



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$$(4R^2 + 6Rr + 4r^2)(4R^2 + 8Rr + 4r^2) \leq 16R^3(4R + r) \Leftrightarrow$$

$$(2R^2 + 3Rr + 2r^2)(R^2 + 2Rr + r^2) \leq R^3(4R + r) \Leftrightarrow$$

$$6R^4 - 5R^3r - 10R^2r^2 - 7Rr^3 - 2r^4 \geq 0 \Leftrightarrow$$

$$(R - 2r)(6R^3 + 7R^2r + 4Rr^2 + r^3) \geq 0 \text{ true from } R \geq 2r(\text{Euler})$$

Equality holds if and only if the triangle is equilateral.

Remark. The inequality it can be developed.

In acute ΔABC the following relationship holds:

$$1 + \sum_{cyc} \frac{m_a}{w_a} \leq \frac{2R}{r} \cdot \frac{r_a + r_b + r_c}{h_a + h_b + h_c} \leq \frac{2R}{r} \cdot \frac{h_a + h_b + h_c}{r_a + r_b + r_c}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Applying the first inequality and we prove that:

$$\frac{2R}{r} \cdot \frac{r_a + r_b + r_c}{h_a + h_b + h_c} \leq \frac{2R}{r} \cdot \frac{h_a + h_b + h_c}{r_a + r_b + r_c} \Leftrightarrow \left(\sum h_a \right)^2 \leq \left(\sum r_a \right)^2 \Leftrightarrow \sum h_a \leq \sum r_a$$

it is suffices to prove that: $\frac{s^2 + r^2 + 4Rr}{2R} \leq 4R + r \Leftrightarrow s^2 \leq 8R^2 - 2Rr - r^2$ which result

from $s^2 \leq 4R^2 + 4Rr + 3r^2$ (*Gerretsen*).

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2 \Leftrightarrow (r - 2r)(2R + r) \geq 0 \text{ true from}$$

$R \geq 2r$ (*Euler*).

Equality holds if and only if the triangle is equilateral.

1768. In any acute – angled ΔABC , holds:

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{4abc}{a^3 + b^3 + c^3 + abc}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Proof : } & \cos(A - B)\cos(B - C)\cos(C - A) \\ &= \left(2\cos^2 \frac{A - B}{2} - 1 \right) \left(2\cos^2 \frac{B - C}{2} - 1 \right) \left(2\cos^2 \frac{C - A}{2} - 1 \right) \end{aligned}$$



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$$\stackrel{(a)}{=} 8 \prod \cos^2 \frac{B-C}{2} - 4 \left(\prod \cos^2 \frac{B-C}{2} \right) \sum \sec^2 \frac{B-C}{2} + 2 \sum \cos^2 \frac{B-C}{2} - 1$$

$$\text{Now, } \sum \cos^2 \frac{B-C}{2} = \sum \frac{(b+c)^2 \sin^2 \frac{A}{2}}{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{16R^2 s} \sum \frac{bc(b+c)^2}{s-a}$$

$$= \frac{1}{16R^2 s} \sum \frac{bc(s+s-a)^2}{s-a}$$

$$= \frac{1}{16R^2 s} \sum \left\{ \frac{bcs^2}{s-a} + 2sbc + bc(s-a) \right\} = \frac{1}{16R^2 s} \left\{ s^3 \sum \sec^2 \frac{A}{2} + 3s \sum ab - 3abc \right\}$$

$$= \frac{1}{16R^2 s} \left[s^3 \left\{ \frac{s^2 + (4R+r)^2}{s^2} \right\} + 3s(s^2 + 4Rr + r^2) - 12Rrs \right] = \frac{4s^2 + (4R+r)^2 + 3r^2}{16R^2}$$

$$\Rightarrow \sum \cos^2 \frac{B-C}{2} \stackrel{(1)}{=} \frac{4s^2 + (4R+r)^2 + 3r^2}{16R^2}$$

$$\text{Again, } \sum \sec^2 \frac{B-C}{2} = \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{(b+c)^2 \sin^2 \frac{A}{2}} = \sum \frac{16R^2 s(s-a)a}{4Rrs(b+c)^2}$$

$$= \frac{2R}{r} \sum \frac{a(b+c-a)}{(b+c)^2} \stackrel{(2)}{=} \frac{2R}{r} \left\{ \sum \frac{a}{b+c} - \sum \frac{a^2}{(b+c)^2} \right\}$$

$$\text{Now, } \sum \frac{a}{b+c} = \frac{\sum a(c+a)(a+b)}{\prod(b+c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2 + 2Rr + r^2)}$$

$$= \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \stackrel{(3)}{=} \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2}$$

$$\text{and, } \sum \frac{a^2}{(b+c)^2} = \sum \frac{(2s-(b+c))^2}{(b+c)^2}$$

$$= \sum \frac{4s^2 - 4s(b+c) + (b+c)^2}{(b+c)^2} \stackrel{(i)}{=} 4s^2 \left[\frac{\sum \{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2} \right]$$

$$- 4s \left[\frac{\sum (c+a)(a+b)}{\prod(b+c)} \right] + 3$$

$$\sum \{(c+a)^2(a+b)^2\} = \sum (\sum ab + a^2)^2 = \sum \left\{ (\sum ab)^2 + 2a^2 \sum ab + a^4 \right\}$$

$$= 3 (\sum ab)^2 + 2 (\sum ab) (\sum a^2) + (\sum a^2)^2 - 2 \sum a^2 b^2$$



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$$\begin{aligned}
 &= \left(\sum ab \right)^2 + 2 \left(\sum ab \right) (\sum a^2) + (\sum a^2)^2 + 2 \sum a^2 b^2 + 4abc(2s) - 2 \sum a^2 b^2 \\
 &= \left(\sum ab + \sum a^2 \right)^2 + 32Rrs^2 \\
 &\quad = (3s^2 - 4Rr - r^2)^2 + 32Rrs^2
 \end{aligned}$$

$$\therefore \sum \{(c+a)^2(a+b)^2\} \stackrel{(ii)}{\cong} (3s^2 - 4Rr - r^2)^2 + 32Rrs^2$$

$$\text{Again, } \sum (c+a)(a+b) = \sum \left(\sum ab + a^2 \right) = 3 \sum ab + \sum a^2$$

$$= \sum a^2 + 2 \sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2$$

$$\therefore \sum (c+a)(a+b) \stackrel{(iii)}{\cong} 5s^2 + 4Rr + r^2$$

$$\because \prod (b+c) = s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum \frac{a^2}{(b+c)^2}$$

$$= \frac{4s^2 \{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2\}}{4s^2(s^2 + 2Rr + r^2)^2} - \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} + 3$$

$$= \frac{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2 - 2(5s^2 + 4Rr + r^2)(s^2 + 2Rr + r^2) + 3(s^2 + 2Rr + r^2)^2}{(s^2 + 2Rr + r^2)^2}$$

$$= \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}$$

$$\Rightarrow \sum \frac{a^2}{(b+c)^2} \stackrel{(4)}{\cong} \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}$$

$$(2), (3), (4) \Rightarrow \sum \sec^2 \frac{B-C}{2}$$

$$= \frac{2R}{r} \left\{ \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} \right.$$

$$\left. - \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \right\}$$

$$\stackrel{(5)}{\cong} \frac{2R}{r} \left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right]$$



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$$\begin{aligned}
 & \text{Also, } 8 \prod \cos^2 \frac{B-C}{2} = 8 \prod \frac{(b+c)^2 \sin^2 \frac{A}{2}}{a^2} \\
 &= 8 \left\{ \frac{4s^2(s^2 + 2Rr + r^2)^2}{16R^2r^2s^2} \right\} \left(\frac{r^2}{16R^2} \right) \stackrel{(6)}{\cong} \frac{(s^2 + 2Rr + r^2)^2}{8R^4} \\
 & \quad (a), (1), (5), (8) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) = \frac{(s^2 + 2Rr + r^2)^2}{8R^4} \\
 & - \left\{ \frac{(s^2 + 2Rr + r^2)^2}{16R^4} \right\} \frac{2R}{r} \left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right] \\
 & \quad + \frac{4s^2 + (4R + r)^2 + 3r^2}{8R^2} - 1 \\
 & \Rightarrow \cos(A-B)\cos(B-C)\cos(C \\
 & - A) \stackrel{(m)}{\cong} \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} \\
 & \quad (\text{where } \sigma = (2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) \\
 & \quad - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}) \\
 & \text{Now, } \frac{4abc}{a^3 + b^3 + c^3 + abc} \geq \frac{2r}{R} \Leftrightarrow \frac{16Rrs}{2s(s^2 - 6Rr - 3r^2) + 4Rrs} \geq \frac{2r}{R} \Leftrightarrow 4R^2 \\
 & \geq s^2 - 4Rr - 3r^2 \rightarrow \text{true(Gerretsen)} \\
 & \therefore \frac{4abc}{a^3 + b^3 + c^3 + abc} \stackrel{(n)}{\geq} \frac{2r}{R} \\
 & \therefore (m), (n) \Rightarrow \text{it suffices to prove} \\
 & : \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} - \frac{2r}{R} \\
 & \leq 0 \\
 & \Leftrightarrow \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r - 16R^3r^2}{8R^4r} \leq 0 \\
 & \Leftrightarrow s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4 \stackrel{(x)}{\leq} 0 \\
 & \because \Delta ABC \text{ is acute - angled, Walker and Gerretsen} \\
 & \Rightarrow (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2) \leq 0 \\
 & \Rightarrow \text{in order to prove (x),} \\
 & \text{it suffices to prove : } s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4
 \end{aligned}$$



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$$\leq (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2)$$

$$\Leftrightarrow (R + 2r)s^2 \stackrel{(y)}{\leq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$$

$$\text{Now, } (R + 2r)s^2 \stackrel{\text{Gerretsen}}{\geq} (R + 2r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$$

$$\Leftrightarrow 4t^3 - 5t^2 - 4t - 4 \stackrel{?}{\geq} 0 \quad (\text{where } t = \frac{R}{r})$$

$$\Leftrightarrow (t - 2)(4t^2 + 3t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (y) \Rightarrow (x) \text{ is true}$$

$$\therefore \cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{4abc}{a^3 + b^3 + c^3 + abc} \quad (\text{Proved})$$

1769. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \frac{h_a}{g_a + s - a} \leq \frac{1}{2} \left(\frac{g_a + g_b + g_c}{r} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \sqrt{4 - \frac{R}{r}} \right)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof : } R(4R + r)^2 \stackrel{(1)}{\leq} (4R - 2r)s^2$$

$$\text{Now, RHS of (1)} \stackrel{\text{Rouche}}{\geq} (4R - 2r)$$

$$- 2r) \left(2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \right) \stackrel{?}{\leq} R(4R + r)^2$$

$$\Leftrightarrow R(4R + r)^2 - (2R^2 + 10Rr - r^2)(4R - 2r) \stackrel{?}{\geq} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow (R - 2r)(8R^2 - 12Rr + r^2) \stackrel{(2)}{\leq} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr}$$

$\because R - 2r \stackrel{\text{Euler}}{\geq} \therefore$ in order to prove (2), it suffices to prove : $8R^2 - 12Rr + r^2$

$$> 2(4R - 2r)\sqrt{R^2 - 2Rr}$$



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$$\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 4(R^2 - 2Rr)(4R - 2r)^2 > 0 \Leftrightarrow r^2(4R + r)^2 > 0 \rightarrow \text{true}$$

$$\Rightarrow (2) \Rightarrow (1) \text{ is true} \therefore \frac{4R - 2r}{R} \leq \frac{(4R + r)^2}{s^2}$$

$$\Rightarrow \left(\sum \frac{a}{b}\right)^2 \left(\frac{4R - 2r}{R}\right) \stackrel{(3)}{\leq} \frac{(4R + r)^2}{s^2} \left(\sum \frac{a}{b}\right)^2 \therefore (3)$$

⇒ in order to prove $\left(\sum \frac{a}{b}\right)^2 \left(\frac{4R - 2r}{R}\right) \leq \frac{s^2}{r^2}$, **it suffices to prove :**

$$\frac{(4R + r)^2}{s^2} \left(\sum \frac{a}{b}\right)^2 \leq \frac{s^2}{r^2} \Leftrightarrow \frac{s^2}{r^2} \stackrel{(4)}{\geq} \left(\sum \frac{a}{b}\right) \left(1 + \frac{4R}{r}\right)$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and c

$$= x + y$$

$$\text{Now, } \frac{s^2}{r^2} = \frac{s^4}{\Delta^2} = \frac{s^4}{s(s-a)(s-b)(s-c)} \stackrel{(m)}{\cong} \frac{(\sum x)^3}{xyz} \text{ and } 1 + \frac{4R}{r}$$

$$= 1 + \frac{4abc}{4s(s-a)(s-b)(s-c)} = 1 + \frac{\prod(y+z)}{xyz} \stackrel{(n)}{\cong} \frac{xyz + \prod(y+z)}{xyz}$$

$$\text{Also, } \sum \frac{a}{b} = \sum \frac{y+z}{z+x} \stackrel{(p)}{\cong} \frac{\sum(x+y)(y+z)^2}{\prod(y+z)} \therefore (m), (n), (p) \Rightarrow (4) \Leftrightarrow \frac{(\sum x)^3}{xyz}$$

$$\geq \left[\frac{xyz + \prod(y+z)}{xyz} \right] \left[\frac{\sum(x+y)(y+z)^2}{\prod(y+z)} \right]$$

$$\Leftrightarrow \{\prod(y+z)\}(\sum x)^3 \geq \{xyz + \prod(y+z)\} \sum(x+y)(y+z)^2$$

$$\Leftrightarrow \sum x^2 y^4 + \sum x^3 y^3 \stackrel{(5)}{\geq} xyz(\sum x^2 y) + 3x^2 y^2 z^2$$

Now, if $u, v, w > 0$, then

$$: v^3 + v^3 + u^3 \stackrel{A-G}{\geq} 3v^2 u, w^3 + w^3 + v^3 \stackrel{A-G}{\geq} 3w^2 v \text{ and } u^3 + u^3$$

$$+ w^3 \stackrel{A-G}{\geq} 3u^2 w \text{ and adding these three :}$$

$\sum u^3 \geq \sum uv^2$ and choosing $u = xy, v = yz$ and $w = zx$, we get

$$: \sum x^3 y^3 \stackrel{(a)}{\geq} xyz(\sum x^2 y) \text{ and } \sum x^2 y^4 \stackrel{(b)}{\geq} 3x^2 y^2 z^2$$



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$$(a) + (b) \Rightarrow (5) \Rightarrow (4) \text{ is true} \Rightarrow \left(\sum \frac{a}{b} \right)^2 \left(\frac{4R - 2r}{R} \right) \leq \frac{s^2}{r^2} \Rightarrow \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \sqrt{4 - \frac{2r}{R}} \stackrel{(6)}{\leq} \frac{s}{r}$$

$$\begin{aligned} & \text{Now, Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) \\ &= an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= ag_a^2 + a(s - b)(s - c) \end{aligned}$$

$$\begin{aligned} & \text{Adding the above two, we get : } (b^2 + c^2)(2s - b - c) \\ &= an_a^2 + ag_a^2 + 2a(s - b)(s - c) \\ &\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\ &= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\ &\Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\ &\Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 = 2(n_a^2 + g_a^2) \\ &\Rightarrow n_a^2 + g_a^2 \stackrel{(i)}{\cong} (b - c)^2 + 2s(s - a) \end{aligned}$$

$$\begin{aligned} & \text{Also, Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= an_a^2 + a(as - s^2) \\ &\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\ &= as^2 - \frac{4as(s - b)(s - c)}{a} \stackrel{(ii)}{\cong} s^2 - \frac{4s(s - b)(s - c)}{a} \therefore (i), (ii) \Rightarrow g_a^2 \\ &= (b - c)^2 + 2s(s - a) - s^2 + \frac{4s(s - b)(s - c)}{a} \\ &= s^2 - 2sa + a^2 + (b - c)^2 - a^2 + \frac{4s(s - b)(s - c)}{a} \\ &= (s - a)^2 + (b - c + a)(b - c - a) + \frac{4s(s - b)(s - c)}{a} \\ &= (s - a)^2 - 4(s - b)(s - c) + \frac{4s(s - b)(s - c)}{a} = (s - a)^2 + 4(s - b)(s - c) \left(\frac{s}{a} - 1 \right) \\ &= (s - a)^2 + \frac{4(s - a)(s - b)(s - c)}{a} \end{aligned}$$



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$$\begin{aligned}
 & \Rightarrow g_a^2 = (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a} \Rightarrow (g_a + s - a)(g_a - s + a) \\
 & = \frac{4(s-a)(s-b)(s-c)}{a} = 2r \left(\frac{2rs}{a} \right) = 2rh_a \\
 \Rightarrow \frac{h_a}{g_a + s - a} &= \frac{g_a - s + a}{2r} \text{ and analogs} \Rightarrow \sum \frac{h_a}{g_a + s - a} = \frac{\sum(g_a - s + a)}{2r} \\
 &= \frac{\sum g_a}{2r} - \frac{s}{2r} \stackrel{\text{by (6)}}{\geq} \frac{g_a + g_b + g_c}{2r} - \frac{1}{2} \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \sqrt{4 - \frac{2r}{R}} \\
 \therefore \sum \frac{h_a}{g_a + s - a} &\leq \frac{1}{2} \left(\frac{g_a + g_b + g_c}{r} - \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \right) \sqrt{4 - \frac{2r}{R}} \right) \text{ (Proved)}
 \end{aligned}$$

1770. 1) In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos B + \cos C}{r_a} \leq \frac{s^2}{m_a m_b m_c}$$

Proposed by Bogdan Fuștei-Romania

Solution by Marin Chirciu-Romania

2) Lemma. In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos B + \cos C}{r_a} = \frac{2}{R}$$

Proof.

$$\begin{aligned}
 \sum_{cyc} \frac{\cos B + \cos C}{r_a} &= \sum_{cyc} \frac{\frac{a^2 + c^2 - b^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab}}{\frac{s}{s-a}} = \\
 &= \frac{1}{2S} \sum_{cyc} \frac{b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2)}{abc} (s-a) = \frac{2}{R}
 \end{aligned}$$

Let's solve the proposed problem.

Using Lemma, the inequality can be written as:



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$\frac{2}{R} \leq \frac{s^2}{m_a m_b m_c} \Leftrightarrow m_a m_b m_c \leq \frac{Rs^2}{2}$ which result from $m_a \leq R(1 + \cos A)$ true in acute

triangle $m_a \leq R \cdot 2\cos^2 \frac{A}{2} \Leftrightarrow m_a \leq 2R\cos^2 \frac{A}{2}$ implies that

$$\prod_{cyc} m_a \leq \prod_{cyc} 2R\cos^2 \frac{A}{2} = 8R^3 \prod_{cyc} \cos^2 \frac{A}{2} = 8R^3 \cdot \left(\frac{s}{4R}\right)^2 = \frac{Rs^2}{2}.$$

Equality holds if and only if the triangle is equilateral.

Remark. In same class of problems.

3) In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos B + \cos C}{h_a} \geq \frac{s^2}{m_a m_b m_c}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

4) Lemma. In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos B + \cos C}{h_a} = \frac{1}{r}$$

Proof.

$$\begin{aligned} \sum_{cyc} \frac{\cos B + \cos C}{h_a} &= \sum_{cyc} \frac{\frac{a^2 + c^2 - b^2}{2ac} + \frac{a^2 + b^2 - c^2}{2ab}}{\frac{2S}{a}} = \\ &= \frac{1}{4S} \sum_{cyc} \frac{b(a^2 + c^2 - b^2) + c(a^2 + b^2 - c^2)}{bc} = \frac{1}{r} \end{aligned}$$

Let's solve the proposed problem.

Using lemma the inequality can be written as:

$\frac{1}{r} \geq \frac{s^2}{m_a m_b m_c} \Leftrightarrow m_a m_b m_c \geq rs^2$ which result from $m_a \geq \sqrt{s(s-a)}$ therefore

$$\prod_{cyc} m_a \geq \prod_{cyc} \sqrt{s(s-a)} = sS = rs^2$$

Equality holds if and only if the triangle is equilateral.

Remark.



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For the sum $\sum_{cyc} \frac{\cos B + \cos C}{r_a} = \frac{2}{R}$ and $\sum_{cyc} \frac{\cos B + \cos C}{h_a} = \frac{1}{r}$ we have:

5) In any ΔABC the following relationship holds:

$$\sum_{cyc} \frac{\cos B + \cos C}{r_a} \leq \sum_{cyc} \frac{\cos B + \cos C}{h_a}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using: $\sum_{cyc} \frac{\cos B + \cos C}{r_a} = \frac{2}{R}$ and $\sum_{cyc} \frac{\cos B + \cos C}{h_a} = \frac{1}{r}$ we have:

$$\frac{2}{R} \leq \frac{1}{r} \Leftrightarrow R \geq 2r (\text{Euler})$$

6) In any ΔABC the following relationship holds:

$$\frac{4r}{R} \leq \sum_{cyc} \frac{\cos B + \cos C}{r_a} \leq \frac{s^2}{m_a m_b m_c} \leq \sum_{cyc} \frac{\cos B + \cos C}{h_a} \leq \frac{R}{2r^2}$$

Proposed by Marin Chirciu-Romania

Solution by proposer

Using inequalities (1),(3) and $R \geq 2r$ (Euler)

1771. In ΔABC the following relationship holds:

$$\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{(m_a + m_b + m_c)^4}{(m_a + m_b + m_c)^4 + 2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)} \geq 2$$

Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} 16S^2 &= 2 \sum a^2 b^2 - \sum a^4 = 2 \sum a^2 b^2 - \left(\left(\sum a^2 \right)^2 - 2 \sum a^2 b^2 \right) \\ &= 4 \sum a^2 b^2 - \left(\sum a^2 \right)^2 \end{aligned}$$



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$$= 4 \cdot \frac{16}{9} \sum m_a^2 m_b^2 - \frac{16}{9} \left(\sum m_a^2 \right)^2 \Rightarrow 9S^2 = 4 \sum m_a^2 m_b^2 - \left(\sum m_a^2 \right)^2$$

$$\stackrel{(1)}{\Rightarrow} 9S^2 \geq 2 \sum m_a^2 m_b^2 - \sum m_a^4$$

Now, $\frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{(m_a + m_b + m_c)^4}{(m_a + m_b + m_c)^4 + 2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)} \geq 2$

$$\Leftrightarrow \frac{\sum a^2 - \sum ab}{\sum ab} + \frac{2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)}{(m_a + m_b + m_c)^4 + 2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)} \geq 0$$

$$\stackrel{\text{by (1)}}{\Leftrightarrow} \frac{\sum a^2 - \sum ab}{\sum ab} + \frac{2(m_a m_b m_c (m_a + m_b + m_c) - 2 \sum m_a^2 m_b^2 + \sum m_a^4)}{(m_a + m_b + m_c)^4 + 2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)} \stackrel{(i)}{\geq} 0$$

Now, Schur $\Rightarrow \sum m_a^4 + m_a m_b m_c (m_a + m_b + m_c)$

$$\geq \sum_{\text{cyc}} m_a^3 m_b + \sum_{\text{cyc}} m_b^3 m_a \stackrel{\text{A-G}}{\geq} 2 \sum m_a^2 m_b^2$$

$$\Rightarrow m_a m_b m_c (m_a + m_b + m_c) - 2 \sum m_a^2 m_b^2 + \sum m_a^4 \stackrel{(2)}{\geq} 0 \text{ and } \because \sum a^2 - \sum ab \stackrel{(3)}{\geq} 0$$

$\therefore (2), (3) \Rightarrow (i)$ is true

$$\therefore \frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{(m_a + m_b + m_c)^4}{(m_a + m_b + m_c)^4 + 2(m_a m_b m_c (m_a + m_b + m_c) - 9S^2)}$$

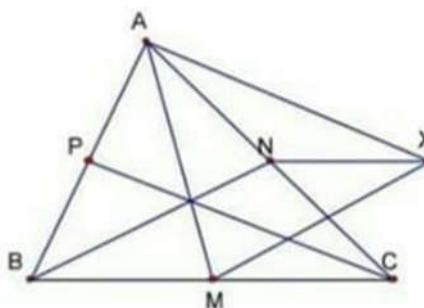
$$\geq 2 \text{ (Proved)}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

We have: $MA = MC, NA = NC, PA = PB$

$BMXN$ – parallelogram $\Rightarrow AM = m_a, BN = MX = m_b, CP = AX = m_c$

Choose: $F \equiv A, D \equiv M, E \equiv X \Rightarrow S_{\Delta DEF} = \frac{3}{4} S_{\Delta ABC}$





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$$m_a^2 + m_b^2 + m_c^2 = \frac{3}{4}(a^2 + b^2 + c^2) \Rightarrow a^2 + b^2 + c^2 = \frac{4}{3}(m_a^2 + m_b^2 + m_c^2)$$

$$m'_a = \frac{3}{4}a \Rightarrow a = \frac{4}{3}m'_a = \frac{2}{3}\sqrt{2(m_b^2 + m_c^2) - m_a^2} \text{ (and analogs)}$$

$$\frac{m_a m_b + m_b m_c + m_c m_a}{a^2 + b^2 + c^2} \stackrel{(1)}{\leq} \frac{5}{8} + \frac{r^2}{2R^2}$$

(By Yin Hua Yan in 2000, Research in Inequalities)

Then, Inequality becomes as:

$$\begin{aligned} & \frac{\frac{4}{3} \sum m_a^2}{\frac{4}{9} \sum \left(\sqrt{2(m_b^2 + m_c^2) - m_a^2} \right) \left(\sqrt{2(m_a^2 + m_c^2) - m_b^2} \right)} + \frac{(\sum m_a)^4}{(\sum m_a)^4 + 2 \left(m_a m_b m_c (m_a + m_b + m_c) - 9 \left(\frac{4}{3} S_{\Delta DEF} \right)^2 \right)} \geq 2 \\ & \frac{3 \sum m_a^2}{\sum \left(\sqrt{2(m_b^2 + m_c^2) - m_a^2} \right) \left(\sqrt{2(m_a^2 + m_c^2) - m_b^2} \right)} + \frac{(\sum m_a)^4}{(\sum m_a)^4 + 2 \left(m_a m_b m_c (m_a + m_b + m_c) - 16 S_{\Delta DEF}^2 \right)} \geq 2; (2) \end{aligned}$$

So, for any ΔABC we need to prove:

$$\begin{aligned} & \frac{3 \sum a^2}{\sum \left(\sqrt{2(b^2 + c^2) - a^2} \right) \left(\sqrt{2(a^2 + c^2) - b^2} \right)} + \frac{(\sum a)^4}{(\sum a)^4 + 2(abc(a+b+c) - 16 S_{\Delta ABC}^2)} \geq 2 \\ & \frac{3}{4} \cdot \frac{\sum a^2}{\sum m_a m_b} + \frac{(2s)^4}{(2s)^4 + 2(4Rrs \cdot 2s - 16s^2 r^2)} \geq 2 \Leftrightarrow \\ & \frac{3}{4} \cdot \frac{\sum a^2}{\sum m_a m_b} + \frac{s^2}{s^2 + Rr - 2r^2} \geq 2; (3) \end{aligned}$$

$$by(1) \Rightarrow \frac{\sum a^2}{\sum m_a m_b} \geq \frac{1}{\frac{5}{8} + \frac{r^2}{2R^2}} = \frac{8R^2}{5R^2 + 4r^2}; (4)$$

From (3), (4) we need to prove:

$$\frac{3}{4} \cdot \frac{8R^2}{5R^2 + 4r^2} + \frac{s^2}{s^2 + Rr - 2r^2} \geq 2 \Leftrightarrow$$

$$24R^2(s^2 + Rr - 2r^2) + 4s^2(5R^2 + 4r^2) \geq 2 \cdot 4(5R^2 + 4r^2)(s^2 + Rr - 2r^2) \Leftrightarrow$$

$$24R^2r^2 + 24rR^3 - 48R^2r^2 + 20R^2s^2 + 16s^2r^2 \geq$$

$$\geq 8(5R^2s^2 + 5rR^3 - 10R^2r^2 + 4r^2s^2 + 4Rr^3 - 8r^4) \Leftrightarrow$$

$$(R^2 - 4r^2)s^2 + 8R^2r^2 - 4rR^3 - 8Rr^3 + 16r^4 \geq 0 \Leftrightarrow$$

But: $R \geq 2r$ (Euler) $\Rightarrow R^2 \geq 4r^2$; $s^2 \geq 16Rr - 5r^2$ (Gerretsen) \Rightarrow

$$(R^2 - 4r^2)(16Rr - 5r^2) + 8R^2r^2 - 4rR^3 - 8Rr^3 + 16r^4 \geq 0$$



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$$(R^2 - 4r^2)(16R - 5r) + 8R^2r - 4R^3 - 8Rr^2 + 16r^3 \geq 0 \left(\therefore t = \frac{R}{r} \geq 2 \right) \Leftrightarrow$$

$$(t^2 - 4)(16t - 5) + 8t^2 - 4t^3 - 8t + 16 \geq 0 \Leftrightarrow$$

$$12t^3 + 3t^2 - 72t + 36 \geq 0 \Leftrightarrow 4t^3 + t^2 - 24t + 12 \geq 0 \Leftrightarrow (t-2)(4t^2 + 9t - 6) \geq 0$$

Which is clearly true by: $t \geq 2 \Rightarrow t-2 \geq 0$; $4t^2 + 9t - 6 \geq 4 \cdot 4 + 9 \cdot 2 - 6 > 0$

Proved.

1772. In ΔABC the following relationship holds:

$$\frac{1}{R + r \cos A} + \frac{1}{R + r \cos B} + \frac{1}{R + r \cos C} \geq \frac{6}{2R + r}$$

Proposed by Florentin Vișescu-Romania

Solution by Marian Ursărescu-Romania

From Bergstrom inequality we have:

$$\frac{1}{R + r \cos A} + \frac{1}{R + r \cos B} + \frac{1}{R + r \cos C} \geq \frac{9}{3R + r(\cos A + \cos B + \cos C)}$$

We must show that:

$$\frac{9}{3R + r(\cos A + \cos B + \cos C)} \geq \frac{6}{2R + r} \Leftrightarrow$$

$6R + 3r \geq 6R + 2r(\cos A + \cos B + \cos C) \Leftrightarrow \cos A + \cos B + \cos C \leq \frac{3}{2}$ true, because

$$\cos A + \cos B + \cos C \leq \frac{3}{2} \Leftrightarrow \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^2 + c^2 - b^2}{2ac} + \frac{b^2 + a^2 - c^2}{2ba} \leq \frac{3}{2} \Leftrightarrow$$

$ab(a+b) + ac(a+c) + bc(b+c) \leq a^3 + b^3 + c^3$ true by Schur's inequality.

1773. In ΔABC the following relationship holds:

$$\sum_{cyc} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)} \geq \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n}; n \in \mathbb{N}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\text{Let: } x = \cos \frac{A}{2}, y = \cos \frac{B}{2}, z = \cos \frac{C}{2}; (x, y, z > 0) \Rightarrow xyz = \frac{s}{4R}$$



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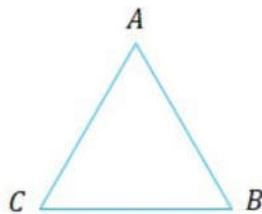
$$\begin{aligned}
 & \sqrt[3]{(y+z)(z+x)(x+y)} \stackrel{AM-GM}{\leq} \frac{y+z+z+x+x+y}{3} = \frac{2(x+y+z)}{3} = \\
 & = \frac{2}{3}(cosA + cosB + cosC) \stackrel{Jensen}{\leq} \frac{2}{3} \cdot 3 \cos\left(\frac{A+B+C}{3}\right) = \frac{2}{3} \cdot 3 \cdot \frac{\sqrt{3}}{2} = \sqrt{3} \Rightarrow \\
 & Lhs = \sum_{cyc} \frac{1}{x^{3n}(y+z)} \stackrel{AM-GM}{\geq} 3 \sqrt[3]{\frac{1}{(xyz)^{3n}(y+z)(z+x)(x+y)}} = \\
 & = \frac{3}{(xyz)^n \sqrt[3]{(x+y)(y+z)(z+x)}} \geq \frac{1}{(xyz)^n} \cdot \frac{3}{\sqrt{3}} = \frac{1}{\left(\frac{s}{4R}\right)^n} \cdot \sqrt{3} = \left(\frac{4R}{s}\right)^n \sqrt{3} \geq \\
 & \geq \left(\frac{2}{3\sqrt{3}}\right)^n \left(\frac{8}{3\sqrt{3}}\right)^n \sqrt{3} = \sqrt{3} \left(\frac{2}{\sqrt{3}}\right)^{3n}. \text{ Proved.}
 \end{aligned}$$

Solution 2 by Asmat Quatea-Kabul-Afghanistan

$$f(A, B, C) = \sum_{cyc} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)}$$

$$k = \text{either } \min(f) \text{ or } \max(f) \Rightarrow \text{when } A = B = C = \frac{\pi}{3}$$

$$k = \sum_{cyc} \frac{1}{\cos^{3n} \frac{\pi}{6} \left(\cos \frac{\pi}{6} + \cos \frac{\pi}{6} \right)} = \frac{3}{\sqrt{3} \left(\frac{\sqrt{3}}{2} \right)^{3n}} = \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n}$$



checking k:

suppose n = 1

*if A approach to π
then B and C approach to 0
and f approach to infinity
hence the k is minimum value of
cyclic trigonometric expression*

$$\sum_{cyc} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)} \geq \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n}; n \in \mathbb{N}$$



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Solution 3 by Soumava Chakraborty-Kolkata-India

Case 1 : $n = 0$ Then, LHS

$$\begin{aligned}
 &= \sum_{\text{cyc}} \frac{1}{\cos \frac{B}{2} + \cos \frac{C}{2}} \stackrel{\text{Bergstrom}}{\geq} \frac{9}{2 \sum \cos \frac{A}{2}} \stackrel{\text{Jensen}}{\geq} \frac{9}{2 \cdot 3 \frac{\sqrt{3}}{2}} \left(\because f(x) = \cos \frac{x}{2} \forall x \in (0, \pi) \text{ is concave} \right) = \sqrt{3} \\
 &= \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n} \left(\because n = 0 \right) \therefore \sum_{\text{cyc}} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)} \stackrel{(i)}{\geq} \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n}
 \end{aligned}$$

Case 2 : $n \in \mathbb{N} - \{0\}$ Let $g(x) = x^n \forall x > 0$ Then, $g''(x) = n(n-1)x^{n-2}$

$$\geq 0 \left(\because n \geq 1 \right) \Rightarrow x^n \text{ is convex} \Rightarrow x^n + y^n + z^n$$

$$\begin{aligned}
 &\stackrel{\text{Jensen}}{\geq} 3 \left(\frac{\sum x}{3} \right)^n \text{ and } \therefore \text{putting } x = \sec \frac{A}{2}, y = \sec \frac{B}{2}, z = \sec \frac{C}{2}, \text{ we have : } \sum_{\text{cyc}} \sec^n \frac{A}{2} \\
 &\geq 3 \left(\frac{\sum \sec \frac{A}{2}}{3} \right)^n \stackrel{\text{Jensen}}{\geq} 3 \left(\frac{3 \sec \frac{\pi}{6}}{3} \right)^n
 \end{aligned}$$

$$\left(\because f(x) = \sec \frac{x}{2} \forall x \in (0, \pi) \text{ is convex} \right) = 3 \left(\frac{2}{\sqrt{3}} \right)^n \therefore \sum_{\text{cyc}} \sec^n \frac{A}{2} \stackrel{(1)}{\geq} 3 \left(\frac{2}{\sqrt{3}} \right)^n$$

$$\text{Now, } \sum_{\text{cyc}} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)}$$

$$= \sum_{\text{cyc}} \frac{\left(\sec^n \frac{A}{2} \right)^3}{\cos \frac{B}{2} + \cos \frac{C}{2}} \stackrel{\text{Holder}}{\geq} \frac{\left(\sum \sec^n \frac{A}{2} \right)^3}{6 \sum \cos \frac{A}{2}} \stackrel{\text{by (1) and Jensen}}{\geq} \frac{\left(3 \left(\frac{2}{\sqrt{3}} \right)^n \right)^3}{6 \cdot 3 \frac{\sqrt{3}}{2}}$$

$$= \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n}$$

$$\therefore \sum_{\text{cyc}} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)} \stackrel{(ii)}{\geq} \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n} \therefore (i), (ii) \Rightarrow \sum_{\text{cyc}} \frac{1}{\cos^{3n} \frac{A}{2} \left(\cos \frac{B}{2} + \cos \frac{C}{2} \right)}$$

$$\geq \sqrt{3} \left(\frac{2}{\sqrt{3}} \right)^{3n} \forall n \in \mathbb{N} (\text{Pd})$$



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1774. In any acute – angled ΔABC ,

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{8abc}{(a + b)(b + c)(c + a)}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 & \text{Proof : } \cos(A - B)\cos(B - C)\cos(C - A) \\
 &= \left(2\cos^2 \frac{A-B}{2} - 1\right) \left(2\cos^2 \frac{B-C}{2} - 1\right) \left(2\cos^2 \frac{C-A}{2} - 1\right) \\
 &\stackrel{(a)}{=} 8 \prod \cos^2 \frac{B-C}{2} - 4 \left(\prod \cos^2 \frac{B-C}{2}\right) \sum \sec^2 \frac{B-C}{2} + 2 \sum \cos^2 \frac{B-C}{2} - 1 \\
 & \text{Now, } \sum \cos^2 \frac{B-C}{2} = \sum \frac{(b+c)^2 \sin^2 \frac{A}{2}}{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{16R^2 s} \sum \frac{bc(b+c)^2}{s-a} \\
 &= \frac{1}{16R^2 s} \sum \frac{bc(s+s-a)^2}{s-a} \\
 &= \frac{1}{16R^2 s} \sum \left\{ \frac{bcs^2}{s-a} + 2sbc + bc(s-a) \right\} = \frac{1}{16R^2 s} \left\{ s^3 \sum \sec^2 \frac{A}{2} + 3s \sum ab - 3abc \right\} \\
 &= \frac{1}{16R^2 s} \left[s^3 \left\{ \frac{s^2 + (4R+r)^2}{s^2} \right\} + 3s(s^2 + 4Rr + r^2) - 12Rrs \right] = \frac{4s^2 + (4R+r)^2 + 3r^2}{16R^2} \\
 &\Rightarrow \sum \cos^2 \frac{B-C}{2} \stackrel{(1)}{=} \frac{4s^2 + (4R+r)^2 + 3r^2}{16R^2} \\
 & \text{Again, } \sum \sec^2 \frac{B-C}{2} = \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{(b+c)^2 \sin^2 \frac{A}{2}} = \sum \frac{16R^2 s(s-a)a}{4Rrs(b+c)^2} \\
 &= \frac{2R}{r} \sum \frac{a(b+c-a)}{(b+c)^2} \stackrel{(2)}{=} \frac{2R}{r} \left\{ \sum \frac{a}{b+c} - \sum \frac{a^2}{(b+c)^2} \right\} \\
 & \text{Now, } \sum \frac{a}{b+c} = \frac{\sum a(c+a)(a+b)}{\prod(b+c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2 + 2Rr + r^2)} \\
 &= \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \stackrel{(3)}{=} \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} \\
 & \text{and, } \sum \frac{a^2}{(b+c)^2} = \sum \frac{(2s-(b+c))^2}{(b+c)^2} \\
 &= \sum \frac{4s^2 - 4s(b+c) + (b+c)^2}{(b+c)^2} \stackrel{(i)}{=} 4s^2 \left[\frac{\sum \{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2} \right] \\
 &- 4s \left[\frac{\sum (c+a)(a+b)}{\prod(b+c)} \right] + 3
 \end{aligned}$$



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$$\begin{aligned}
 \sum \{(c+a)^2(a+b)^2\} &= \sum \left(\sum ab + a^2 \right)^2 = \sum \left\{ \left(\sum ab \right)^2 + 2a^2 \sum ab + a^4 \right\} \\
 &= 3 \left(\sum ab \right)^2 + 2 \left(\sum ab \right) (\sum a^2) + (\sum a^2)^2 - 2 \sum a^2 b^2 \\
 &= \left(\sum ab \right)^2 + 2 \left(\sum ab \right) (\sum a^2) + (\sum a^2)^2 + 2 \sum a^2 b^2 + 4abc(2s) - 2 \sum a^2 b^2 \\
 &= \left(\sum ab + \sum a^2 \right)^2 + 32Rrs^2 \\
 &\quad = (3s^2 - 4Rr - r^2)^2 + 32Rrs^2 \\
 &\stackrel{(ii)}{\therefore} \sum \{(c+a)^2(a+b)^2\} \cong (3s^2 - 4Rr - r^2)^2 + 32Rrs^2 \\
 \text{Again, } \sum (c+a)(a+b) &= \sum \left(\sum ab + a^2 \right) = 3 \sum ab + \sum a^2 \\
 &= \sum a^2 + 2 \sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2 \\
 &\stackrel{(iii)}{\therefore} \sum (c+a)(a+b) \cong 5s^2 + 4Rr + r^2 \\
 \because \prod (b+c) &= s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum \frac{a^2}{(b+c)^2} \\
 &= \frac{4s^2 \{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2\}}{4s^2(s^2 + 2Rr + r^2)^2} - \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} + 3 \\
 &= \frac{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2 - 2(5s^2 + 4Rr + r^2)(s^2 + 2Rr + r^2) + 3(s^2 + 2Rr + r^2)^2}{(s^2 + 2Rr + r^2)^2} \\
 &= \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \\
 &\Rightarrow \sum \frac{a^2}{(b+c)^2} \stackrel{(4)}{\cong} \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \\
 &\quad (2), (3), (4) \Rightarrow \sum \sec^2 \frac{B-C}{2} \\
 &= \frac{2R}{r} \left\{ \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} \right. \\
 &\quad \left. - \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \right\} \\
 &\stackrel{(5)}{\cong} \frac{2R}{r} \left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right] \\
 \text{Also, } 8 \prod \cos^2 \frac{B-C}{2} &= 8 \prod \frac{(b+c)^2 \sin^2 \frac{A}{2}}{a^2} \\
 &= 8 \left\{ \frac{4s^2(s^2 + 2Rr + r^2)^2}{16R^2r^2s^2} \right\} \left(\frac{r^2}{16R^2} \right) \stackrel{(6)}{\cong} \frac{(s^2 + 2Rr + r^2)^2}{8R^4}
 \end{aligned}$$



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$$\begin{aligned}
 & (\text{a}), (\text{1}), (\text{5}), (\text{8}) \Rightarrow \cos(A - B)\cos(B - C)\cos(C - A) = \frac{(s^2 + 2Rr + r^2)^2}{8R^4} \\
 & - \frac{\left((s^2 + 2Rr + r^2)^2\right)}{16R^4} \cdot \frac{2R}{r} \left[\frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right] \\
 & + \frac{4s^2 + (4R + r)^2 + 3r^2}{8R^2} - 1 \\
 & \Rightarrow \cos(A - B)\cos(B - C)\cos(C - A) \stackrel{(\text{m})}{=} \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} \\
 & \quad (\text{where } \sigma = (2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) \\
 & \quad - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}) \\
 & \text{Now, } \frac{8abc}{(a+b)(b+c)(c+a)} \stackrel{\text{Gerretsen}}{\geq} \frac{2r}{R} \Leftrightarrow \frac{32Rrs}{2s(s^2 + 2Rr + r^2)} \stackrel{?}{\geq} \frac{2r}{R} \Leftrightarrow 8R^2 \stackrel{(\text{iv})}{\geq} s^2 + 2Rr + r^2 \\
 & \text{Now, RHS of (iv)} \stackrel{?}{\geq} 4R^2 + 6Rr + 4r^2 \stackrel{?}{\geq} 8R^2 \Leftrightarrow (R - 2r)(2R + r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \\
 & \Rightarrow (\text{iv) is true} \Leftrightarrow \frac{8abc}{(a+b)(b+c)(c+a)} \stackrel{(\text{n})}{\geq} \frac{2r}{R} \\
 & \quad \therefore (\text{m), (n) } \Rightarrow \text{it suffices to prove} \\
 & : \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} - \frac{2r}{R} \\
 & \leq 0 \\
 & \Leftrightarrow \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r - 16R^3r^2}{8R^4r} \stackrel{(\text{x})}{\leq} 0 \\
 & \Leftrightarrow s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4 \stackrel{(\text{x})}{\leq} 0 \\
 & \quad \because \Delta ABC \text{ is acute - angled, Walker and Gerretsen} \\
 & \Rightarrow (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2) \leq 0 \\
 & \Rightarrow \text{in order to prove (x),} \\
 & \text{it suffices to prove : } s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4 \\
 & \leq (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2) \\
 & \stackrel{(\text{y})}{\Leftrightarrow} (R + 2r)s^2 \stackrel{\text{Gerretsen}}{\leq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3 \\
 & \text{Now, } (R + 2r)s^2 \stackrel{?}{\geq} (R + 2r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3 \\
 & \Leftrightarrow 4t^3 - 5t^2 - 4t - 4 \stackrel{?}{\geq} 0 \quad \left(\text{where } t = \frac{R}{r} \right) \\
 & \quad \stackrel{\text{Euler}}{\Leftrightarrow} (t - 2)(4t^2 + 3t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \quad \because t \stackrel{?}{\geq} 2 \Rightarrow (\text{x}) \Rightarrow (\text{y) is true} \\
 & \therefore \cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{8abc}{(a+b)(b+c)(c+a)} \quad (\text{Proved})
 \end{aligned}$$



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1775. In any ΔABC ,

$$\sum \frac{h_b h_c}{a^2} \left\{ \sum \left(\frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 \leq \frac{s_a + s_b + s_c}{r}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Proof : } b + c - a &= 4R \cos \frac{A}{2} \cos \frac{B-C}{2} - 4R \sin \frac{A}{2} \cos \frac{A}{2} \\ &= 4R \cos \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) = 8R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &\Rightarrow s - a \stackrel{(1)}{\cong} 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{aligned}$$

$$\begin{aligned} \text{Also, } a \cos A + b \cos B + c \cos C &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= R\{2\sin(A+B)\cos(A-B) + 2\sin C \cos C\} \\ &= 2R \sin C \{\cos(A-B) - \cos(A+B)\} = 4R \prod \sin A = 4R \left(\frac{abc}{8R^3} \right) \end{aligned}$$

$$= \frac{4Rrs}{2R^2} = \frac{2rs}{R} \Rightarrow \sum a \cos A \stackrel{(2)}{\cong} \frac{2rs}{R}$$

$$\begin{aligned} \text{Now, } \sum \left(\frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) &= \sum \left\{ \frac{a \left(\frac{rs}{s-a} - \frac{rs}{s} \right) (b+c)}{2abccos \frac{A}{2}} \sqrt{\frac{2rs}{4R \tan \frac{A}{2} \sin \frac{A}{2} \cos \frac{A}{2}}} \right\} \\ &= \sqrt{\frac{r}{2R}} \sum \left[\frac{4R \sin \frac{A}{2} \cos \frac{A}{2} \left\{ \frac{ars}{s(s-a)} \right\} (b+c)}{8Rrscos \frac{A}{2} \sin \frac{A}{2}} \right] \end{aligned}$$



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$$\begin{aligned}
 & \stackrel{\text{by (1)}}{\cong} \sqrt{\frac{r}{2R}} \sum \left[\frac{4R \sin \frac{A}{2} \cos \frac{A}{2} \left\{ \frac{4R r \sin \frac{A}{2} \cos \frac{A}{2}}{4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right\} (\mathbf{b} + \mathbf{c})}{8R r s \cos \frac{A}{2} \sin \frac{A}{2}} \right] \\
 &= \left(\frac{1}{2s} \right) \sqrt{\frac{r}{2R}} \sum \left[\left\{ \frac{4R \sin^2 \frac{A}{2} \cos \frac{A}{2}}{4R \cos \frac{A}{2} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \right\} (\mathbf{b} + \mathbf{c}) \right] \\
 &= \left(\frac{1}{2s} \right) \sqrt{\frac{r}{2R}} \sum \left[\left\{ \frac{4R \sin^2 \frac{A}{2}}{4R \left(\frac{r}{4R} \right)} \right\} (\mathbf{b} + \mathbf{c}) \right] \\
 &= \left(\frac{R}{rs} \right) \sqrt{\frac{r}{2R}} \sum \left\{ 2 \sin^2 \frac{A}{2} (\mathbf{b} + \mathbf{c}) \right\} = \left(\frac{R}{rs} \right) \sqrt{\frac{r}{2R}} \sum \{ 1 - \cos A \} (\mathbf{b} + \mathbf{c}) \\
 &= \left(\frac{R}{rs} \right) \sqrt{\frac{r}{2R}} [4s - \sum \{ (2s - a) \cos A \}] \\
 &= \left(\frac{R}{rs} \right) \sqrt{\frac{r}{2R}} \{ 4s - 2s \left(1 + \frac{r}{R} \right) + \sum a \cos A \} \\
 &\stackrel{\text{by (2)}}{\cong} \left(\frac{R}{rs} \right) \sqrt{\frac{r}{2R}} \left(2s - \frac{2rs}{R} + \frac{2rs}{R} \right) = \left(\frac{2R}{r} \right) \sqrt{\frac{r}{2R}} = \sqrt{\frac{2R}{r}} \Rightarrow \left\{ \sum \left(\frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 = \frac{2R}{r} \\
 &\Rightarrow \sum \frac{h_b h_c}{a^2} \left\{ \sum \left(\frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 = \frac{2R}{r} \sum \frac{ca \cdot ab}{4R^2 a^2} = \frac{1}{r} \sum \frac{bc}{2R} \\
 &= \frac{1}{r} \sum h_a \leq \frac{s_a + s_b + s_c}{r} \quad (\text{Proved})
 \end{aligned}$$

1776. In } \Delta ABC, K -\text{Lemoine's point, } G -\text{centroid. Prove that:}

$$KG \perp BC \Leftrightarrow (b = c) \vee (5a^2 = b^2 + c^2)$$

$$KG \parallel BC \Leftrightarrow 2a^2 = b^2 + c^2$$

Proposed by Gheorghe Alexe, George Florin Serban-Romania



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Solution by Marian Ursărescu-Romania

$$a) \overrightarrow{KG} \perp \overrightarrow{BC} \Leftrightarrow \overrightarrow{KG} \cdot \overrightarrow{BC} = \mathbf{0} \Leftrightarrow \overrightarrow{KG} \cdot \overrightarrow{CB} = \mathbf{0}; \quad (1)$$

$$\text{But } \overrightarrow{PK} = \frac{a^2 \cdot \overrightarrow{PA} + b^2 \cdot \overrightarrow{PB} + c^2 \cdot \overrightarrow{PC}}{a^2 + b^2 + c^2}, \forall P \in G \Rightarrow \overrightarrow{KG} = \frac{a^2 \cdot \overrightarrow{GA} + b^2 \cdot \overrightarrow{GB} + c^2 \cdot \overrightarrow{GC}}{a^2 + b^2 + c^2}; \quad (2)$$

$\overrightarrow{CB} = \overrightarrow{GB} - \overrightarrow{GC}$; (3). From (1),(2),(3) we have:

$$\frac{a^2 \cdot \overrightarrow{GA} + b^2 \cdot \overrightarrow{GB} + c^2 \cdot \overrightarrow{GC}}{a^2 + b^2 + c^2} \cdot (\overrightarrow{GB} - \overrightarrow{GC}) = \mathbf{0} \Leftrightarrow$$

$$(a^2 \cdot \overrightarrow{GA} + b^2 \cdot \overrightarrow{GB} + c^2 \cdot \overrightarrow{GC}) \cdot (\overrightarrow{GB} - \overrightarrow{GC}) = \mathbf{0}; \quad (4)$$

But in any ΔABC : $\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} = \mathbf{0}$; (5)

From (4),(5) we have: $[(b^2 - a^2)\overrightarrow{GB} + (c^2 - a^2)\overrightarrow{GC}] \cdot (\overrightarrow{GB} - \overrightarrow{GC}) = \mathbf{0} \Leftrightarrow$

$$(b^2 - a^2)\overrightarrow{GB}^2 - (c^2 - a^2)\overrightarrow{GC}^2 + (c^2 - b^2)\overrightarrow{GB} \cdot \overrightarrow{GC} = \mathbf{0}; \quad (6)$$

$$\overrightarrow{GB}^2 = \frac{4}{9}m_b^2 = \frac{1}{9}(2a^2 + 2c^2 - b^2); \quad \overrightarrow{GC}^2 = \frac{4}{9}m_c^2 = \frac{1}{9}(2a^2 + 2b^2 - c^2); \quad (7)$$

$$\overrightarrow{GB} \cdot \overrightarrow{GC} = \frac{\overrightarrow{GB}^2 + \overrightarrow{GC}^2 - \overrightarrow{BC}^2}{2} = \frac{1}{4}\left(\frac{4}{9}m_b^2 + \frac{4}{9}m_c^2 - a^2\right) = \frac{1}{18}(b^2 + c^2 - 5a^2); \quad (8)$$

From (6),(7),(8) we get:

$$(b^2 - a^2)(2a^2 + 2c^2 - b^2) - (c^2 - a^2)(2a^2 + 2b^2 - c^2) + \frac{1}{2}(c^2 - b^2)(b^2 + c^2 - 5a^2) = 0$$

$$(c^2 - b^2)(b^2 + c^2 - 5a^2) \Leftrightarrow \begin{cases} b = c \\ b^2 + c^2 = 5a^2 \end{cases}$$

$$b) \overrightarrow{KG} \parallel \overrightarrow{BC} \Leftrightarrow \exists \alpha \in \mathbb{R}^* \text{ such that } \overrightarrow{KG} = \alpha \cdot \overrightarrow{BC} \Leftrightarrow \frac{a^2 \cdot \overrightarrow{GA} + b^2 \cdot \overrightarrow{GB} + c^2 \cdot \overrightarrow{GC}}{a^2 + b^2 + c^2} = \alpha \cdot (\overrightarrow{GB} - \overrightarrow{GC}) \Leftrightarrow$$

$$(b^2 - c^2)\overrightarrow{GB} + (c^2 - a^2)\overrightarrow{GC} = \alpha(a^2 + b^2 + c^2)\overrightarrow{GC} - \alpha(a^2 + b^2 + c^2)\overrightarrow{GB} \Leftrightarrow$$

$$(b^2 - a^2 + \alpha(a^2 + b^2 + c^2))\overrightarrow{GB} = (\alpha(a^2 + b^2 + c^2) - c^2 + a^2)\overrightarrow{GC}; \quad (9)$$

But if $b^2 - a^2 + \alpha(a^2 + b^2 + c^2) \neq 0$ and $\alpha(a^2 + b^2 + c^2) - c^2 + a^2 \neq 0 \Rightarrow$

$$\overrightarrow{KG} = k \cdot \overrightarrow{GC} \text{ false.} \quad (10)$$

From (9),(10) we get: $\begin{cases} b^2 - a^2 + \alpha(a^2 + b^2 + c^2) = 0 \\ \alpha(a^2 + b^2 + c^2) - c^2 + a^2 = 0 \end{cases} \Leftrightarrow b^2 - a^2 - a^2 + c^2 = 0 \Leftrightarrow 2a^2 = b^2 + c^2$



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$$1777. \forall a, b, c \geq 1, \sqrt[15]{8192(a^5 + b^5)(b^5 + c^5)(c^5 + a^5)} \leq a^2 + b^2 + c^2$$

Proposed by Jalil Hajimir – Canada

Solution by Soumava Chakraborty-Kolkata-India

$$2(a^2 - ab + b^2)^2 - (a^4 + b^4) - (a - b)^4 = 0 \Rightarrow 2(a^2 - ab + b^2)^2 - (a^4 + b^4)$$

$$= (a - b)^4 \geq 0 \Rightarrow 2(a^2 - ab + b^2)^2 \stackrel{(i)}{\geq} a^4 + b^4$$

$$\text{Now, } a^5 + b^5 = (a + b) \left(a^4 + b^4 - ab(a^2 - ab + b^2) \right) \stackrel{\text{by (i)}}{\geq} (a$$

$$+ b) \left(2(a^2 - ab + b^2)^2 - ab(a^2 - ab + b^2) \right)$$

$$= (a + b)(a^2 - ab + b^2)(2a^2 - 3ab + 2b^2) \Rightarrow (a^5 + b^5)^2$$

$\leq (a + b)^2(a^2 - ab + b^2)^2(2a^2 - 3ab + 2b^2)^2$ and analogs

$$\therefore \sqrt[15]{(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \leq$$

$$\sqrt[15]{(a + b)^2(a^2 - ab + b^2)^2(2a^2 - 3ab + 2b^2)^2(b + c)^2(b^2 - bc + c^2)^2(2b^2 - 3bc + 2c^2)^2(c + a)^2(c^2 - ca + a^2)^2(2c^2 - 3ca + 2a^2)^2}$$

$$\stackrel{\text{weighted AM-GM}}{\geq} \frac{\sum(a + b)^2 + 2(a^2 - ab + b^2 + 2a^2 - 3ab + 2b^2 + b^2 - bc + c^2 + 2b^2 - 3bc + 2c^2 + c^2 - ca + a^2 + 2c^2 - 3ca + 2a^2)}{15}$$

$$= \frac{14a^2 + 14b^2 + 14c^2 - 6ab - 6bc - 6ca}{15}$$

$$\therefore \sqrt[15]{(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \stackrel{(1)}{\geq} \frac{14 \sum a^2 - 6 \sum ab}{15} \text{ and } \because \sqrt[15]{8192}$$

$$= \sqrt[15]{2^{13}} \stackrel{(2)}{\geq} 1$$

$$\therefore \sqrt[15]{8192(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \stackrel{\text{by (1) and (2)}}{\geq} \frac{14 \sum a^2 - 6 \sum ab}{15} \stackrel{?}{\geq} \sum a^2$$

$$\Leftrightarrow 14 \sum a^2 - 6 \sum ab \stackrel{?}{\geq} 15 \sum a^2$$

$$\Leftrightarrow \sum a^2 + 6 \sum ab \stackrel{?}{>} 0 \rightarrow \text{true}$$

$$\therefore \sqrt[15]{8192(a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2} \stackrel{(3)}{\geq} a^2 + b^2 + c^2$$



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$$\text{Now } \because a, b \geq 1 \therefore a^5 + b^5 \geq 2 \Rightarrow (a^5 + b^5)^2 \geq 2(a^5 + b^5) > a^5 + b^5 \Rightarrow a^5 + b^5 < (a^5 + b^5)^2 \text{ and analogs}$$

$$\begin{aligned} & \therefore (a^5 + b^5)(b^5 + c^5)(c^5 + a^5) < (a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2 \\ & \Rightarrow \left(\frac{1}{15}\right) \ln((a^5 + b^5)(b^5 + c^5)(c^5 + a^5)) < \left(\frac{1}{15}\right) \ln((a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2) \\ & \Rightarrow \ln \sqrt[15]{(a^5 + b^5)(b^5 + c^5)(c^5 + a^5)} < \ln \sqrt[15]{((a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2)} \\ & \Rightarrow \sqrt[15]{8192(a^5 + b^5)(b^5 + c^5)(c^5 + a^5)} \\ & \quad < \sqrt[15]{8192((a^5 + b^5)^2(b^5 + c^5)^2(c^5 + a^5)^2)} \stackrel{\text{by (3)}}{\lesssim} a^2 + b^2 + c^2 \text{ (Proved)} \end{aligned}$$

1778. Let DEF be the orthic triangle in acute $\triangle ABC$. Then :

$$\frac{4[\text{DEF}]}{[\text{ABC}]} \leq \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2}\right)^2$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Using } \triangle ABD, \cos B = \frac{BD}{AB} \stackrel{(i)}{\hat{=}} c \cos B \text{ and using } \triangle CBF, \cos B = \frac{BF}{BC}$$

$$\Rightarrow BF \stackrel{(ii)}{\hat{=}} a \cos B \text{ and } \therefore [BDF] = \frac{1}{2} BD \cdot BF \cdot \sin B$$

$$\stackrel{\text{by (i),(ii)}}{=} \frac{1}{2} a c \sin B \cos^2 B = \frac{abc}{8R} (1 + \cos 2B) = \frac{4Rrs}{8R} (1 + \cos 2B) = \frac{rs}{2} (1 + \cos 2B)$$

$$\Rightarrow [BDF] \stackrel{(1)}{\hat{=}} \frac{rs}{2} (1 + \cos 2B) \text{ and similarly,}$$

$$[\text{CDE}] \stackrel{(2)}{\hat{=}} \frac{rs}{2} (1 + \cos 2C) \text{ and } [\text{AEF}] \stackrel{(3)}{\hat{=}} \frac{rs}{2} (1 + \cos 2A) \text{ and } \therefore (1) + (2) + (3)$$

$$\Rightarrow [\text{ABC}] - [\text{DEF}] = \frac{rs}{2} \left(3 + \sum \cos 2A \right)$$



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$$\begin{aligned}
 &= \frac{rs}{2} \left(3 - 1 - 4 \prod \cos A \right) = rs \left(1 - 2 \prod \cos A \right) \Rightarrow [\text{DEF}] = rs - rs \left(1 - 2 \prod \cos A \right) \\
 &= 2rs \prod \cos A \Rightarrow \frac{4[\text{DEF}]}{[\text{ABC}]} \stackrel{(m)}{\cong} 8 \prod \cos A
 \end{aligned}$$

$$\begin{aligned}
 &\text{Now, } \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2 \geq \frac{2r}{R} \Leftrightarrow R(s^2 + 4Rr + r^2)^2 \geq 8r(s^2 - 4Rr - r^2)^2 \\
 &\Leftrightarrow (R - 2r)s^4 + s^2(8R^2r + 66Rr^2 + 16r^3) + 16R^3r^2 - 120R^2r^3 - 63Rr^4 \\
 &\quad \stackrel{(iii)}{- 8r^5} \stackrel{\cong}{\cong} 6rs^4
 \end{aligned}$$

$$\begin{aligned}
 &\text{Now, LHS of (iii) } \underbrace{\stackrel{\cong}{\cong}}_{(a)} (R - 2r)(16Rr - 5r^2)s^2 + s^2(8R^2r + 66Rr^2 + 16r^3) \\
 &\quad + 16R^3r^2 - 120R^2r^3 - 63Rr^4 - 8r^5
 \end{aligned}$$

$$\begin{aligned}
 &\text{and RHS of (iii) } \underbrace{\stackrel{\cong}{\cong}}_{(b)} 6rs^2(4R^2 + 4Rr + 3r^2) \therefore (a), (b)
 \end{aligned}$$

\Rightarrow in order to prove (iii), it suffices to prove :

$$\begin{aligned}
 &(R - 2r)(16Rr - 5r^2)s^2 + s^2(8R^2r + 66Rr^2 + 16r^3) + 16R^3r^2 - 120R^2r^3 - 63Rr^4 \\
 &- 8r^5 \geq 6rs^2(4R^2 + 4Rr + 3r^2) \\
 &\Leftrightarrow s^2(5R + 8r) + 16R^3 - 120R^2r - 63Rr^2 - 8r^3 \stackrel{(iv)}{\cong} 0
 \end{aligned}$$

$$\begin{aligned}
 &\text{Now, LHS of (iv) } \underbrace{\stackrel{\cong}{\cong}}_{(a)} (16Rr - 5r^2)(5R + 8r) + 16R^3 - 120R^2r - 63Rr^2 \\
 &- 8r^3 \stackrel{?}{\cong} 0 \Leftrightarrow 2t^3 - 5t^2 + 5t - 6 \stackrel{?}{\cong} 0 \left(\text{where } t = \frac{R}{r} \right) \\
 &\Leftrightarrow (t - 2)\{(t - 2)(2t + 3) + 9t^2\} \stackrel{?}{\cong} 0 \rightarrow \text{true} \because t \stackrel{?}{\cong} 2 \Rightarrow (iv) \Rightarrow (iii) \text{ is true}
 \end{aligned}$$

$$\therefore \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2 \stackrel{(n)}{\cong} \frac{2r}{R}$$

$$\begin{aligned}
 &\therefore (\text{m}), (\text{n}) \Rightarrow \text{it suffices to prove : } 8 \prod \cos A \leq \frac{2r}{R} \Leftrightarrow \frac{s^2 - 4R^2 - 4Rr - r^2}{R^2} \leq \frac{r}{R} \Leftrightarrow s^2 \\
 &\leq 4R^2 + 5Rr + r^2
 \end{aligned}$$



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$$\Leftrightarrow s^2 - 4R^2 - 4Rr - 3r^2 - r(R - 2r) \leq 0 \rightarrow \text{true}$$

$$\because s^2 - 4R^2 - 4Rr - 3r^2 \stackrel{\text{Gerretsen}}{\leq} 0 \text{ and } R - 2r \stackrel{\text{Euler}}{\geq} 0 \therefore \frac{4[\text{DEF}]}{[\text{ABC}]}$$

$$\leq \left(\frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2$$

1779. In ΔABC the following relationship holds:

$$\left(\sum_{cyc} \sqrt{m_a} \right) \left(\sum_{cyc} \sqrt{w_a} \right) \left(\sum_{cyc} m_a \right) \left(\sum_{cyc} w_a \right) \geq \frac{27}{2R + 5r} \left(\sum_{cyc} h_a h_b \right)^2$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution by Marian Ursărescu-Romania

$$\sum_{cyc} \sqrt{m_a} \geq 3\sqrt[3]{\sqrt{m_a m_b m_c}} \text{ and } \sum_{cyc} m_a \geq 3\sqrt[3]{m_a m_b m_c} \Rightarrow$$

$$\left(\sum_{cyc} \sqrt{m_a} \right) \left(\sum_{cyc} m_a \right) \geq 9\sqrt{m_a m_b m_c}$$

Similarly: $(\sum_{cyc} \sqrt{w_a})(\sum_{cyc} w_a) \geq 9\sqrt{w_a w_b w_c}$. Therefore,

$$\left(\sum_{cyc} \sqrt{m_a} \right) \left(\sum_{cyc} \sqrt{w_a} \right) \left(\sum_{cyc} m_a \right) \left(\sum_{cyc} w_a \right) > 81\sqrt{m_a m_b m_c w_a w_b w_c}$$

We must show that:

$$3\sqrt{m_a m_b m_c w_a w_b w_c} \geq \frac{1}{2R + 5r} \cdot \left(\sum_{cyc} h_a h_b \right)^2 ; \quad (1)$$

But in any ΔABC we have: $m_a \cdot w_a \geq s(s - a)$; $(\therefore s = \frac{a+b+c}{2}) \Rightarrow$

$$\sqrt{m_a m_b m_c w_a w_b w_c} \geq sS = s^2 r; \quad (2)$$

From (1),(2) we must show:



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$$3s^2r \geq \frac{1}{2R+5r} \cdot \left(\sum_{cyc} h_a h_b \right)^2; \quad (3)$$

$$\text{But } \sum_{cyc} h_a h_b = \frac{2s^2r}{R}; \quad (4)$$

$$\text{From (3),(4) we must show: } 3s^2r \geq \frac{1}{2R+5r} \cdot \frac{4s^4r^2}{R^2} \Leftrightarrow 3R^2(2R+5r) \geq 4s^2r; \quad (5)$$

$$\text{From Mitrinovic inequality: } s^2 \leq \frac{27R^2}{4}; \quad (6)$$

From (5),(6) we must show: $3(2R+5r) \geq 27r \Leftrightarrow 2R+5r \geq 9r \Leftrightarrow 2R \geq 4r \Leftrightarrow R \geq 2r$ (*Euler*) true. Proved.

1780. In acute ΔABC the following relationship holds:

$$\sum_{cyc} \frac{6 + \tan^2 A + \tan^2 B}{6 + \sqrt{3}\tan A + \tan^2 B + \tan^2 C} \geq \frac{12}{5}$$

Proposed by Gheorghe Alexe and George Florin Ţerban-Romania

Solution by Tran Hong-Dong Thap-Vietnam

Since: ΔABC -acute, then $\tan A, \tan B, \tan C > 0$

$$\sqrt{3}\tan A \stackrel{AM-GM}{\leq} \frac{(\sqrt{3})^2 + \tan^2 A}{2} = \frac{3 + \tan^2 A}{2} \Rightarrow$$

$$\begin{aligned} 6 + \sqrt{3}\tan A + \tan^2 B + \tan^2 C &\leq \frac{15 + \tan^2 A}{2} + \tan^2 B + \tan^2 C = \\ &= \frac{15 + \tan^2 A + 2(\tan^2 B + \tan^2 C)}{2}; \text{ (and analogs)} \end{aligned}$$

Let: $x = \tan^2 A; y = \tan^2 B; z = \tan^2 C \Rightarrow x, y, z > 0$

Then, inequality becomes as:

$$2 \sum_{cyc} \frac{6 + x + y}{15 + x + 2(y+z)} \geq \frac{12}{5} \Leftrightarrow \sum_{cyc} \frac{6 + x + y}{15 + x + 2(y+z)} \geq \frac{6}{5}; (1) \Leftrightarrow$$

$$5 \sum_{cyc} (6 + x + y)(15 + y + 2(x+z))(15 + z + 2(x+y)) \geq \sum_{cyc} (15 + x + 2(x+y)) \Leftrightarrow$$



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$$6 \sum_{cyc} x^3 + (x^2y + y^2z + z^2y) + 45 \sum_{cyc} x^2 \geq \\ \geq 4(xy^2 + yz^2 + zx^2) + 9xyz + 45(xy + yz + zx)$$

Because:

$$3(x^3 + y^3 + z^3) \stackrel{AM-GM}{\geq} 3 \cdot 3 \cdot \sqrt[3]{(xyz)^3} = 9xyz; (2)$$

$$x^2 + y^2 + z^2 \stackrel{AM-GM}{\geq} xy + yz + zx \Rightarrow 45(x^2 + y^2 + z^2) \geq 45(xy + yz + zx); (3)$$

$$x^3 + x^3 + y^3 + x^2y \stackrel{AM-GM}{\geq} 4\sqrt[4]{x^8 \cdot y^4} = 4x^2y$$

$$y^3 + y^3 + z^3 + y^2z \stackrel{AM-GM}{\geq} 4\sqrt[4]{y^8 \cdot z^4} = 4y^2z$$

$$z^3 + z^3 + x^3 + z^2x \stackrel{AM-GM}{\geq} 4\sqrt[4]{z^8 \cdot x^4} = 4z^2x$$

$$3(x^3 + y^3 + z^3) + x^2y + y^2z + z^2x \geq 4(xy^2 + yz^2 + zx^2) \quad (4)$$

From (2),(3),(4) we get (1) true. Proved.

Equality holds if $x^2 = y^2 = z^2 \Leftrightarrow A = B = C = \frac{\pi}{3}$

1781. In any ΔABC , holds

$$4(R + r)^3 \geq \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27R^2r$$

Proposed by Alex Szoros – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} &= \frac{a^2(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} + \frac{2bc\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}} \\ &= \frac{a^2(s-a) + 2(b\sqrt{s-b})(c\sqrt{s-c})}{\sqrt{s(s-a)(s-b)(s-c)}} \\ &\stackrel{A-G}{\geq} \frac{a^2(s-a) + b^2(s-b) + c^2(s-c)}{rs} = \frac{s(\sum a^2) - \sum a^3}{rs} \\ &= \frac{2s(s^2 - 4Rr - r^2) - 2s(s^2 - 6Rr - 3r^2)}{rs} = \frac{2s(2Rr + 2r^2)}{rs} = 4(R+r) \end{aligned}$$



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$$\begin{aligned} \because \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \leq 4(R+r) \text{ and analogs} & \stackrel{\text{multiplying together}}{\Rightarrow} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \\ & + \frac{2ab}{\sqrt{r_a r_b}} \end{aligned}$$

$$\Rightarrow 4(R+r)^3 \stackrel{(1)}{\geq} \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right)$$

$$\begin{aligned} \text{Also, } \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} &= \frac{a^2}{r_a} + \frac{bc}{\sqrt{r_b r_c}} + \frac{bc}{\sqrt{r_b r_c}} \stackrel{A-G}{\geq} 3 \sqrt[3]{\left(\frac{a^2}{r_a} \right) \left(\frac{bc}{\sqrt{r_b r_c}} \right) \left(\frac{bc}{\sqrt{r_b r_c}} \right)} = 3 \sqrt[3]{\frac{16R^2 r^2 s^2}{rs^2}} \\ \therefore \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} &\geq 3 \sqrt[3]{16R^2 r} \text{ and analogs} \end{aligned}$$

$$\begin{aligned} \stackrel{\text{multiplying together}}{\Rightarrow} & \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27 \cdot 16R^2 r \\ \Rightarrow & \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \stackrel{(2)}{\geq} 27R^2 r \end{aligned}$$

$$(1), (2) \Rightarrow 4(R+r)^3 \geq \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27R^2 r \text{ (Proved)}$$

1782. In any ΔABC , holds:

$$3(a^2 + b^2 + c^2) \geq \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2 l_a^2 + 3l_a^4} \geq (a + b + c)^2$$

Proposed by Alex Szoros – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \frac{(x+y)^6}{3x^4 + 10x^2y^2 + 3y^4} &= \frac{(x+y)^2(x+y)^4}{3x^4 + 10x^2y^2 + 3y^4} \leq \frac{2(x^2 + y^2)(x+y)^4}{3x^4 + 10x^2y^2 + 3y^4} \stackrel{?}{\geq} 2(x^2 + y^2) \\ \Leftrightarrow 3x^4 + 10x^2y^2 + 3y^4 &\stackrel{?}{\geq} (x+y)^4 \\ \Leftrightarrow x^4 + 2x^2y^2 + y^4 &\stackrel{?}{\geq} 2xy(x^2 + y^2) \Leftrightarrow (x^2 + y^2)^2 \stackrel{?}{\geq} 2xy(x^2 + y^2) \\ \Leftrightarrow (x^2 + y^2)(x-y)^2 &\stackrel{?}{\geq} 0 \rightarrow \text{true} \end{aligned}$$



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$\therefore \frac{(x+y)^6}{3x^4 + 10x^2y^2 + 3y^4} \leq 2(x^2 + y^2)$ and choosing $x = m_a, y = l_a$, we get

$$\therefore \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4} \stackrel{l_a \leq m_a}{\stackrel{\text{summing up}}{\geq}} 4m_a^2$$

$$\text{and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4} \leq 4 \sum m_a^2 = 3 \sum a^2$$

$$\therefore 3(a^2 + b^2 + c^2) \stackrel{(1)}{\geq} \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4}$$

Again, $\frac{(x+y)^6}{3x^4 + 10x^2y^2 + 3y^4} \geq 4xy \Leftrightarrow (x+y)^6 - 4xy(3x^4 + 10x^2y^2 + 3y^4) \geq 0$

$$\Leftrightarrow (x-y)^6 \geq 0 \rightarrow \text{true}$$

$\therefore \frac{(x+y)^6}{3x^4 + 10x^2y^2 + 3y^4} \geq 4xy$ and choosing $x = m_a, y = l_a$, we get

$$\therefore \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4} \stackrel{\text{loscu}}{\geq} 4m_a l_a$$

$$= 4bc \frac{s(s-a)}{bc} = 4s(s-a) \Rightarrow \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4}$$

$$\geq 4s(s-a) \text{ and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4}$$

$$\geq 4s \sum (s-a)$$

$$= 4s^2 = (a+b+c)^2 \Rightarrow \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4} \stackrel{(2)}{\geq} (a+b+c)^2$$

$$\therefore (1), (2) \Rightarrow 3(a^2 + b^2 + c^2) \geq \sum \frac{(m_a + l_a)^6}{3m_a^4 + 10m_a^2l_a^2 + 3l_a^4} \geq (a+b+c)^2 \text{ (Proved)}$$

1783. Generalization for Marian Ursărescu problem

In ΔABC the following relationship holds:

$$\left(\frac{m_a m_b}{m_a + m_b} \right)^n + \left(\frac{m_b m_c}{m_b + m_c} \right)^n + \left(\frac{m_c m_a}{m_c + m_a} \right)^n \geq 3 \cdot \left(\frac{3r}{2} \right)^n, n \in \mathbb{N}$$

Proposed by Marin Chirciu – Romania



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Solution by Tran Hong-Dong Thap-Vietnam

Let $x = m_a$, $y = m_b$, $z = m_c \rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$

$$\rightarrow LHS = \sum \left(\frac{xy}{x+y} \right)^n \geq \frac{\left(\sum \frac{xy}{x+y} \right)^n}{3^{n-1}} = \frac{\left(\sum \frac{1}{\frac{x}{y} + \frac{y}{x}} \right)^n}{3^{n-1}}$$

$$\stackrel{\text{Schwarz}}{\geq} \frac{\left(\frac{9}{2(\frac{1}{x} + \frac{1}{y} + \frac{1}{z})} \right)^n}{3^{n-1}} \geq \frac{\left(\frac{9}{2} \cdot \frac{1}{r} \right)^n}{3^{n-1}} = \frac{\left(\frac{9r}{2} \right)^n}{3^{n-1}} = 3 \cdot \left(\frac{3r}{2} \right)^n. \text{ Proved.}$$

1784. If $a, b > 0$, then :

$$\frac{4\sqrt{ab}}{a+b} + \frac{(a+b)^2}{4ab} \geq \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

Let $A = \frac{a+b}{2}$, $G = \sqrt{ab}$ and of course : $A \geq G$

$$\frac{4\sqrt{ab}}{a+b} + \frac{(a+b)^2}{4ab} \geq \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} \Leftrightarrow \frac{2G}{A} + \frac{A^2}{G^2} \geq \frac{2A}{G} + \frac{G^2}{A^2} \Leftrightarrow \frac{A^2}{G^2} - \frac{G^2}{A^2} \geq \frac{2A}{G} - \frac{2G}{A}$$

$$\Leftrightarrow \frac{A^4 - G^4}{A^2 G^2} \geq \frac{2(A^2 - G^2)}{AG}$$

$$\Leftrightarrow \left(\frac{A^2 - G^2}{AG} \right) \left(\frac{A^2 + G^2}{AG} - 2 \right) \geq 0 \Leftrightarrow \left(\frac{A^2 - G^2}{AG} \right) \frac{(A-G)^2}{AG} \geq 0 \rightarrow \text{true}$$

$$\therefore \frac{4\sqrt{ab}}{a+b} + \frac{(a+b)^2}{4ab} \geq \frac{a+b}{\sqrt{ab}} + \frac{4ab}{(a+b)^2} \text{ (Proved)}$$

1785. In any ΔABC , holds:

$$S \leq \frac{abc(b+c)w_a}{(a+b)(a+c)(b+c-a)}$$

Proposed by Alex Szoros – Romania



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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 b + c - a &= 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\cos\frac{A}{2}\sin\frac{A}{2} = 4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) \\
 &= 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
 &\stackrel{(1)}{\Rightarrow} b + c - a \stackrel{(1)}{\cong} 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}
 \end{aligned}$$

Now, $S(a+b)(a+c)(b+c-a)$

$$\begin{aligned}
 &\stackrel{\text{by (1)}}{\cong} S\left(4R\cos\frac{C}{2}\cos\frac{A-B}{2}\right)\left(4R\cos\frac{B}{2}\cos\frac{A-C}{2}\right)8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
 &\leq 2rs(4R)^3\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \left(\because 0 < \cos\frac{A-B}{2}, \cos\frac{A-C}{2} \leq 1\right) \\
 &= 2\left(4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)\left(4R\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)(4R)^3\cos\frac{C}{2}\cos\frac{B}{2}\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \\
 &= 2\left(4R\sin\frac{A}{2}\cos\frac{A}{2}\right)\left(4R\sin\frac{B}{2}\cos\frac{B}{2}\right)^2\left(4R\sin\frac{C}{2}\cos\frac{C}{2}\right)^2\cos\frac{A}{2} \\
 &= 2ab^2c^2\left(\frac{\cos\frac{A}{2}}{b+c}\right)(b+c) = abc(b+c)\left(\frac{2bcc\cos\frac{A}{2}}{b+c}\right) = abc(b+c)w_a \\
 &\Rightarrow S \leq \frac{abc(b+c)w_a}{(a+b)(a+c)(b+c-a)} \quad (\text{Proved})
 \end{aligned}$$

1786.

In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$2 \sum \frac{g_a^2}{h_a^2} \geq 2 + \sum \frac{b^2 + c^2}{n_a^2 + g_a^2}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) \\
 &= a n_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\
 &= a g_a^2 + a(s-b)(s-c)
 \end{aligned}$$



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$$\therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2$$

$$\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s - b)(s - c)\} \stackrel{(a)}{\leq} a^2 s^2 (s - a)^2$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and $c = x + y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \stackrel{(1)}{\leq} s(s - a) \text{ and analogs}$$

$$= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c)$$

$$= a g_a^2 + a(s - b)(s - c)$$

Adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$= a n_a^2 + a g_a^2 + 2a(s - b)(s - c)$$

$$\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2)$$

$$= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2$$

$$\Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2)$$

$$\Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 \stackrel{(i)}{\equiv} 2(n_a^2 + g_a^2)$$

$$\Rightarrow 2(b^2 + c^2) - 2(n_a^2 + g_a^2) = 2(b^2 + c^2) - 2(b - c)^2 - (b + c)^2 + a^2$$

$$= a^2 - ((b + c)^2 - 4bc) = a^2 - (b - c)^2$$

$$= (a + b - c)(c + a - b) = 4(s - b)(s - c) \Rightarrow \frac{(b^2 + c^2) - (n_a^2 + g_a^2)}{n_a^2 + g_a^2}$$

$$= \frac{2(s - b)(s - c)}{n_a^2 + g_a^2} \stackrel{A-G}{\leq} \frac{2b c \sin^2 \frac{A}{2}}{2n_a g_a} \stackrel{\text{by (1)}}{\leq} \frac{2b c \sin^2 \frac{A}{2}}{2b c \cos^2 \frac{A}{2}} = \frac{s^2 \tan^2 \frac{A}{2}}{s^2}$$



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$$\Rightarrow \frac{b^2 + c^2}{n_a^2 + g_a^2} \leq 1 + \frac{r_a^2}{s^2} \text{ and analogs} \stackrel{\text{summing up}}{\Rightarrow} 2 + \sum \frac{b^2 + c^2}{n_a^2 + g_a^2} \leq 5 + \frac{\sum r_a^2}{s^2}$$

$$= 5 + \frac{(4R+r)^2 - 2s^2}{s^2} = \frac{(4R+r)^2 + 3s^2}{s^2}$$

$$\Rightarrow 2 + \sum \frac{b^2 + c^2}{n_a^2 + g_a^2} \stackrel{(2)}{\leq} \frac{(4R+r)^2 + 3s^2}{s^2}$$

Again, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c)$

$$\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc$$

$$= an_a^2 + a(as - s^2)$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc)$$

$$= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}$$

$$= as^2 - \frac{4as(s - b)(s - c)}{a} \stackrel{(ii)}{\equiv} s^2 - \frac{4s(s - b)(s - c)}{a} \therefore (i), (ii) \Rightarrow g_a^2$$

$$= (b - c)^2 + 2s(s - a) - s^2 + \frac{4s(s - b)(s - c)}{a}$$

$$= s^2 - 2sa + a^2 + (b - c)^2 - a^2 + \frac{4s(s - b)(s - c)}{a}$$

$$= (s - a)^2 + (b - c + a)(b - c - a) + \frac{4s(s - b)(s - c)}{a}$$

$$= (s - a)^2 - 4(s - b)(s - c) + \frac{4s(s - b)(s - c)}{a} = (s - a)^2 + 4(s - b)(s - c) \left(\frac{s}{a} - 1\right)$$

$$= (s - a)^2 + \frac{4(s - a)(s - b)(s - c)}{a}$$

$$\Rightarrow g_a^2 = (s - a)^2 + \frac{4(s - a)(s - b)(s - c)}{a} = (s - a)^2 + (s - a) \left(\frac{a^2 - (b - c)^2}{a}\right)$$

$$= (s - a) \left(s - a + a - \frac{(b - c)^2}{a}\right) = \left(\frac{s - a}{a}\right) (as - (b - c)^2)$$

$$\Rightarrow \frac{g_a^2}{h_a^2} = \left(\frac{s - a}{a}\right) (as - (b - c)^2) \left(\frac{a^2}{4s^2 r^2}\right) = \frac{a(s - a)(as - (b - c)^2)}{4s^2 r^2} \text{ and analogs}$$

$$\Rightarrow 2 \sum \frac{g_a^2}{h_a^2} \stackrel{(3)}{\equiv} \frac{1}{2s^2 r^2} \sum (a(s - a)(as - (b - c)^2))$$



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$$\begin{aligned}
 \text{Now, } \sum (a(s-a)(as-(b-c)^2)) &= s \sum a^2(s-a) - \sum a(s-a)(b^2+c^2-2bc) \\
 &= s^2 \sum a^2 - s \sum a^3 + 2abc \sum (s-a) - \sum a(s-a)(b^2+c^2) \\
 &= 2s^2(s^2 - 4Rr - r^2) - 2s^2(s^2 - 6Rr - 3r^2) + 2sabc - s \sum a(b^2+c^2) \\
 &\quad + \sum a^2(b^2+c^2) \\
 &= 2s^2(2Rr+2r^2) + 2 \sum a^2b^2 + 2sabc - s \sum ab(2s-c) \\
 &= 2 \left(\sum a^2b^2 + 4sabc \right) - 3sabc + 4s^2(Rr+r^2) - 2s^2 \sum ab \\
 &= 2 \left(\sum ab \right)^2 - 2s^2 \sum ab + 4s^2(Rr+r^2) - 12Rrs^2 \\
 &= 2(4Rr+r^2)(s^2+4Rr+r^2) + 4s^2(Rr+r^2) - 12Rrs^2 \\
 \therefore \sum (a(s-a)(as-(b-c)^2)) &\stackrel{(4)}{\cong} 2(4Rr+r^2)(s^2+4Rr+r^2) + 4s^2(Rr+r^2) \\
 &\quad - 12Rrs^2 \\
 (3), (4) \Rightarrow 2 \sum \frac{g_a^2}{h_a^2} &= \frac{(4R+r)(s^2+4Rr+r^2) + 2s^2(R+r) - 6Rs^2}{s^2r} \\
 &= \frac{(4R+r)^2 + 3s^2}{s^2} \stackrel{\text{by (2)}}{\geq} 2 + \sum \frac{b^2+c^2}{n_a^2+g_a^2} \text{ (Proved)}
 \end{aligned}$$

1787. In any ΔABC , n_a – Nagel's cevian, holds:

$$\sqrt{4 - \frac{2r}{R} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right)} \leq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } \Rightarrow b^2(s-c) + c^2(s-b) &= an_a^2 + a(s-b)(s-c) \\
 \Rightarrow s(b^2+c^2) - bc(2s-a) &= an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc \\
 &= an_a^2 + a(as-s^2)
 \end{aligned}$$



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$$\begin{aligned}
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bcc\cos A - 2bc) \\
 &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 + 2h_a r_a = s^2 \\
 &\Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s+n_a} = \frac{sn_a + n_a^2 + 2h_a r_a}{h_a(s+n_a)} = \frac{sn_a + s^2}{h_a(s+n_a)} \\
 &= \frac{s(s+n_a)}{h_a(s+n_a)} \Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s+n_a} = \frac{s}{h_a} \text{ and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) = s \sum \frac{1}{h_a} \\
 &\Rightarrow \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) \stackrel{(1)}{\cong} \frac{s}{r}
 \end{aligned}$$

We shall now prove : $R(4R+r)^2 \stackrel{(2)}{\geq} (4R-2r)s^2$

Rouche
Now, RHS of (2) $\stackrel{(2)}{\geq} (4R-2r)$

$$- 2r) \left(2R^2 + 10Rr - r^2 + 2(R-2r)\sqrt{R^2 - 2Rr} \right) \stackrel{?}{\geq} R(4R+r)^2$$

$$\begin{aligned}
 \Leftrightarrow R(4R+r)^2 - (2R^2 + 10Rr - r^2)(4R-2r) &\stackrel{?}{\geq} 2(4R-2r)(R-2r)\sqrt{R^2 - 2Rr} \\
 \Leftrightarrow (R-2r)(8R^2 - 12Rr + r^2) &\stackrel{?}{\geq} 2(4R-2r)(R-2r)\sqrt{R^2 - 2Rr} \quad \text{(i)}
 \end{aligned}$$

Euler
 $\because R-2r \stackrel{?}{\geq} \therefore$ in order to prove (i), it suffices to prove : $8R^2 - 12Rr + r^2$

$$> 2(4R-2r)\sqrt{R^2 - 2Rr}$$

$$\begin{aligned}
 \Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 4(R^2 - 2Rr)(4R-2r)^2 &> 0 \Leftrightarrow r^2(4R+r)^2 > 0 \rightarrow \text{true} \\
 \Rightarrow (i) \Rightarrow (2) \text{ is true} \therefore R(4R+r)^2 &\geq (4R-2r)s^2
 \end{aligned}$$

$$\Rightarrow \frac{R(4R+r)^2}{r^2} \geq \frac{(4R-2r)s^2}{r^2} \stackrel{\text{by (1)}}{\Rightarrow} \frac{R(4R+r)^2}{r^2} \geq (4R-2r) \left(\sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) \right)^2$$

$$\Rightarrow \frac{4R-2r}{R} \left(\sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) \right)^2 \leq \frac{(4R+r)^2}{r^2}$$

$$\Rightarrow \sqrt{4 - \frac{2r}{R} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right)} \leq 1 + \frac{4R}{r} \quad (\text{Proved})$$



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1788. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\prod \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) \geq \frac{1}{8} \prod \frac{n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{rr_a}}{h_a - r}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

Proof : Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b)$

$$= an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c)$$

$$= ag_a^2 + a(s - b)(s - c)$$

Adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$= an_a^2 + ag_a^2 + 2a(s - b)(s - c)$$

$$\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2)$$

$$= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2$$

$$\Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2)$$

$$\Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 = 2(n_a^2 + g_a^2)$$

$$\Rightarrow n_a^2 + g_a^2 = (b - c)^2 + 2s(s - a) \quad CBS$$

$$\Rightarrow n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{rr_a} \stackrel{(i)}{\leq} \sqrt{4\sqrt{n_a^2 + g_a^2 + 2r_b r_c + 4rr_a}}$$

$$= 2 \sqrt{(b - c)^2 + 2s(s - a) + 2s(s - a) + 4 \left(\frac{r^2 s}{s - a} \right)}$$

$$= 2 \sqrt{(b - c)^2 + 4s(s - a) + 4 \left(\frac{(s - a)(s - b)(s - c)}{s - a} \right)}$$

$$= 2 \sqrt{(b - c)^2 + (b + c + a)(b + c - a) + (c + a - b)(a + b - c)}$$

$$= 2\sqrt{(b - c)^2 + (b + c)^2 - a^2 + a^2 - (b - c)^2} = 2(b + c)$$

$$\Rightarrow \frac{n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{rr_a}}{2(h_a - r)} \leq \frac{b + c}{\frac{2rs}{a} - r} = \frac{a(b + c)}{r(2s - a)} = \frac{a}{r}$$

$$\therefore \frac{n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{rr_a}}{2(h_a - r)} \stackrel{(ii)}{\leq} \frac{a}{r}$$

Now, $r_b + r_c = s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$

$$\therefore r_b + r_c \stackrel{(ii)}{\leq} 4R \cos^2 \frac{A}{2}$$



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$$\begin{aligned}
 \prod \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right) &= \prod \left(\frac{s}{r_b} + \frac{s}{r_c} \right) = \prod \frac{s(r_b + r_c)}{r_b r_c} \stackrel{\text{by (ii) and analogs}}{\cong} \frac{\prod \left(4R s \cos^2 \frac{A}{2} \right)}{(r_a r_b r_c)^2} \\
 &= \frac{64R^3 s^3 \left(\frac{s^2}{16R^2} \right)}{r^2 s^4} = \frac{4Rs}{r^2} = \frac{4Rrs}{r^3} = \frac{abc}{r^3} = \prod \frac{a}{r} \\
 &\stackrel{\text{by (i) and analogs}}{\cong} \frac{1}{8} \prod \frac{n_a + g_a + \sqrt{2r_b r_c} + 2\sqrt{rr_a}}{h_a - r} \quad (\text{Proved})
 \end{aligned}$$

1789. In any } ABC, n_a – Nagel's cevian, holds:

$$\sqrt{4 - \frac{2r}{R} \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right)} \leq 1 + \frac{4R}{r}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= a n_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\
 &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s - a} \right) = as^2 - 2ah_a r_a \therefore n_a^2 + 2h_a r_a = s^2 \\
 \Rightarrow \frac{n_a}{r_a} + \frac{2h_a}{s + n_a} &= \frac{sn_a + n_a^2 + 2h_a r_a}{r_a(s + n_a)} = \frac{sn_a + s^2}{r_a(s + n_a)} \\
 = \frac{s(s + n_a)}{r_a(s + n_a)} &\Rightarrow \frac{n_a}{r_a} + \frac{2h_a}{s + n_a} = \frac{s}{r_a} \text{ and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right) = s \sum \frac{1}{r_a} \\
 \Rightarrow \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right) &\stackrel{(1)}{\cong} \frac{s}{r} \\
 \text{We shall now prove : } R(4R + r)^2 &\stackrel{(2)}{\cong} (4R - 2r)s^2
 \end{aligned}$$



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Rouche

Now, RHS of (2) $\stackrel{?}{\leq}$ (4R

$$- 2r) \left(2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R^2 - 2Rr} \right) \stackrel{?}{\leq} R(4R + r)^2$$

$$\Leftrightarrow R(4R + r)^2 - (2R^2 + 10Rr - r^2)(4R - 2r) \stackrel{?}{\geq} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow (R - 2r)(8R^2 - 12Rr + r^2) \stackrel{?}{\geq} 2(4R - 2r)(R - 2r)\sqrt{R^2 - 2Rr}$$

(i)

Euler

$\therefore R - 2r \stackrel{?}{\geq} \therefore$ in order to prove (i), it suffices to prove : $8R^2 - 12Rr + r^2$

$$> 2(4R - 2r)\sqrt{R^2 - 2Rr}$$

$$\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 4(R^2 - 2Rr)(4R - 2r)^2 > 0 \Leftrightarrow r^2(4R + r)^2 > 0 \rightarrow \text{true}$$

$$\Rightarrow (i) \Rightarrow (2) \text{ is true} \therefore R(4R + r)^2 \geq (4R - 2r)s^2$$

$$\Rightarrow \frac{R(4R + r)^2}{r^2} \geq \frac{(4R - 2r)s^2}{r^2} \stackrel{\text{by (1)}}{\Rightarrow} \frac{R(4R + r)^2}{r^2} \geq (4R - 2r) \left(\sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right) \right)^2$$

$$\Rightarrow \frac{4R - 2r}{R} \left(\sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right) \right)^2 \leq \frac{(4R + r)^2}{r^2}$$

$$\Rightarrow \sqrt{4 - \frac{2r}{R} \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s + n_a} \right)} \leq 1 + \frac{4R}{r} \quad (\text{Proved})$$

1790. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\frac{\sqrt{R}}{2r} \sum \sqrt{m_a + h_a} \geq \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s + n_a} \right)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

Proof : Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c)$
 $\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc$
 $= an_a^2 + a(as - s^2)$



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$$\begin{aligned}
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 + 2h_a r_a = s^2 \\
 &\Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s+n_a} = \frac{sn_a + n_a^2 + 2h_a r_a}{h_a(s+n_a)} = \frac{sn_a + s^2}{h_a(s+n_a)} \\
 &= \frac{s(s+n_a)}{h_a(s+n_a)} \Rightarrow \frac{n_a}{h_a} + \frac{2r_a}{s+n_a} = \frac{s}{h_a} \text{ and analogs} \quad \stackrel{\text{summing up}}{\Rightarrow} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) = s \sum \frac{1}{h_a} \\
 &\Rightarrow \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) \stackrel{(1)}{\cong} \frac{s}{r} \\
 \frac{\sqrt{R}}{2r} \sum \sqrt{m_a + h_a} &\stackrel{\text{Tereshin}}{\geq} \frac{\sqrt{R}}{2r} \sum \sqrt{\frac{b^2 + c^2}{4R} + \frac{2bc}{4R}} = \frac{\sqrt{R}}{2r} \sum \sqrt{\frac{(b+c)^2}{4R}} = \frac{\sum(b+c)}{4r} = \frac{4s}{4r} \\
 &= \frac{s}{r} \stackrel{\text{by (1)}}{\cong} \sum \left(\frac{n_a}{h_a} + \frac{2r_a}{s+n_a} \right) \text{ (Proved)}
 \end{aligned}$$

1791. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\frac{\sqrt{R}}{2r} \sum \sqrt{m_a + h_a} \geq \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s+n_a} \right)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } \Rightarrow b^2(s - c) + c^2(s - b) &= an_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= an_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)}
 \end{aligned}$$



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$$\begin{aligned}
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 + 2h_a r_a = s^2 \\
 &\Rightarrow \frac{n_a}{r_a} + \frac{2h_a}{s+n_a} = \frac{sn_a + n_a^2 + 2h_a r_a}{r_a(s+n_a)} = \frac{sn_a + s^2}{r_a(s+n_a)} \\
 &= \frac{s(s+n_a)}{r_a(s+n_a)} \Rightarrow \frac{n_a}{r_a} + \frac{2h_a}{s+n_a} = \frac{s}{r_a} \text{ and analogs} \stackrel{\text{summing up}}{\Rightarrow} \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s+n_a} \right) = s \sum \frac{1}{r_a} \\
 &\Rightarrow \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s+n_a} \right) \stackrel{(1)}{\cong} \frac{s}{r} \\
 &\frac{\sqrt{R}}{2r} \sum \sqrt{m_a + h_a} \stackrel{\text{Tereshin}}{\geq} \frac{\sqrt{R}}{2r} \sum \sqrt{\frac{b^2 + c^2}{4R} + \frac{2bc}{4R}} = \frac{\sqrt{R}}{2r} \sum \sqrt{\frac{(b+c)^2}{4R}} = \frac{\sum(b+c)}{4r} = \frac{4s}{4r} \\
 &= \frac{s}{r} \stackrel{\text{by (1)}}{\cong} \sum \left(\frac{n_a}{r_a} + \frac{2h_a}{s+n_a} \right) \text{ (Proved)}
 \end{aligned}$$

1792. In any ΔABC , n_a – Nagel's cevian, holds:

$$(1 - \cos A) \left(1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}} \right) \leq \frac{r_a}{h_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= a n_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(1)}{\cong} s^2 - 2h_a r_a
 \end{aligned}$$



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$$\begin{aligned}
 \text{Now, } \frac{n_a}{h_a} \leq \frac{R}{r} - 1 &\Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{n_a^2}{h_a^2} \stackrel{\text{by (1)}}{\Leftrightarrow} \frac{R^2}{r^2} - \frac{2R}{r} + 1 \geq \frac{s^2 - 2h_a r_a}{h_a^2} = \frac{s^2 a^2}{4r^2 s^2} - \frac{2r_a}{h_a} \\
 &= \frac{a^2}{4r^2} - \left(\frac{2rs}{s-a}\right)\left(\frac{a}{2rs}\right) = \frac{a^2}{4r^2} - \frac{(a-s)+s}{s-a} \\
 &= \frac{a^2}{4r^2} + 1 - \frac{s}{s-a} = 1 + \frac{a^2(s-a) - 4(sr^2)}{4(s-a)r^2} = 1 + \frac{a^2(s-a) - 4(s-a)(s-b)(s-c)}{4(s-a)r^2} \\
 &= 1 + \frac{a^2 - (a^2 - (b-c)^2)}{4r^2} = 1 + \frac{(b-c)^2}{4r^2} \\
 \Leftrightarrow \frac{R^2}{r^2} - \frac{2R}{r} &\geq \frac{(b-c)^2}{4r^2} \Leftrightarrow \frac{R(R-2r)}{r^2} \geq \frac{b^2 + c^2 - 2bc}{4r^2} \Leftrightarrow R-2r \geq \frac{b^2 + c^2}{4R} - \frac{bc}{2R} \\
 \Leftrightarrow R\left(1 - \frac{2r}{R}\right) &\geq \frac{4R^2(\sin^2 B + \sin^2 C)}{4R} - \frac{4R^2 \sin B \sin C}{2R} \Leftrightarrow 1 - \frac{8R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R} \\
 &\geq \sin^2 B + \sin^2 C - 2 \sin B \sin C = (\sin B - \sin C)^2 \\
 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(2 \sin \frac{B}{2} \sin \frac{C}{2}\right) &\geq \left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}\right)^2 \\
 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \left(\cos \frac{B-C}{2} - \cos \frac{B+C}{2}\right) &\geq 4 \sin^2 \frac{A}{2} \left(1 - \cos^2 \frac{B-C}{2}\right) \\
 \Leftrightarrow 1 - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 4 \sin^2 \frac{A}{2} &\geq 4 \sin^2 \frac{A}{2} - 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} \\
 \Leftrightarrow 4 \sin^2 \frac{A}{2} \cos^2 \frac{B-C}{2} - 4 \sin \frac{A}{2} \cos \frac{B-C}{2} + 1 &\geq 0 \\
 \Leftrightarrow \left(2 \sin \frac{A}{2} \cos \frac{B-C}{2} - 1\right)^2 &\geq 0 \rightarrow \text{true} \Rightarrow \frac{n_a}{h_a} \leq \frac{R}{r} - 1 \text{ and analogs} \Rightarrow \frac{n_a n_b n_c}{h_a h_b h_c} \\
 &\leq \left(\frac{R}{r} - 1\right)^3 \Rightarrow \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}} \leq \frac{R}{r} - 1 \Rightarrow 1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}} \leq \frac{R}{r} \\
 \Rightarrow \frac{(1 - \cos A) h_a}{r_a} \left(1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}}\right) &\leq \left(\frac{2 \sin^2 \frac{A}{2} \cdot 2rs}{4R \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2}}\right) \left(\frac{R}{r}\right) = 1 \\
 \Rightarrow (1 - \cos A) \left(1 + \sqrt[3]{\frac{n_a n_b n_c}{h_a h_b h_c}}\right) &\leq \frac{r_a}{h_a} \text{ (Proved)}
 \end{aligned}$$



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1793. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$2 \sum \frac{n_a g_a}{h_a w_a} \geq \sum \frac{m_b + m_c}{m_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= a n_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc \sin^2 \frac{A}{2} \stackrel{(i)}{\cong} as^2 - 4s(s - b)(s - c) \\
 &= as^2 - \frac{as(c + a - b)(a + b - c)}{a} = as^2 - as \left(\frac{a^2 - (b - c)^2}{a} \right) \Rightarrow n_a^2 \\
 &= s \left(s - \frac{a^2 - (b - c)^2}{a} \right) \Rightarrow n_a^2 \stackrel{(1)}{\cong} s \left(s - a + \frac{(b - c)^2}{a} \right)
 \end{aligned}$$

Also, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b)$

$$\begin{aligned}
 &= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\
 &= a g_a^2 + a(s - b)(s - c),
 \end{aligned}$$

and adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$\begin{aligned}
 &= a n_a^2 + a g_a^2 + 2a(s - b)(s - c) \\
 \Rightarrow 2a(b^2 + c^2) &= 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\
 &= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\
 \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 &= 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 \Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 &= 2(n_a^2 + g_a^2) \\
 \Rightarrow n_a^2 + g_a^2 \stackrel{(ii)}{\cong} (b - c)^2 + 2s(s - a) &\because (i), (ii) \Rightarrow g_a^2 \\
 &= (b - c)^2 + 2s(s - a) - s^2 + \frac{4s(s - b)(s - c)}{a}
 \end{aligned}$$



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$$\begin{aligned}
 &= s^2 - 2sa + a^2 + (b - c)^2 - a^2 + \frac{4s(s - b)(s - c)}{a} \\
 &= (s - a)^2 + (b - c + a)(b - c - a) + \frac{4s(s - b)(s - c)}{a} \\
 &= (s - a)^2 - 4(s - b)(s - c) + \frac{4s(s - b)(s - c)}{a} = (s - a)^2 + 4(s - b)(s - c) \left(\frac{s}{a} - 1 \right) \\
 &= (s - a)^2 + \frac{4(s - a)(s - b)(s - c)}{a} \\
 &= (s - a) \left(s - a + \frac{a^2 - (b - c)^2}{a} \right) \stackrel{(2)}{\cong} (s - a) \left(s - \frac{(b - c)^2}{a} \right) \therefore (1), (2) \Rightarrow n_a^2 g_a^2 \\
 &= s(s - a) \left(s - a + \frac{(b - c)^2}{a} \right) \left(s - \frac{(b - c)^2}{a} \right) \\
 &= s(s - a) \left(s(s - a) + s \frac{(b - c)^2}{a} - \frac{(b - c)^2}{a} (s - a) - \frac{(b - c)^4}{a^2} \right) \\
 &\stackrel{(a)}{\cong} n_a^2 g_a^2 \stackrel{(a)}{\cong} s(s - a) \left(s(s - a) + (b - c)^2 - \frac{(b - c)^4}{a^2} \right) \\
 \text{Again, } m_a^2 w_a^2 &= \frac{(b - c)^2 + 4s(s - a)}{4} \cdot \frac{4bc s(s - a)}{(b + c)^2} \\
 &\stackrel{(b)}{\cong} m_a^2 w_a^2 \stackrel{(b)}{\cong} s(s - a) \frac{bc}{(b + c)^2} ((b - c)^2 + 4s(s - a)) \therefore (a), (b) \\
 &\Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2 \\
 &= s(s - a) \left(s(s - a) + (b - c)^2 - \frac{(b - c)^4}{a^2} - \frac{bc}{(b + c)^2} ((b - c)^2 + 4s(s - a)) \right) \\
 &= s(s - a) \left(s(s - a) + (b - c)^2 \left(\frac{a^2 - (b - c)^2}{a^2} \right) - \frac{bc}{(b + c)^2} ((b - c)^2 + (b + c)^2 - a^2) \right) \\
 &= s(s - a) \left(s(s - a) - bc + (a^2 - (b - c)^2) \left(\frac{(b - c)^2}{a^2} + \frac{bc}{(b + c)^2} \right) \right) \\
 &= \frac{s(s - a)}{4} \left(((b + c)^2 - a^2 - 4bc) + (a^2 - (b - c)^2) \left(\frac{4(b - c)^2}{a^2} + \frac{4bc}{(b + c)^2} \right) \right)
 \end{aligned}$$



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$$\begin{aligned}
 &= \frac{s(s-a)}{4} \left((b-c)^2 - a^2 + (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} \right) \right) \\
 &= \frac{s(s-a)}{4} (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} - 1 \right) \\
 &= \frac{s(s-a)}{4} \cdot 4(s-b)(s-c) \left(\frac{4(b-c)^2}{a^2} - \frac{(b-c)^2}{(b+c)^2} \right) = r^2 s^2 (b-c)^2 \left(\frac{4}{a^2} - \frac{1}{(b+c)^2} \right) \\
 &= r^2 s^2 (b-c)^2 \left(\frac{2}{a} + \frac{1}{b+c} \right) \left(\frac{2b+2c-a}{a(b+c)} \right) \geq 0 \\
 \Rightarrow n_a^2 g_a^2 &\geq m_a^2 w_a^2 \Rightarrow \frac{2n_a g_a}{h_a w_a} \stackrel{(m)}{\geq} \frac{2m_a}{h_a w_a} \Rightarrow \frac{2n_a g_a}{h_a w_a} \geq \frac{2m_a}{h_a} \text{ and analogs} \\
 \Rightarrow 2 \sum \frac{n_a g_a}{h_a w_a} &\stackrel{(m)}{\geq} \sum \frac{2m_a}{h_a} \\
 \text{Now, Tereshin} \Rightarrow m_a &\geq \frac{b^2 + c^2}{4R} \Rightarrow \frac{4RSm_a}{S} \geq b^2 + c^2 \Rightarrow \frac{abcm_a}{S} \geq b^2 + c^2 \Rightarrow \frac{am_a}{S} \\
 &\geq \frac{b}{c} + \frac{c}{b}
 \end{aligned}$$

applying which on a triangle with sides $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ whose area of course

$$= \frac{S}{3} \text{ and medians of course} = \frac{a}{2}, \frac{b}{2}, \frac{c}{2} \text{ we get :}$$

$$\begin{aligned}
 \frac{\left(\frac{2m_a}{3}\right)\left(\frac{a}{2}\right)}{\frac{S}{3}} &\geq \frac{\frac{2m_b}{3}}{\frac{2m_c}{3}} + \frac{\frac{2m_c}{3}}{\frac{2m_b}{3}} \Rightarrow \frac{2m_a}{\left(\frac{2S}{a}\right)} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \Rightarrow \frac{2m_a}{h_a} \geq \frac{m_b}{m_c} + \frac{m_c}{m_b} \text{ and analogs} \\
 \Rightarrow \sum \frac{2m_a}{h_a} &\geq \sum \left(\frac{m_b}{m_c} + \frac{m_c}{m_b} \right) = \sum \left(\frac{m_c}{m_a} + \frac{m_b}{m_a} \right) \\
 \Rightarrow \sum \frac{2m_a}{h_a} &\stackrel{(n)}{\geq} \sum \frac{m_b + m_c}{m_a} \therefore (m), (n) \Rightarrow 2 \sum \frac{n_a g_a}{h_a w_a} \geq \sum \frac{m_b + m_c}{m_a} \text{ (Proved)}
 \end{aligned}$$

1794. In any ΔABC , , n_a – Nagel's cevian, holds:

$$2 \sum \frac{h_a}{s-n_a} = \frac{s}{r} + \sum \frac{n_a}{r_a}$$

Proposed by Bogdan Fuștei – Romania



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Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= an_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\
 &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_ar_a \therefore n_a^2 \stackrel{(1)}{\cong} s^2 - 2h_ar_a \\
 \text{Now, } 2 \sum \frac{h_a}{s - n_a} &= 2 \sum \frac{h_a(s + n_a)}{(s - n_a)(s + n_a)} \\
 &= 2 \sum \frac{h_a(s + n_a)}{s^2 - n_a^2} \stackrel{\text{by (1) and analogs}}{\cong} 2 \sum \frac{h_a(s + n_a)}{2h_ar_a} = \sum \frac{s + n_a}{r_a} \\
 &= s \sum \frac{1}{r_a} + \sum \frac{n_a}{r_a} \\
 &= \frac{s}{r} + \sum \frac{n_a}{r_a} \text{ (Proved)}
 \end{aligned}$$

1795. In any } ABC, , n_a – Nagel's cevian, holds:

$$\sum \cot^2 \frac{A}{2} = \sum \frac{n_a^2}{r_a^2} + 2 \sum \frac{h_a}{r_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= an_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}
 \end{aligned}$$



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$$\begin{aligned}
 &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a\left(\frac{2\Delta}{a}\right)\left(\frac{\Delta}{s-a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 = s^2 - 2h_a r_a \Rightarrow \frac{n_a^2}{r_a^2} \\
 &= \frac{s^2}{s^2 \tan^2 \frac{A}{2}} - 2 \frac{h_a}{r_a} \text{ and analogs} \\
 \Rightarrow \sum \frac{n_a^2}{r_a^2} &= \sum \cot^2 \frac{A}{2} - 2 \sum \frac{h_a}{r_a} \Rightarrow \sum \cot^2 \frac{A}{2} = \sum \frac{n_a^2}{r_a^2} + 2 \sum \frac{h_a}{r_a} \text{ (Proved)}
 \end{aligned}$$

1796. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \frac{n_a g_a}{w_a} \geq \sqrt{\sum h_a h_b} \sum \tan \frac{A}{2}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : Stewart's theorem } &\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\
 \Rightarrow s(b^2 + c^2) - bc(2s-a) &= an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\
 &= an_a^2 + a(as - s^2) \\
 \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\
 &= as^2 - 4sbc \sin^2 \frac{A}{2} \stackrel{(i)}{\cong} as^2 - 4s(s-b)(s-c) \\
 &= as^2 - \frac{as(c+a-b)(a+b-c)}{a} = as^2 - as \left(\frac{a^2 - (b-c)^2}{a} \right) \Rightarrow n_a^2 \\
 &= s \left(s - \frac{a^2 - (b-c)^2}{a} \right) \Rightarrow n_a^2 \stackrel{(1)}{\cong} s \left(s - a + \frac{(b-c)^2}{a} \right)
 \end{aligned}$$

Also, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b)$

$$= an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c)$$

$$= ag_a^2 + a(s-b)(s-c),$$

and adding the above two, we get : $(b^2 + c^2)(2s - b - c)$

$$= an_a^2 + ag_a^2 + 2a(s-b)(s-c)$$

$$\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a+b-c)(c+a-b) \Rightarrow 2(b^2 + c^2)$$

$$= 2(n_a^2 + g_a^2) + a^2 - (b-c)^2$$



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$$\begin{aligned}
 & \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 & \Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 & \stackrel{(ii)}{\Rightarrow} n_a^2 + g_a^2 \stackrel{(ii)}{\cong} (b - c)^2 + 2s(s - a) \therefore (i), (ii) \Rightarrow g_a^2 \\
 & = (b - c)^2 + 2s(s - a) - s^2 + \frac{4s(s - b)(s - c)}{a} \\
 & = s^2 - 2sa + a^2 + (b - c)^2 - a^2 + \frac{4s(s - b)(s - c)}{a} \\
 & = (s - a)^2 + (b - c + a)(b - c - a) + \frac{4s(s - b)(s - c)}{a} \\
 & = (s - a)^2 - 4(s - b)(s - c) + \frac{4s(s - b)(s - c)}{a} = (s - a)^2 + 4(s - b)(s - c) \left(\frac{s}{a} - 1 \right) \\
 & = (s - a)^2 + \frac{4(s - a)(s - b)(s - c)}{a} \\
 & = (s - a) \left(s - a + \frac{a^2 - (b - c)^2}{a} \right) \Rightarrow g_a^2 \stackrel{(2)}{\cong} (s - a) \left(s - \frac{(b - c)^2}{a} \right) \therefore (1), (2) \Rightarrow n_a^2 g_a^2 \\
 & = s(s - a) \left(s - a + \frac{(b - c)^2}{a} \right) \left(s - \frac{(b - c)^2}{a} \right) \\
 & = s(s - a) \left(s(s - a) + s \frac{(b - c)^2}{a} - \frac{(b - c)^2}{a} (s - a) - \frac{(b - c)^4}{a^2} \right) \\
 & \Rightarrow n_a^2 g_a^2 \stackrel{(a)}{\cong} s(s - a) \left(s(s - a) + (b - c)^2 - \frac{(b - c)^4}{a^2} \right) \\
 & \text{Again, } m_a^2 w_a^2 = \frac{(b - c)^2 + 4s(s - a)}{4} \cdot \frac{4bc s(s - a)}{(b + c)^2} \\
 & \Rightarrow m_a^2 w_a^2 \stackrel{(b)}{\cong} s(s - a) \frac{bc}{(b + c)^2} ((b - c)^2 + 4s(s - a)) \therefore (a), (b) \\
 & \Rightarrow n_a^2 g_a^2 - m_a^2 w_a^2 \\
 & = s(s - a) \left(s(s - a) + (b - c)^2 - \frac{(b - c)^4}{a^2} - \frac{bc}{(b + c)^2} ((b - c)^2 + 4s(s - a)) \right) \\
 & = s(s - a) \left(s(s - a) + (b - c)^2 \left(\frac{a^2 - (b - c)^2}{a^2} \right) - \frac{bc}{(b + c)^2} ((b - c)^2 + (b + c)^2 - a^2) \right)
 \end{aligned}$$



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$$\begin{aligned}
 &= s(s-a) \left(s(s-a) - bc + (a^2 - (b-c)^2) \left(\frac{(b-c)^2}{a^2} + \frac{bc}{(b+c)^2} \right) \right) \\
 &= \frac{s(s-a)}{4} \left(((b+c)^2 - a^2 - 4bc) + (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} \right) \right) \\
 &= \frac{s(s-a)}{4} \left((b-c)^2 - a^2 + (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} \right) \right) \\
 &= \frac{s(s-a)}{4} (a^2 - (b-c)^2) \left(\frac{4(b-c)^2}{a^2} + \frac{4bc}{(b+c)^2} - 1 \right) \\
 &= \frac{s(s-a)}{4} \cdot 4(s-b)(s-c) \left(\frac{4(b-c)^2}{a^2} - \frac{(b-c)^2}{(b+c)^2} \right) = r^2 s^2 (b-c)^2 \left(\frac{4}{a^2} - \frac{1}{(b+c)^2} \right) \\
 &= r^2 s^2 (b-c)^2 \left(\frac{2}{a} + \frac{1}{b+c} \right) \left(\frac{2b+2c-a}{a(b+c)} \right) \geq 0
 \end{aligned}$$

$$\Rightarrow n_a^2 g_a^2 \geq m_a^2 w_a^2 \Rightarrow \frac{n_a g_a}{w_a} \geq m_a \text{ and analogs} \Rightarrow \sum \frac{n_a g_a}{w_a} \stackrel{(m)}{\geq} \sum m_a$$

$$\begin{aligned}
 \text{Now, } r_b + r_c &= s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \\
 \therefore r_b + r_c &\stackrel{(iii)}{\cong} 4R \cos^2 \frac{A}{2}
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } (b+c)^2 &\geq 32R \cos^2 \frac{A}{2} \stackrel{\text{by (iii)}}{\cong} 8r(r_b + r_c) = 8r^2 s \left(\frac{1}{s-b} + \frac{1}{s-c} \right)
 \end{aligned}$$

$$= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c$$

$$\geq 4\sqrt{2R} \cos \frac{A}{2} \Rightarrow \sum m_a \stackrel{\text{loscu}}{\geq} \sum \left(\frac{b+c}{2} \cos \frac{A}{2} \right)$$

$$\geq \sqrt{2Rr} \sum 2 \cos^2 \frac{A}{2} = \sqrt{2Rr} \sum (1 + \cos A) = \sqrt{2Rr} \left(4 + \frac{r}{R} \right) = \sqrt{\frac{2r}{R}} \left(\sum r_a \right)$$

$$= \sqrt{\frac{2r}{R}} \left(\sum s \tan \frac{A}{2} \right) \Rightarrow \sum m_a \stackrel{(n)}{\geq} s \sqrt{\frac{2r}{R}} \sum \tan \frac{A}{2}$$



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$$\text{Now, } \sqrt{\sum h_a h_b} = \sqrt{\sum \frac{bc \cdot ca}{4R^2}} = \sqrt{\frac{4Rrs(2s)}{4R^2}} \Rightarrow \sqrt{\sum h_a h_b} \stackrel{(t)}{\cong} s \sqrt{\frac{2r}{R}} \therefore (m), (n), (t)$$

$$\Rightarrow \sum \frac{n_a g_a}{w_a} \geq \sqrt{\sum h_a h_b} \sum \tan \frac{A}{2} \text{ (Proved)}$$

1797. In any } \Delta ABC, n_a - \text{Nagel's cevian, } g_a - \text{Gergonne's cevian holds:}

$$\sum \frac{h_a}{g_a + s - a} \leq \frac{1}{2} \left(\frac{g_a + g_b + g_c}{r} - \frac{1}{\sqrt{2}} \sum \frac{n_a}{r_a} - \sum \sqrt{\frac{h_a}{r_a}} \right)$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Proof : Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) \\ &= an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= ag_a^2 + a(s - b)(s - c) \end{aligned}$$

$$\begin{aligned} \text{Adding the above two, we get : } &(b^2 + c^2)(2s - b - c) \\ &= an_a^2 + ag_a^2 + 2a(s - b)(s - c) \\ \Rightarrow 2a(b^2 + c^2) &= 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\ &= 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\ \Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 &= 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\ &\Rightarrow (b - c)^2 + 4s(s - a) + (b - c)^2 = 2(n_a^2 + g_a^2) \\ &\stackrel{(i)}{\Rightarrow} n_a^2 + g_a^2 \cong (b - c)^2 + 2s(s - a) \end{aligned}$$

$$\begin{aligned} \text{Also, Stewart's theorem } &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\ \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= an_a^2 + a(as - s^2) \\ \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \end{aligned}$$



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$$\begin{aligned}
 &= as^2 - \frac{4as(s-b)(s-c)}{a} \Rightarrow n_a^2 \stackrel{(ii)}{\cong} s^2 - \frac{4s(s-b)(s-c)}{a} \therefore (i), (ii) \Rightarrow g_a^2 \\
 &= (b-c)^2 + 2s(s-a) - s^2 + \frac{4s(s-b)(s-c)}{a} \\
 &= s^2 - 2sa + a^2 + (b-c)^2 - a^2 + \frac{4s(s-b)(s-c)}{a} \\
 &= (s-a)^2 + (b-c+a)(b-c-a) + \frac{4s(s-b)(s-c)}{a} \\
 &= (s-a)^2 - 4(s-b)(s-c) + \frac{4s(s-b)(s-c)}{a} = (s-a)^2 + 4(s-b)(s-c)\left(\frac{s}{a} - 1\right) \\
 &= (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a} \\
 \Rightarrow g_a^2 &= (s-a)^2 + \frac{4(s-a)(s-b)(s-c)}{a} \Rightarrow (g_a + s - a)(g_a - s + a) \\
 &= \frac{4(s-a)(s-b)(s-c)}{a} = 2r\left(\frac{2rs}{a}\right) = 2rh_a \\
 \Rightarrow \frac{h_a}{g_a + s - a} &= \frac{g_a - s + a}{2r} \text{ and analogs} \Rightarrow \sum \frac{h_a}{g_a + s - a} = \frac{\sum(g_a - s + a)}{2r} = \frac{\sum g_a}{2r} - \frac{\sum(s-a)}{2r} \\
 &= \frac{g_a + g_b + g_c}{2r} - \frac{s}{2r} \\
 \Rightarrow \sum \frac{h_a}{g_a + s - a} - \frac{g_a + g_b + g_c}{2r} &\stackrel{(1)}{\cong} -\frac{s}{2r} \\
 \text{Now, } \frac{1}{\sqrt{2}} \sum \frac{n_a}{r_a} + \sum \sqrt{\frac{h_a}{r_a}} &= \sum \left(\frac{1}{\sqrt{2}} \frac{n_a}{r_a} + \sqrt{\frac{h_a}{r_a}} \right) \stackrel{\text{CBS}}{\geq} \sum \sqrt{2} \sqrt{\left(\frac{n_a^2}{2r_a^2} + \frac{h_a}{r_a} \right)} \stackrel{\text{by (ii)}}{\cong} \sum \sqrt{2} \sqrt{\frac{s^2 - \frac{4s(s-b)(s-c)}{a} + 2h_ar_a}{2r_a^2}} \\
 &= \sum \sqrt{2} \sqrt{\frac{s^2 - \frac{4s(s-b)(s-c)}{a} + \frac{4(s-a)s(s-b)(s-c)}{a(s-a)}}{2r_a^2}} = s \sum \frac{1}{r_a} = \frac{s}{r} \\
 \Rightarrow \frac{1}{2} \left(-\frac{1}{\sqrt{2}} \sum \frac{n_a}{r_a} - \sum \sqrt{\frac{h_a}{r_a}} \right) &\stackrel{(2)}{\cong} -\frac{s}{2r}
 \end{aligned}$$



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$$\begin{aligned}
 (1), (2) \Rightarrow \sum \frac{h_a}{g_a + s - a} - \frac{g_a + g_b + g_c}{2r} &\leq \frac{1}{2} \left(-\frac{1}{\sqrt{2}} \sum \frac{n_a}{r_a} - \sum \sqrt{\frac{h_a}{r_a}} \right) \Rightarrow \sum \frac{h_a}{g_a + s - a} \\
 &\leq \frac{1}{2} \left(\frac{g_a + g_b + g_c}{r} - \frac{1}{\sqrt{2}} \sum \frac{n_a}{r_a} - \sum \sqrt{\frac{h_a}{r_a}} \right)
 \end{aligned}$$

1798. In any ΔABC , n_a – Nagel's cevian, g_a – Gergonne's cevian holds:

$$\sum \frac{n_a - g_a}{r_b + r_c - 2h_a} \leq \sum \frac{r_a}{n_a}$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Triangle inequality } \Rightarrow g_a &\leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abc \cos \frac{A}{2}}{a(b+c)} \\
 &\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rr \sin \frac{A}{2} \cos \frac{A}{2}}{4R(b+c) \sin \frac{A}{2} \cos \frac{A}{2}} \\
 &\Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a+b+c}{(b+c) \sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} 1 + \frac{a}{(b+c) \sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c) \sin \frac{A}{2} \stackrel{?}{\leq} a \\
 &\Leftrightarrow 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leq} 4R \sin \frac{A}{2} \cos \frac{A}{2} \\
 &\Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true } \because g_a \leq w_a \leq m_a \Rightarrow m_a \stackrel{(1)}{\geq} g_a
 \end{aligned}$$

Now, Stewart's theorem $\Rightarrow b^2(s-c) + c^2(s-b) = a n_a^2 + a(s-b)(s-c)$

$$\begin{aligned}
 &\Rightarrow 4an_a^2 - 4am_a^2 \\
 &= 4b^2(s-c) + 4c^2(s-b) - 4a(s-b)(s-c) - a(2b^2 + 2c^2 - a^2) \\
 &= 2b^2(a+b-c) + 2c^2(c+a-b) - a(c+a-b)(a+b-c) - a(2b^2 + 2c^2 - a^2) \\
 &\quad = a(b-c)^2 + 2b^3 + 2c^3 - 2b^2c - 2bc^2 \\
 &= a(b-c)^2 + 2(b+c)(b^2 + c^2 - bc) - 2bc(b+c) = a(b-c)^2 + 2(b+c)(b-c)^2 \\
 &\quad = (a+2b+2c)(b-c)^2 \geq 0 \Rightarrow 4an_a^2 \geq 4am_a^2
 \end{aligned}$$

$$\Rightarrow n_a \stackrel{(2)}{\geq} m_a \because (1), (2) \Rightarrow n_a \stackrel{(3)}{\geq} g_a$$



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$$\begin{aligned} \text{Proof : } r_b + r_c &= s \left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left(\frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2} \\ &\therefore r_b + r_c \stackrel{(4)}{\cong} 4R \cos^2 \frac{A}{2} \end{aligned}$$

Now, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b)$

$$\begin{aligned} &= a n_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= a g_a^2 + a(s - b)(s - c) \end{aligned}$$

$$\therefore a n_a^2 \cdot a g_a^2 \geq a^2 s^2 (s - a)^2$$

$$\begin{aligned} &\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) \\ &\quad - a(s - b)(s - c)\} \stackrel{(a)}{\sum} a^2 s^2 (s - a)^2 \end{aligned}$$

Let $s - a = x, s - b = y$ and $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$ and c

$$= x + y$$

Using these substitutions, (a)

$$\begin{aligned} &\Leftrightarrow \{z(z + x)^2 + y(x + y)^2 - yz(y + z)\} \{y(z + x)^2 + z(x + y)^2 \\ &\quad - yz(y + z)\} \geq x^2(y + z)^2(x + y + z)^2 \end{aligned}$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y + z) \Leftrightarrow x(y - z)^2 + (y + z)(y - z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow (a) \text{ is true} \Rightarrow n_a g_a \stackrel{(5)}{\cong} s(s - a) \text{ and analogs}$$

Also, Stewart's theorem $\Rightarrow b^2(s - c) + c^2(s - b) = a n_a^2 + a(s - b)(s - c)$

$$\begin{aligned} &\Rightarrow s(b^2 + c^2) - bc(2s - a) = a n_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= a n_a^2 + a(as - s^2) \end{aligned}$$

$$\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = a n_a^2 - as^2 \Rightarrow a n_a^2 = as^2 + s(2bccosA - 2bc)$$

$$= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)}$$

$$= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left(\frac{2\Delta}{a} \right) \left(\frac{\Delta}{s - a} \right) = as^2 - 2ah_a r_a \therefore n_a^2 = s^2 - 2h_a r_a$$

$$\Rightarrow n_a^2 - n_a g_a \stackrel{\text{by (5)}}{\cong} s^2 - 2h_a r_a - s(s - a)$$

$$\begin{aligned} &= as - 2h_a r_a \Rightarrow n_a^2 - n_a g_a \stackrel{(6)}{\cong} as - 2h_a r_a \end{aligned}$$



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$$\begin{aligned}
 \text{Again, } r_a(r_b + r_c - 2h_a) &\stackrel{\text{by (4)}}{=} s \tan \frac{A}{2} \left(4R \cos^2 \frac{A}{2} \right) - 2h_a r_a = s \cdot 4R \cos \frac{A}{2} \sin \frac{A}{2} - 2h_a r_a \\
 &= as - 2h_a r_a \stackrel{\text{by (6)}}{\geq} n_a(n_a - g_a) \\
 \Rightarrow \frac{n_a - g_a}{r_b + r_c - 2h_a} &\leq \frac{r_a}{n_a} \left(\because n_a \geq g_a \text{ by (3)} \right) \text{ and analogs} \stackrel{\text{summing up}}{\Rightarrow} \sum \frac{n_a - g_a}{r_b + r_c - 2h_a} \\
 &\leq \sum \frac{r_a}{n_a} \text{ (Proved)}
 \end{aligned}$$

1799. In any ΔABC , holds:

$$4(R+r)^3 \geq \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27R^2 r$$

Proposed by Alex Szoros – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned}
 \text{Proof : } \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} &= \frac{a^2(s-a)}{\sqrt{s(s-a)(s-b)(s-c)}} + \frac{2bc\sqrt{(s-b)(s-c)}}{\sqrt{s(s-a)(s-b)(s-c)}} \\
 &= \frac{a^2(s-a) + 2(b\sqrt{s-b})(c\sqrt{s-c})}{\sqrt{s(s-a)(s-b)(s-c)}} \\
 &\stackrel{A-G}{\geq} \frac{a^2(s-a) + b^2(s-b) + c^2(s-c)}{rs} = \frac{s(\sum a^2) - \sum a^3}{rs} \\
 &= \frac{2s(s^2 - 4Rr - r^2) - 2s(s^2 - 6Rr - 3r^2)}{rs} = \frac{2s(2Rr + 2r^2)}{rs} = 4(R+r) \\
 \therefore \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} &\leq 4(R+r) \text{ and analogs} \stackrel{\text{multiplying together}}{\Rightarrow} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \leq 64(R+r)^3 \\
 \Rightarrow 4(R+r)^3 &\stackrel{(1)}{\geq} \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right)
 \end{aligned}$$



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$$\text{Also, } \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} = \frac{a^2}{r_a} + \frac{bc}{\sqrt{r_b r_c}} + \frac{bc}{\sqrt{r_b r_c}} \stackrel{\text{A-G}}{\geq} 3 \sqrt[3]{\left(\frac{a^2}{r_a}\right)\left(\frac{bc}{\sqrt{r_b r_c}}\right)\left(\frac{bc}{\sqrt{r_b r_c}}\right)} = 3 \sqrt[3]{\frac{16R^2 r^2 s^2}{rs^2}}$$

$$\therefore \frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \geq 3 \sqrt[3]{16R^2 r} \text{ and analogs}$$

$$\text{multiplying together} \quad \Rightarrow \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27 \cdot 16R^2 r$$

$$\Rightarrow \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \stackrel{(2)}{\geq} 27R^2 r$$

$$(1), (2) \Rightarrow 4(R+r)^3 \geq \frac{1}{16} \left(\frac{a^2}{r_a} + \frac{2bc}{\sqrt{r_b r_c}} \right) \left(\frac{b^2}{r_b} + \frac{2ca}{\sqrt{r_c r_a}} \right) \left(\frac{c^2}{r_c} + \frac{2ab}{\sqrt{r_a r_b}} \right) \geq 27R^2 r \text{ (Proved)}$$

1800. In any ΔABC , holds:

$$8 \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2} \leq \cos^4 \frac{A-B}{2} + 3 \cos^2 \frac{A-B}{2} + 4$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Proof : } \cos \frac{A-B}{2} \cos \frac{B-C}{2} \cos \frac{C-A}{2} = \prod \left(\frac{b+c}{a} \sin \frac{A}{2} \right) = \frac{2s(s^2 + 2Rr + r^2) \left(\frac{r}{4R} \right)}{4Rrs}$$

$$= \frac{s^2 + 2Rr + r^2}{8R^2} \Rightarrow \text{LHS} \stackrel{(i)}{\equiv} s^2 + 2Rr + r^2$$

$$\text{Now, } r_a + r_b = s \left(\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} + \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} \right) = \frac{s \sin \left(\frac{A+B}{2} \right) \cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{C}{2}}{\left(\frac{s}{4R} \right)} = 4R \cos^2 \frac{C}{2}$$

$$\therefore r_a + r_b \stackrel{(ii)}{\equiv} 4R \cos^2 \frac{C}{2}$$

$$\text{Now, } (a+b)^2 \geq 32Rr \cos^2 \frac{C}{2} \stackrel{\text{by (ii)}}{\equiv} 8r(r_a + r_b) = 8r^2 s \left(\frac{1}{s-a} + \frac{1}{s-b} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{c}{(s-a)(s-b)} = 4c(a+b-c)$$



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$$\begin{aligned} \Leftrightarrow (a+b)^2 + 4c^2 - 4c(a+b) &\geq 0 \Leftrightarrow (a+b-2c)^2 \geq 0 \rightarrow \text{true} \therefore a+b \\ &\geq 4\sqrt{2Rr}\cos\frac{C}{2} \Rightarrow 4R\cos\frac{C}{2}\cos\frac{A-B}{2} \geq 4\sqrt{2Rr}\cos\frac{C}{2} \\ &\Rightarrow \cos^2\frac{A-B}{2} \geq \frac{2r}{R} \Rightarrow \cos^4\frac{A-B}{2} + 3\cos^2\frac{A-B}{2} + 4 \\ &\geq \frac{4r^2}{R^2} + \frac{6r}{R} + 4 \stackrel{?}{\geq} \text{LHS} \stackrel{\text{by (i)}}{\cong} s^2 + 2Rr + r^2 \\ &\Leftrightarrow s^2 + 2Rr + r^2 \stackrel{?}{\leq} 4R^2 + 6Rr + 4r^2 \\ \Leftrightarrow s^2 \stackrel{?}{\leq} 4R^2 + 4Rr + 3r^2 &\rightarrow \text{true (Gerretsen)} \therefore 8\cos\frac{A-B}{2}\cos\frac{B-C}{2}\cos\frac{C-A}{2} \\ &\leq \cos^4\frac{A-B}{2} + 3\cos^2\frac{A-B}{2} + 4 \text{ (Proved)} \end{aligned}$$



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It's nice to be important but more important it's to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru