

The background of the cover is a vibrant space scene. It features a large, bright yellow and orange sun or star in the upper center, casting a glow over the scene. To the left, a large, reddish planet with a dark, cratered surface is visible. In the lower left, a smaller, similar planet is shown. The right side of the image is filled with a field of dark, irregularly shaped asteroids or meteoroids, set against a deep blue and purple nebula-like background.

*RMM - Triangle Marathon 1601 - 1700*

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1601. In any  $\Delta ABC$  holds :

$$\frac{ab^2 + bc^2 + ca^2}{a + b + c} \leq 2R(2R - r)$$

Proposed by Marian Ursărescu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$b^3 + b^3 + a^3 \stackrel{A-G}{\geq} 3ab^2, c^3 + c^3 + b^3 \stackrel{A-G}{\geq} 3bc^2 \text{ and}$$

$$a^3 + a^3 + c^3 \stackrel{A-G}{\geq} 3ca^2 \text{ and adding these 3 inequalities, we get :}$$

$$3 \sum ab^2 \leq 3 \sum a^3 \Rightarrow \frac{ab^2 + bc^2 + ca^2}{a + b + c} \leq \frac{\sum a^3}{2s} = \frac{2s(s^2 - 6Rr - 3r^2)}{2s} \leq 2R(2R - r)$$

$$\Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \rightarrow \text{true (Gerretsen)}$$

$$\Rightarrow \frac{ab^2 + bc^2 + ca^2}{a + b + c} \leq 2R(2R - r) \text{ (Proved)}$$

1602. In any  $\Delta ABC$  holds:

$$\mu + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} \stackrel{(1)}{\geq} \frac{4\mu + 1}{9} (\sin^2 A + \sin^2 B + \sin^2 C)$$

$$\forall \mu \geq 2$$

Proposed by Marin Chirciu – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$(1) \Leftrightarrow \mu \left( 1 - \frac{4}{9} \sum \sin^2 A \right) + \frac{\sin A \sin B \sin C}{\sin A + \sin B + \sin C} - \frac{\sum \sin^2 A}{9} \stackrel{(2)}{\geq} 0$$

$$\text{Now, } \frac{4}{9} \sum \sin^2 A \stackrel{\text{Leibnitz}}{\leq} 1 \Rightarrow 1 - \frac{4}{9} \sum \sin^2 A \geq 0 \text{ and } \therefore \mu \geq 2$$

$$\therefore \text{LHS of (2)} \geq 2 \left( 1 - \frac{4}{9} \sum \sin^2 A \right) + \frac{\sin A \sin B \sin C}{\sum \sin A} - \frac{\sum \sin^2 A}{9} \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 2 \left( \sum \sin A \right) \left( 9 - 4 \sum \sin^2 A \right) + 9 \sin A \sin B \sin C - \left( \sum \sin^2 A \right) \left( \sum \sin A \right) \stackrel{?}{\geq} 0$$

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$$\Leftrightarrow 2 \sum \sin A + \sin A \sin B \sin C - \left( \sum \sin^2 A \right) \left( \sum \sin A \right) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow \frac{2s}{R} + \frac{4Rrs}{8R^3} - \frac{s}{R} \left( \frac{s^2 - 4Rr - r^2}{2R^2} \right) \stackrel{?}{\geq} 0$$

$$\Leftrightarrow 4sR^2 + Rrs - s(s^2 - 4Rr - r^2) \stackrel{?}{\geq} 0 \Leftrightarrow s^2 - 4Rr - r^2 \stackrel{?}{\geq} 4R^2 + Rr$$

$$\Leftrightarrow s^2 \stackrel{?}{\geq} 4R^2 + 5Rr + r^2 \quad (3)$$

Now, LHS of (3)  $\stackrel{\text{Gerretsen}}{\geq} 4R^2 + 4Rr + 3r^2 \stackrel{?}{\geq} 4R^2 + 5Rr + r^2 \Leftrightarrow Rr \stackrel{?}{\geq} 2r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r$   
 $\rightarrow$  true (Euler)

$\Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$  is true (Proved)

1603. In  $\triangle ABC$  the following relationship holds:

$$\frac{\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2}}{\sin^2 A + \sin^2 B + \sin^2 C} \leq \frac{R}{2r}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution by Daniel Sitaru – Romania

$$\begin{aligned} & \left( \sum_{cyc} \cos^2 \frac{A}{2} \right) \left( \sum_{cyc} \sin^2 A \right)^{-1} = \\ & = \left( 2 + \frac{r}{2R} \right) \left( \frac{1}{4R^2} \sum_{cyc} a^2 \right)^{-1} = \\ & = 4R^2 \left( 2 + \frac{r}{2R} \right) \cdot \frac{1}{s^2 - r^2 - 4Rr} \stackrel{\text{GERRETSEN}}{\geq} \\ & = 4R^2 \cdot \frac{4R + r}{2R} \cdot \frac{1}{16Rr - 5r^2 - r^2 - 4Rr} = \frac{2R(4R + r)}{12Rr - 6r^2} = \\ & = \frac{R}{6r} \cdot \frac{4R + r}{2R - r} \stackrel{\text{EULER}}{\geq} \frac{R}{6r} \cdot \frac{4R + \frac{R}{2}}{2R - \frac{R}{2}} = \frac{R}{6r} \cdot \frac{9R}{3R} = \frac{R}{2r} \end{aligned}$$

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**1604. In  $\triangle ABC$  the following relationship holds:**

$$\left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n + 9 \left(\frac{ab + bc + ca}{4s^2}\right)^2 \geq 4, n \in \mathbb{N}, n \geq 2$$

*Proposed by Marin Chirciu-Romania*

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

For  $n \in \mathbb{N}, n \geq 2$ :

$$\begin{aligned} \left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n &\geq \frac{\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^n}{3^{n-1}} = \frac{\left(\frac{a^2}{ab} + \frac{b^2}{bc} + \frac{c^2}{ca}\right)^n}{3^{n-1}} \\ &\geq \frac{\left(\frac{(a+b+c)^2}{ab+bc+ca}\right)^n}{3^{n-1}} = \frac{\left(\frac{4s^2}{ab+bc+ca}\right)^n}{3^{n-1}} \end{aligned}$$

$$\text{Let } u = \frac{(a+b+c)^2}{ab+bc+ca} = \frac{4s^2}{ab+bc+ca} \geq 3$$

We need to prove:

$$\frac{u^n}{3^{n-1}} + 9 \cdot \left(\frac{1}{u}\right)^2 \geq 4 \Leftrightarrow u^{n+2} - 4 \cdot 3^{n-1}u^2 + 3^{n+1} \stackrel{(*)}{\geq} 0$$

$$\text{Let } \varphi(u) = u^{n+2} - 4 \cdot 3^{n-1}u^2 + 3^{n+1}, \forall u \geq 3, n \in \mathbb{N}, n \geq 2$$

$$\varphi'(u) = (n+2)u^{n+1} - 8 \cdot 3^{n-1}u = u[(n+2)u^n - 8 \cdot 3^{n-1}]$$

$$\geq 3[(n+2)3^n - 8 \cdot 3^{n-1}] = 3^n[3(n+2) - 8] \stackrel{n \geq 2}{\geq} 9(3 \cdot 5 - 8) = 63 > 0$$

$$\Rightarrow \varphi(u) \uparrow [3, \infty) \Rightarrow \varphi(u) \geq \varphi(3) = 0 \Rightarrow (*) \text{ is true. Proved.}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\text{Let } f(x) = x^n \forall n \in \mathbb{N} - \{0\} \text{ and } \forall x > 0 \therefore f''(x) = n(n-1)x^{n-2} \geq 0$$

$$\Rightarrow f(x) \text{ is convex } \therefore \left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n$$

$$\stackrel{\text{Jensen}}{\underset{(i)}{\geq}} 3 \left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right)^n$$

$$\text{Now, } \frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \stackrel{\text{A-G}}{\geq} 1 \Rightarrow \ln \left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) \geq 0 \text{ and } \therefore n-1 \geq 0$$

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$$\begin{aligned} \therefore (n-1)\ln\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) &\geq 0 \Rightarrow n\ln\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) \geq \ln\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) \\ \Rightarrow \ln\left\{\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right)^n\right\} &\geq \ln\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) \Rightarrow \left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right)^n \stackrel{(ii)}{\geq} \frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3} \\ \therefore (i), (ii) &\Rightarrow \left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n \geq 3\left(\frac{\frac{a}{b} + \frac{b}{c} + \frac{c}{a}}{3}\right) \\ &= \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \Rightarrow \left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n + 9\left(\frac{ab + bc + ca}{4s^2}\right)^2 \\ &\geq \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 9\left(\frac{ab + bc + ca}{4s^2}\right)^2 \geq \frac{9(a^2 + b^2 + c^2)}{(a+b+c)^2} + 9\left(\frac{ab + bc + ca}{(a+b+c)^2}\right)^2 \\ &= \frac{9x}{x+2y} + \frac{9y^2}{(x+2y)^2} \quad (\text{where } x = \sum a^2 \text{ and } y = \sum ab) = \frac{9x(x+2y) + 9y^2}{(x+2y)^2} \geq 4 \\ &\Leftrightarrow 9x^2 + 18xy + 9y^2 \geq 4x^2 + 16xy + 16y^2 \\ &\Leftrightarrow 5x^2 + 2xy - 7y^2 \geq 0 \Leftrightarrow (x-y)(5x+7y) \geq 0 \rightarrow \text{true} \because x \geq y \\ &\Rightarrow \left(\frac{a}{b}\right)^n + \left(\frac{b}{c}\right)^n + \left(\frac{c}{a}\right)^n + 9\left(\frac{ab + bc + ca}{4s^2}\right)^2 \geq 4 \text{ (Proved)} \end{aligned}$$

### Solution 3 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \boxed{\text{For } n=1} \text{ LHS} &= \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 9\left(\frac{ab + bc + ca}{4s^2}\right)^2 \\ &\geq \frac{9(a^2 + b^2 + c^2)}{(a+b+c)^2} + 9\left(\frac{ab + bc + ca}{(a+b+c)^2}\right)^2 \\ &= \frac{9x}{x+2y} + \frac{9y^2}{(x+2y)^2} \quad (\text{where } x = \sum a^2 \text{ and } y = \sum ab) = \frac{9x(x+2y) + 9y^2}{(x+2y)^2} \geq 4 \\ &\Leftrightarrow 9x^2 + 18xy + 9y^2 \geq 4x^2 + 16xy + 16y^2 \\ &\Leftrightarrow 5x^2 + 2xy - 7y^2 \geq 0 \Leftrightarrow (x-y)(5x+7y) \geq 0 \rightarrow \text{true} \because x \geq y \\ &\Rightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} + 9\left(\frac{ab + bc + ca}{4s^2}\right)^2 \geq 4 \Rightarrow \boxed{(1) \text{ is true}} \end{aligned}$$

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$$\begin{aligned}
 \boxed{\text{For } n=2} \text{ LHS} &= \left(\frac{a}{b}\right)^2 + \left(\frac{b}{c}\right)^2 + \left(\frac{c}{a}\right)^2 + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \\
 &\geq \frac{1}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)^2 + 9\left(\frac{ab+bc+ca}{(a+b+c)^2}\right)^2 \\
 &\geq \frac{1}{3}\left\{\frac{9(a^2+b^2+c^2)}{(a+b+c)^2}\right\}^2 + 9\left(\frac{ab+bc+ca}{(a+b+c)^2}\right)^2 = \frac{27x^2}{(x+2y)^2} + \frac{9y^2}{(x+2y)^2} = \frac{27x^2+9y^2}{(x+2y)^2} \\
 &\geq 4 \Leftrightarrow 27x^2 + 9y^2 \geq 4x^2 + 16xy + 16y^2 \\
 &\Leftrightarrow 23x^2 - 16xy - 7y^2 \geq 0(x-y)(23x+7y) \geq 0 \rightarrow \text{true} \because x \geq y \\
 &\Rightarrow \left(\frac{a}{b}\right)^2 + \left(\frac{b}{c}\right)^2 + \left(\frac{c}{a}\right)^2 + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \geq 4 \Rightarrow \boxed{\text{(2) is true}}
 \end{aligned}$$

Let us assume that (1) holds true  $n = k$  where  $k \in \mathbb{N}$  and  $k \geq 3$

$$\Rightarrow \left(\frac{a}{b}\right)^k + \left(\frac{b}{c}\right)^k + \left(\frac{c}{a}\right)^k + 9\left(\frac{ab+bc+ca}{4s^2}\right)^{2(i)} \stackrel{(i)}{\geq} 4$$

We shall show that (1) holds true for  $n = k + 1$  as well, meaning,

$$\begin{aligned}
 &\left(\frac{a}{b}\right)^{k+1} + \left(\frac{b}{c}\right)^{k+1} + \left(\frac{c}{a}\right)^{k+1} + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \geq 4 \\
 \text{Now, } &\left(\frac{a}{b}\right)^{k+1} + \left(\frac{b}{c}\right)^{k+1} + \left(\frac{c}{a}\right)^{k+1} + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \stackrel{\text{Chebyshev}}{\geq} \\
 &\geq \frac{1}{3}\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right)\left\{\left(\frac{a}{b}\right)^k + \left(\frac{b}{c}\right)^k + \left(\frac{c}{a}\right)^k\right\} + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \\
 &\stackrel{\text{A-G}}{\geq} \frac{1}{3}(3)\left\{\left(\frac{a}{b}\right)^k + \left(\frac{b}{c}\right)^k + \left(\frac{c}{a}\right)^k\right\} + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \\
 &= \left(\frac{a}{b}\right)^k + \left(\frac{b}{c}\right)^k + \left(\frac{c}{a}\right)^k + 9\left(\frac{ab+bc+ca}{4s^2}\right)^2 \stackrel{\text{by (i)}}{\geq} 4 \\
 &\Rightarrow \text{(1) holds true for } n = k + 1
 \end{aligned}$$

$\therefore$  we conclude via the principle of mathematical induction that (1) holds true  $\forall n \in \mathbb{N} - \{0\}$  (QED)

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**1605. In any  $\Delta ABC$   $n_a$  –Nagel's cevian holds:**

$$\frac{m_a}{n_a} + \frac{m_b}{n_b} + \frac{m_c}{n_c} \geq \sum \frac{m_b + m_c - m_a}{\sqrt{n_b n_c}}$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

Let  $x = m_b + m_c - m_a$ ,  $y = m_c + m_a - m_b$  and  $z = m_a + m_b - m_c$

$$\therefore \sum x = \sum m_a \Rightarrow m_a = \frac{y+z}{2}, m_b = \frac{z+x}{2}, m_c = \frac{x+y}{2}$$

$\therefore$  sum of any 2 medians  $>$  third  $\therefore x, y, z > 0$  and also, let  $\frac{1}{\sqrt{n_a}} = u$ ,  $\frac{1}{\sqrt{n_b}} = v$ ,

$$\frac{1}{\sqrt{n_c}} = w \therefore \text{proposed inequality} \Leftrightarrow$$

$$\frac{y+z}{2}u^2 + \frac{z+x}{2}v^2 + \frac{x+y}{2}w^2 \geq xvw + ywu + zuv$$

$$\Leftrightarrow x\left(\frac{v^2+w^2}{2}\right) + y\left(\frac{w^2+u^2}{2}\right) + z\left(\frac{u^2+v^2}{2}\right) \geq xvw + ywu + zuv \rightarrow \text{true}$$

$$\therefore x\left(\frac{v^2+w^2}{2}\right) \stackrel{A-G}{\geq} xvw \text{ and analogs (Proved)}$$

**1606. In any  $\Delta ABC$ ,  $n_a$  –Nagel's cevian,  $g_a$  –Gergonne's cevian, holds:**

$$\frac{g_a h_a}{w_a} + \frac{g_b h_b}{w_b} + \frac{g_c h_c}{w_c} \geq \sqrt{\frac{2r}{R} \left( \frac{r_b r_c}{n_a} + \frac{r_c r_a}{n_b} + \frac{r_a r_b}{n_c} \right)}$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$r_b + r_c = s \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left( \frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left( \frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{\cong} 4R \cos^2 \frac{A}{2}$$

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$$\text{Now, } (b+c)^2 \geq 32Rrcos^2 \frac{A}{2} \stackrel{\text{by (i)}}{\cong} 8r(r_b + r_c) = 8r^2s \left( \frac{1}{s-b} + \frac{1}{s-c} \right)$$

$$= 8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)} = 4a(b+c-a)$$

$$\Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) \geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true}$$

$$\therefore b+c \stackrel{\text{(ii)}}{\geq} 4\sqrt{2Rrcos} \frac{A}{2} \text{ and analogs}$$

$$\text{Stewart's theorem} \Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c)$$

$$\text{and } b^2(s-b) + c^2(s-c) = ag_a^2 + a(s-b)(s-c)$$

$$\therefore an_a^2 \cdot ag_a^2 \geq a^2s^2(s-a)^2$$

$$\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c)$$

$$- a(s-b)(s-c)\} \stackrel{\text{(a)}}{\geq} a^2s^2(s-a)^2$$

$$\text{Let } s-a = x, s-b = y \text{ and } s-c = z \therefore s = x+y+z \Rightarrow a = y+z,$$

$$b = z+x \text{ and } c = x+y$$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\} \{y(z+x)^2 + z(x+y)^2$$

$$- yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0$$

$$\rightarrow \text{true} \Rightarrow \text{(a) is true} \Rightarrow n_a g_a \stackrel{\text{(iii)}}{\geq} s(s-a)$$

$$\text{Now, } \frac{n_a g_a h_a}{w_a r_b r_c} \stackrel{\text{by (iii)}}{\geq} \frac{s(s-a)}{\left\{ \frac{s(s-a)(s-b)(s-c)}{(s-b)(s-c)} \right\}} \left( \frac{2rs}{a} \right) \left( \frac{b+c}{2bccos \frac{A}{2}} \right) \stackrel{\text{by (ii)}}{\geq}$$

$$\geq \left( \frac{rs}{4Rrs} \right) \left( \frac{4\sqrt{2Rrcos} \frac{A}{2}}{\cos \frac{A}{2}} \right) = \sqrt{\frac{2r}{R}} \Rightarrow \frac{g_a h_a}{w_a} \geq \sqrt{\frac{2r}{R}} \left( \frac{r_b r_c}{n_a} \right) \text{ and analogs}$$

$$\frac{g_a h_a}{w_a} + \frac{g_b h_b}{w_b} + \frac{g_c h_c}{w_c} \geq \sqrt{\frac{2r}{R}} \left( \frac{r_b r_c}{n_a} + \frac{r_c r_a}{n_b} + \frac{r_a r_b}{n_c} \right) \text{ (Proved)}$$

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1607. In any  $\Delta ABC$  holds,  $n_a$  – Nagel's cevian, holds:

$$\sum \frac{n_a}{h_a} \geq \frac{3\sqrt{2}}{2} + \frac{1}{2\sqrt{2}r} \sum |a - b|$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\ &\Rightarrow s(b^2 + c^2) - bc(2s - a) = an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow \\ &\Rightarrow s(b^2 + c^2) - 2sbc = an_a^2 + a(as - s^2) \\ &\Rightarrow s(b^2 + c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc\sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\ &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left( \frac{2\Delta}{a} \right) \left( \frac{\Delta}{s - a} \right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(i)}{=} s^2 - 2h_a r_a \end{aligned}$$

$$\text{Now, } \frac{n_a}{h_a} \geq \frac{\sqrt{2}}{2} + \frac{1}{2\sqrt{2}r} |b - c|$$

$$\Leftrightarrow \frac{n_a^2}{h_a^2} \geq \frac{1}{2} + \frac{(b - c)^2}{8r^2} + 2 \left( \frac{\sqrt{2}}{2} \right) \left( \frac{1}{2\sqrt{2}r} \right) |b - c| \stackrel{\text{by (i)}}{\Leftrightarrow} \frac{s^2 - 2h_a r_a}{h_a^2} \geq \frac{1}{2} + \frac{(b - c)^2}{8r^2} + \frac{|b - c|}{2r}$$

$$\Leftrightarrow \frac{s^2 a^2}{4r^2 s^2} - 2 \left( \frac{rs}{s - a} \right) \left( \frac{a}{2rs} \right) - \frac{1}{2} - \frac{(b - c)^2}{8r^2} \geq \frac{|b - c|}{2r}$$

$$\Leftrightarrow \frac{a^2}{4r^2} - \frac{a(s - b)(s - c)}{sr^2} - \frac{1}{2} - \frac{(b - c)^2}{8r^2} \geq \frac{|b - c|}{2r}$$

$$\Leftrightarrow \frac{2(a + b + c)a^2 - 4a(c + a - b)(a + b - c) - (b + c - a)(c + a - b)(a + b - c) - (a + b + c)(b - c)^2}{16sr^2} \geq$$

$$\geq \frac{|b - c|}{2r} \Leftrightarrow \frac{a(b - c)^2 + (s - a)a^2}{4sr^2} \geq \frac{|b - c|}{r}$$

$$\Leftrightarrow \{a(b - c)^2 + (s - a)a^2\}^2 \stackrel{(m)}{\geq} 16s(s - a)(s - b)(s - c)(b - c)^2$$

$$\text{Let } x = s - a, y = s - b, z = s - c \therefore \sum x = s \Rightarrow a = y + z, b = z + x, c = x + y$$

$\therefore$  via these substitutions, (m)  $\Leftrightarrow$

$$\{(y + z)(y - z)^2 + x(y + z)^2\}^2 \geq 16xyz(x + y + z)(y - z)^2$$

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$$\begin{aligned}
 &\Leftrightarrow x^2\{y^4 + z^4 + 38y^2z^2 - 12yz(y^2 + z^2)\} \\
 &+ 2x\{y^5 + z^5 - 7yz(y^3 + z^3) + 6y^2z^2(y + z)\} \\
 &+ y^6 + z^6 - 2yz(y^4 + z^4 - 2y^2z^2) - y^2z^2(y^2 + z^2) \geq 0 \\
 &\Leftrightarrow x^2\{(y^2 + z^2)^2 + 36y^2z^2 - 12yz(y^2 + z^2)\} \\
 &+ 2x(y + z)(y^4 - y^3z + y^2z^2 - yz^3 + z^4 - 7y^3z - yz^3 + 7y^2z^2 + 6y^2z^2) \\
 &+ (y^2 + z^2)(y^4 + z^4 - y^2z^2) - y^2z^2(y^2 + z^2) - 2yz(y^4 + z^4 - 2y^2z^2) \geq 0 \\
 &\Leftrightarrow x^2(y^2 + z^2 - 6yz)^2 + 2x(y + z)\{y^4 + z^4 + 14y^2z^2 - 8yz(y^2 + z^2)\} \\
 &+ (y^4 + z^4 - 2y^2z^2)(y^2 + z^2 - 2yz) \geq 0 \\
 &\Leftrightarrow x^2(y^2 + z^2 - 6yz)^2 + 2x(y + z)\{(y^2 + z^2)^2 - 8yz(y^2 + z^2) + 12y^2z^2\} \\
 &+ (y - z)^2(y^2 - z^2)^2 \geq 0 \\
 &\Leftrightarrow x^2(y^2 + z^2 - 6yz)^2 + 2x(y + z)(y^2 + z^2 - 6yz)(y^2 + z^2 - 2yz) + (y + z)^2(y - z)^4 \geq 0 \\
 &\Leftrightarrow x^2(y^2 + z^2 - 6yz)^2 + 2x(y^2 + z^2 - 6yz)(y + z)(y - z)^2 + (y + z)^2(y - z)^4 \geq 0 \\
 &\Leftrightarrow \{x(y^2 + z^2 - 6yz) + (y + z)(y - z)^2\}^2 \geq 0 \\
 &\rightarrow \text{true} \Rightarrow \text{(m) is true} \Rightarrow \frac{n_a}{h_a} \geq \frac{\sqrt{2}}{2} + \frac{1}{2\sqrt{2}r} |b - c| \text{ and analogs} \Rightarrow \\
 &\sum \frac{n_a}{h_a} \geq \frac{3\sqrt{2}}{2} + \frac{1}{2\sqrt{2}r} \sum |b - c| = \frac{3\sqrt{2}}{2} + \frac{1}{2\sqrt{2}r} \sum |a - b| \quad (\text{QED})
 \end{aligned}$$

1608. In any  $\triangle ABC$ ,  $n_a$  –Nagel’s cevian,  $g_a$  –Gergonne’s cevian, holds:

$$\frac{n_a g_a}{a} + \frac{n_b g_b}{b} + \frac{n_c g_c}{c} \geq \frac{S}{r_a - r} + \frac{S}{r_b - r} + \frac{S}{r_c - r}$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c)$$

$$\text{and } b^2(s - b) + c^2(s - c) = ag_a^2 + a(s - b)(s - c)$$

$$\therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s - a)^2$$

$$\Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c)$$

$$- a(s - b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2$$

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Let  $s - a = x, s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$  and  
 $c = x + y$

Using these substitutions, (a)

$$\begin{aligned} &\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2 \\ &\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true} \\ &\Rightarrow \text{(a) is true} \Rightarrow n_a g_a \geq s(s-a) \\ &\Rightarrow n_a g_a (r_a - r) \geq s(s-a) \left( \frac{rs}{s-a} - \frac{rs}{s} \right) = \left\{ \frac{s(s-a)}{s(s-a)} \right\} rsa = rsa \Rightarrow \frac{n_a g_a (r_a - r)}{a s} \geq \frac{rs}{rs} \\ &= 1 \Rightarrow \frac{n_a g_a}{a} \geq \frac{S}{r_a - r} \text{ and analogs} \\ &\therefore \frac{n_a g_a}{a} + \frac{n_b g_b}{b} + \frac{n_c g_c}{c} \geq \frac{S}{r_a - r} + \frac{S}{r_b - r} + \frac{S}{r_c - r} \text{ (Proved)} \end{aligned}$$

**1609. In any  $\Delta ABC$ ,  $n_a$  –Nagel’s cevian,  $g_a$  –Gergonne’s cevian, holds:**

$$\frac{n_a g_a (r_a - r)}{h_b h_c} + \frac{n_b g_b (r_b - r)}{h_c h_a} + \frac{n_c g_c (r_c - r)}{h_a h_b} \geq 3R$$

*Proposed by Bogdan Fuștei – Romania*

**Solution by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s-c) + c^2(s-b) \\ &= an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\ &= ag_a^2 + a(s-b)(s-c) \\ \therefore an_a^2 \cdot ag_a^2 &\geq a^2 s^2 (s-a)^2 \\ \Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} &\{b^2(s-b) + c^2(s-c) \\ &- a(s-b)(s-c)\} \stackrel{(a)}{\geq} a^2 s^2 (s-a)^2 \end{aligned}$$

Let  $s - a = x, s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$  and  
 $c = x + y$

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Using these substitutions, (a)

$$\begin{aligned} &\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 \\ &\quad - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2 \\ \Leftrightarrow &xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true} \\ &\Rightarrow \text{(a) is true} \Rightarrow n_a g_a \geq s(s-a) \\ &\Rightarrow n_a g_a (r_a - r) \geq s(s-a) \left( \frac{rs}{s-a} - \frac{rs}{s} \right) = \left\{ \frac{s(s-a)}{s(s-a)} \right\} rsa = rsa \Rightarrow \frac{n_a g_a (r_a - r)}{h_b h_c} \\ &\geq \frac{4R^2 rsa}{ca \cdot ab} = \frac{4R^2 rs}{4Rrs} = R \therefore \frac{n_a g_a (r_a - r)}{h_b h_c} \geq R \text{ and analogs} \\ &\Rightarrow \frac{n_a g_a (r_a - r)}{h_b h_c} + \frac{n_b g_b (r_b - r)}{h_c h_a} + \frac{n_c g_c (r_c - r)}{h_a h_b} \geq 3R \text{ (Proved)} \end{aligned}$$

**1610. In any  $\Delta ABC$  holds:**

$$\frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r} - 2} \right) \leq \sum \sqrt{\frac{b+c-a}{a}} \leq \frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r} - 2} \right)$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} \sqrt{\frac{b+c-a}{a}} &\leq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r} - 2} \right) \Leftrightarrow \frac{b+c-a}{a} \leq \frac{1}{2} \left( \frac{R}{r} + \frac{R}{r} - 2 + \frac{2\sqrt{R(R-2r)}}{r} \right) \\ &= \frac{R}{r} - 1 + \frac{\sqrt{R(R-2r)}}{r} \\ \Leftrightarrow \frac{b+c-a}{a} + 1 &\leq \frac{AO + IO}{r} \Leftrightarrow \frac{4R \cos \frac{A}{2} \cos \frac{B-C}{2}}{4R \cos \frac{A}{2} \sin \frac{A}{2}} \leq \frac{AO + IO}{r} \\ &\Leftrightarrow AI \cdot \cos \frac{B-C}{2} \stackrel{(i)}{\leq} AO + IO \\ \text{Now, } \because -\frac{\pi}{2} &< \frac{B-C}{2} < \frac{\pi}{2} \therefore 0 < \cos \frac{B-C}{2} \leq 1 \Rightarrow \end{aligned}$$

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triangle inequality

$$\text{Al. } \cos \frac{B-C}{2} \leq \text{Al} \stackrel{\text{triangle inequality}}{\geq} \text{AO} + \text{IO} \Rightarrow \text{(i) is true}$$

$$\Rightarrow \sqrt{\frac{b+c-a}{a}} \leq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r}-2} \right) \text{ and analogs}$$

$$\Rightarrow \sum \sqrt{\frac{b+c-a}{a}} \stackrel{(1)}{\geq} \frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r}-2} \right)$$

$$\text{Again, } \sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right) \Leftrightarrow \frac{b+c-a}{a}$$

$$\geq \frac{1}{2} \left( \frac{R}{r} + \frac{R}{r} - 2 - \frac{2\sqrt{R(R-2r)}}{r} \right) = \frac{R}{r} - 1 - \frac{\sqrt{R^2 - 2Rr}}{r}$$

$$\Leftrightarrow \frac{b+c}{a} \geq \frac{R - \sqrt{R^2 - 2Rr}}{r} = \frac{(R + \sqrt{R^2 - 2Rr})(R - \sqrt{R^2 - 2Rr})}{r(R + \sqrt{R^2 - 2Rr})} = \frac{2Rr}{r(R + \sqrt{R^2 - 2Rr})}$$

$$\Leftrightarrow \frac{2a}{b+c} - 1 \leq \frac{R + \sqrt{R^2 - 2Rr}}{R} - 1$$

$$\Leftrightarrow \frac{2a - b - c}{b+c} \stackrel{(ii)}{\geq} \frac{\sqrt{R^2 - 2Rr}}{R}$$

$$\text{If } 2a - b - c \leq 0, \text{ then, LHS of (ii)} \leq 0 \stackrel{\text{Euler}}{\geq} \frac{\sqrt{R^2 - 2Rr}}{R} \Rightarrow$$

$$\text{(ii) is true} \Rightarrow \sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right)$$

Let us now consider the case when  $2a - b - c > 0$

$$\therefore \text{(ii)} \Leftrightarrow \left( \frac{2a - b - c}{b+c} \right)^2 \leq \frac{R^2 - 2Rr}{R^2} \Leftrightarrow \frac{2r}{R} \leq 1 - \left( \frac{2a - b - c}{b+c} \right)^2$$

$$\Leftrightarrow \frac{2r}{R} \leq \left( 1 - \frac{2a - b - c}{b+c} \right) \left( 1 + \frac{2a - b - c}{b+c} \right) = \frac{4a(b+c-a)}{(b+c)^2} = \frac{8a(s-a)}{(b+c)^2} \Leftrightarrow$$

$$\left( \frac{r}{R} \right) (b+c)^2 \leq 4a(s-a)$$

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$$\Leftrightarrow 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\left(4R\cos\frac{A}{2}\cos\frac{B-C}{2}\right)^2 \leq 4 \cdot 4R\sin\frac{A}{2}\cos\frac{A}{2} \cdot 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Leftrightarrow \cos^2\frac{B-C}{2} \leq 1 \rightarrow \text{true} \because 0 < \cos\frac{B-C}{2} \leq 1$$

$$\Rightarrow \text{(ii) is true} \Rightarrow \sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right)$$

$$\therefore \text{ combining both cases, } \sqrt{\frac{b+c-a}{a}} \geq \frac{\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right) \text{ and analogs}$$

$$\Rightarrow \sum \sqrt{\frac{b+c-a}{a}} \stackrel{(2)}{\geq} \frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right) \because (1), (2) \Rightarrow \frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} - \sqrt{\frac{R}{r}-2} \right)$$

$$\leq \sum \sqrt{\frac{b+c-a}{a}} \leq \frac{3\sqrt{2}}{2} \left( \sqrt{\frac{R}{r}} + \sqrt{\frac{R}{r}-2} \right) \text{ (Proved)}$$

**1611. In  $\triangle ABC$  the following relationship holds:**

$$\frac{b+c}{2s+a} + \frac{c+a}{2s+b} + \frac{a+b}{2s+c} \geq \frac{3}{2} + \frac{\mu r(R-2r)}{Rs}, \mu \leq \frac{5}{64}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\text{Let: } \Omega = \frac{b+c}{2s+a} + \frac{c+a}{2s+b} + \frac{a+b}{2s+c}$$

$$= \frac{(2s-a)(2s+b)(2s+c) + (2s+a)(2s-b)(2s+c) + (2s+a)(2s+b)(2s-c)}{(2s+a)(2s+b)(2s+c)}$$

$$= \frac{24s^3 + 4(a+b+c)s^2 - 2(ab+bc+ca)s - 3abc}{8s^3 + 4(a+b+c)s^2 + 2(ab+bc+ca)s + abc}$$

$$= \frac{24s^3 + 8s^3 - 2(s^2 + 4Rr + r^2)s - 12Rrs}{8s^3 + 8s^3 + 2(s^2 + 4Rr + r^2)s + 4Rrs}$$

$$= \frac{2s(16s^2 - s^2 - 4Rr - r^2 - 6Rr)}{2s(8s^2 + s^2 + 4Rr + r^2 + 2Rr)} = \frac{15s^2 - 10Rr - r^2}{9s^2 + 6Rr + r^2}$$

So, we need to prove:

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$$\Omega \geq \frac{3}{2} + \frac{\mu r(R-2r)}{Rs}, \mu \leq \frac{5}{64} \Leftrightarrow \frac{15s^2 - 10Rr - r^2}{9s^2 + 6Rr + r^2} \stackrel{(*)}{\geq} \frac{3}{2} + \frac{\mu r(R-2r)}{Rs}$$

$$\text{If } \mu \leq 0 \Rightarrow \frac{3}{2} \geq \frac{3}{2} + \frac{\mu r(R-2r)}{Rs}$$

We just check:

$$\frac{15s^2 - 10Rr - r^2}{9s^2 + 6Rr + r^2} \geq \frac{3}{2} \Leftrightarrow 30s^2 - 20Rr - 2r^2 \geq 27s^2 + 18Rr + 3r^2$$

$$\Leftrightarrow 3s^2 \geq 38Rr + 5r^2$$

But:  $s^2 \geq 16Rr - 5r^2$  (Mitrinovic), then we have:

$$3s^2 \geq 48Rr - 15r^2 \stackrel{(**)}{\geq} 38Rr + 5r^2$$

$$(**) \Leftrightarrow 10Rr \geq 20r^2 \Leftrightarrow R \geq 2r \text{ (Euler)} \Rightarrow (1) \text{ is true} \Rightarrow (2) \text{ is true.}$$

If  $0 < \mu \leq \frac{5}{64}$  then:

$$\frac{15s^2 - 10Rr - r^2}{9s^2 + 6Rr + r^2} \geq \frac{3}{2} + \frac{\mu r(R-2r)}{Rs}$$

$$\text{But: } s \geq 3\sqrt{3}r \Rightarrow \frac{r}{s} \leq \frac{1}{3\sqrt{3}}$$

So, we just check:

$$\frac{15s^2 - 10Rr - r^2}{9s^2 + 6Rr + r^2} \stackrel{(1)}{\geq} \frac{3}{2} + \frac{\mu(R-2r)}{3\sqrt{3}R} \Leftrightarrow$$

$$6\sqrt{3}R(15s^2 - 10Rr - r^2) \geq (9s^2 + 6Rr + r^2)(9\sqrt{3}R + 2\mu(R-r))$$

$$\Leftrightarrow 6\sqrt{3}R(15s^2 - 10Rr - r^2) \geq (9s^2 + 6Rr + r^2)((9\sqrt{3} + 2\mu)R - 4\mu r)$$

$$\Leftrightarrow 9(\sqrt{3} - 2\mu)Rs^2 + 36\mu rs^2$$

$$\geq 6\sqrt{3}R(10Rr + r^2) + (6Rr + r^2)((9\sqrt{3} + 2\mu)R - 4\mu r)$$

$$\Leftrightarrow [9(\sqrt{3} - 2\mu)R + 36\mu r]s^2$$

$$\geq 6\sqrt{3}Rr(10R + r) + r(6R + r)((9\sqrt{3} + 2\mu)R - 4\mu r)$$

From  $s^2 \geq 16Rr - 5r^2$  (Mitrinovic), we need to prove:

$$9[(\sqrt{3} - 2\mu)R + 4\mu r](16R - 5r) \stackrel{(2)}{\geq}$$

$$6\sqrt{3}R(10R + r) + (6R + r)((9\sqrt{3} + 2\mu)R - 4\mu r)$$

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$$\begin{aligned}
 & \stackrel{t \geq 2}{\iff} 9[(\sqrt{3} - 2\mu)t + 4\mu](16t - 5) \\
 & \geq 6\sqrt{3}t(10t + 1) + (6t + 1)((9\sqrt{3} + 2\mu)t - 4\mu) \\
 & \Leftrightarrow 2(t - 2)[15(\sqrt{3} - 10\mu)t + 44\mu] \geq 0 \text{ true, because} \\
 & \quad t \geq 2, 0 < \mu \leq \frac{5}{64} \Rightarrow t - 2 \geq 0, \\
 & 15(\sqrt{3} - 10\mu)t + 44\mu > 15\left(\sqrt{3} - \frac{50}{64}\right) + 0 > 15\left(1 - \frac{50}{64}\right) = \frac{105}{32} > 0 \\
 & \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true. Proved.}
 \end{aligned}$$

**1612. In  $\triangle ABC$  the following relationship holds:**

$$\left(\frac{h_a}{r_a} + \frac{h_b}{r_b} + \frac{h_c}{r_c}\right) \left(\sqrt[3]{\frac{h_a}{r_a}} + \sqrt[3]{\frac{h_b}{r_b}} + \sqrt[3]{\frac{h_c}{r_c}}\right) \geq 9$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned}
 h_a &= \frac{2S}{a}; r_a = \frac{S}{s-a}; h_b = \frac{2S}{b}; r_b = \frac{S}{s-b}; h_c = \frac{2S}{c}; r_c = \frac{S}{s-c} \\
 \text{Let: } x &= \frac{h_a}{r_a} = \frac{2(s-a)}{a} > 0; y = \frac{h_b}{r_b} = \frac{2(s-b)}{b} > 0; z = \frac{h_c}{r_c} = \frac{2(s-c)}{c} > 0 \\
 \Rightarrow \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} &= \frac{1}{2 \cdot \frac{s}{a} - 2 + 2} + \frac{1}{2 \cdot \frac{s}{b} - 2 + 2} + \frac{1}{2 \cdot \frac{s}{c} - 2 + 2} \\
 &= \frac{a+b+c}{2s} = 1 \Leftrightarrow \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = 1; (*)
 \end{aligned}$$

Now,

$$\begin{aligned}
 LHS &= (x+y+z)(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z}) \\
 &= (\sqrt{x^2} + \sqrt{y^2} + \sqrt{z^2}) (\sqrt[6]{x^2} + \sqrt[6]{y^2} + \sqrt[6]{z^2}) \stackrel{CBS}{\geq} (\sqrt[3]{x^2} + \sqrt[3]{y^2} + \sqrt[3]{z^2})^2 \\
 &= (\sqrt[3]{x \cdot 1 \cdot 1} + \sqrt[3]{y \cdot 1 \cdot 1} + \sqrt[3]{z \cdot 1 \cdot 1})^2
 \end{aligned}$$

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$$\begin{aligned} \stackrel{Am-Hm}{\geq} \left( \frac{3}{\frac{1}{x} + \frac{1}{x} + 1} + \frac{3}{\frac{1}{y} + \frac{1}{y} + 1} + \frac{3}{\frac{1}{z} + \frac{1}{z} + 1} \right)^2 &= \left( \frac{3x}{x+2} + \frac{3y}{y+2} + \frac{3z}{z+2} \right)^2 \\ &= 3^2 \left( \frac{x}{x+2} + \frac{y}{y+2} + \frac{z}{z+2} \right)^2 \stackrel{(*)}{=} 9 \end{aligned}$$

**1613. In  $\triangle ABC$  the following relationship holds:**

$$\left( \frac{a}{m_a} \right)^4 + \left( \frac{b}{m_b} \right)^4 + \left( \frac{c}{m_c} \right)^4 \geq \frac{16}{3}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Bogdan Fuștei-Romania**

$\forall P$  – point of plan the  $\triangle ABC$ , we have:

$$\frac{PA}{a} + \frac{PB}{b} + \frac{PC}{c} \geq \sqrt{3}; GA = \frac{2}{3}m_a \text{ and analogs}$$

$$\text{If } P = G \rightarrow \frac{GA}{a} + \frac{GB}{b} + \frac{GC}{c} \geq \sqrt{3} \Leftrightarrow \frac{2}{3} \left( \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \right) \geq \sqrt{3} \Leftrightarrow \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \geq \frac{3\sqrt{3}}{2}$$

$$\left( \frac{m_a}{a} + \frac{m_b}{b} + \frac{m_c}{c} \right) \left( \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \right) \stackrel{Am-Hm}{\geq} 9 = \frac{3\sqrt{3}}{2} \cdot 2\sqrt{3}$$

$$\text{So, we have: } \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq 2\sqrt{3} \Rightarrow \left( \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \right)^2 \geq 12$$

$$3 \left( \frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \right) \stackrel{C.B.S}{\geq} \left( \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \right)^2$$

$$\frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \geq 4 \Leftrightarrow \left( \frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \right)^2 \geq 16$$

$$3 \left( \frac{a^4}{m_a^4} + \frac{b^4}{m_b^4} + \frac{c^4}{m_c^4} \right) \stackrel{C.B.S}{\geq} \left( \frac{a^2}{m_a^2} + \frac{b^2}{m_b^2} + \frac{c^2}{m_c^2} \right)^2 \geq 16$$

$$\frac{a^4}{m_a^4} + \frac{b^4}{m_b^4} + \frac{c^4}{m_c^4} \geq \frac{16}{3}$$

**Solution 2 by Marian Ursărescu-Romania**

$$\text{From Holder inequality, we have: } \frac{a^4}{m_a^4} + \frac{b^4}{m_b^4} + \frac{c^4}{m_c^4} \geq \frac{\left( \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \right)^4}{3^3}$$

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We must show:  $\left(\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c}\right)^4 \geq 3^3 \cdot 16 \Leftrightarrow \frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq 2\sqrt{3}$ ; (1)

$$\text{Lemma: } m_a \leq \frac{a^2 + b^2 + c^2}{2\sqrt{3}a}$$

Proof:  $4m_a^2 + 3a^2 \geq 4\sqrt{3}a \Rightarrow 2b^2 + 2c^2 - a^2 + 3a^2 \geq 4\sqrt{3}m_a \Rightarrow m_a \leq \frac{a^2 + b^2 + c^2}{2\sqrt{3}a} \Rightarrow$

$$\frac{1}{m_a} \geq \frac{2\sqrt{3}a}{a^2 + b^2 + c^2} \Rightarrow \frac{a}{m_a} \geq \frac{2\sqrt{3}a^2}{a^2 + b^2 + c^2} \text{ and simillary, then}$$

$$\frac{a}{m_a} + \frac{b}{m_b} + \frac{c}{m_c} \geq \frac{2\sqrt{3}(a^2 + b^2 + c^2)}{a^2 + b^2 + c^2} = 2\sqrt{3} \rightarrow (1) \text{ it's true}$$

1614. In  $\triangle ABC$  the following relationship holds:

$$\frac{4R + r}{6} \sqrt{\frac{4R + r}{r}} \geq \frac{h_a^2}{w_a + m_a} + \frac{h_b^2}{w_b + m_b} + \frac{h_c^2}{w_c + m_c} \geq \frac{r^2}{R}$$

Proposed by Mokhtar Khassani-Mostaganem-Algerie

Solution 1 by Bogdan Fuștei-Romania

$$\sum_{cyc} \frac{h_a^2}{w_a + m_a} \geq \sum_{cyc} \frac{h_a^2}{2m_a} = \frac{1}{2} \cdot \sum_{cyc} \frac{h_a^2}{m_a}; (1)$$

From  $\frac{R}{2r} \geq \frac{m_a}{h_a}$  (Panaitol inequality) we have:  $\frac{h_a}{2m_a} \geq \frac{r}{R} \Leftrightarrow \frac{h_a^2}{2m_a} \geq \frac{r}{R} \cdot h_a$

and analogs.

$$\frac{1}{2} \cdot \sum_{cyc} \frac{h_a^2}{m_a} \geq \frac{r}{R} (h_a + h_b + h_c) \geq \frac{r}{R} \cdot 9r = \frac{9r^2}{R} > \frac{r^2}{R}; (2)$$

$h_a^2 \leq s(s-a)$  and analogs (because:  $h_a \leq w_a \leq \sqrt{s(s-a)}$ )

$$m_a \geq \frac{b+c}{2} \cdot \cos \frac{A}{2} \text{ and analogs, then we have:}$$

$$m_a \cdot w_a \geq s(s-a) \Rightarrow 2m_a \cdot w_a \geq 2s(s-a) \text{ and with}$$

$$m_a^2 + w_a^2 \geq 2m_a w_a \Rightarrow (m_a + w_a)^2 = m_a^2 + w_a^2 + 2m_a w_a \geq 4s(s-a)$$

$$m_a + w_a \geq 2\sqrt{s(s-a)} \text{ and analogs.}$$

$$\frac{s(s-a)}{2\sqrt{s(s-a)}} \geq \frac{h_a^2}{m_a + w_a} \Rightarrow \frac{1}{2} \sqrt{s(s-a)} \geq \frac{h_a^2}{m_a + w_a} \text{ and analogs.}$$

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So, we have:  $\frac{1}{2} \sum_{cyc} \sqrt{s(s-a)} \geq \sum_{cyc} \frac{h_a^2}{m_a + w_a}$

Applying C-B-S and we have:  $s\sqrt{3} \geq \sum_{cyc} \sqrt{s(s-a)}$

$$\frac{s\sqrt{3}}{2} \geq \frac{1}{2} \sum_{cyc} \sqrt{s(s-a)}$$

$$\text{But: } 4R + r \geq s\sqrt{3} \Rightarrow \frac{4R+r}{2} \geq \sum_{cyc} \frac{h_a^2}{m_a + w_a}$$

We must show that:  $\frac{4R+r}{6} \sqrt{\frac{4R+r}{r}} \geq \frac{4R+r}{2} \Leftrightarrow$

$$\sqrt{\frac{4R+r}{r}} \geq 3 \Leftrightarrow \frac{4R+r}{r} \geq 9 \Leftrightarrow R \geq 2r \text{ (Euler). Proved.}$$

### Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \Omega &= \frac{h_a^2}{w_a + m_a} + \frac{h_b^2}{w_b + m_b} + \frac{h_c^2}{w_c + m_c} \stackrel{C-B-S}{\geq} \\ &\frac{(h_a + h_b + h_c)^2}{(m_a + m_b + m_c) + (w_a + w_b + w_c)} \stackrel{\sum w_a \leq \sum m_a \leq \frac{9R}{2}}{\geq} \\ &\geq \frac{(h_a + h_b + h_c)^2}{2 \cdot \frac{9R}{2}} = \frac{\left(\frac{s^2 + r^2 + 4Rr}{2R}\right)^2}{9R} = \frac{(s^2 + r^2 + 4Rr)^2}{36R^2} \stackrel{(*)}{\geq} \frac{r^2}{R} \end{aligned}$$

$$(*) \Leftrightarrow (s^2 + r^2 + 4Rr)^2 \geq 36r^2R^2 \Leftrightarrow s^2 + r^2 + 4Rr \geq 6Rr$$

$$\Leftrightarrow s^2 \geq 2Rr - r^2$$

But:  $s^2 \geq 16Rr - 5r^2$  (Mitrinovic inequality), then

$$16Rr - 5r^2 \stackrel{(**)}{\geq} 2Rr - r^2$$

$$(**) \Leftrightarrow 14Rr \geq 4r^2 \Leftrightarrow R \geq \frac{2}{7}r \text{ (true because: } R \geq 2r \geq \frac{2}{7}r \text{ Euler)}$$

$\Leftrightarrow (**)$  – is true  $\Leftrightarrow (*)$  – is true.

$w_a, m_a \geq h_a \Rightarrow w_a + m_a \geq 2h_a$  and analogs.

$$\Omega = \sum_{cyc} \frac{h_a^2}{w_a + m_a} \leq \sum_{cyc} \frac{h_a^2}{2h_a} = \frac{1}{2} (h_a + h_b + h_c) = \frac{s^2 + r^2 + 4Rr}{4R} \leq$$

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$$\leq \frac{4R^2 + 4Rr + 3r^2 + 4Rr + r^2}{4R} = \frac{R^2 + 2Rr + r^2}{R} = \frac{(R+r)^2}{R}$$

We need to prove:

$$\frac{(R+r)^2}{R} \stackrel{(1)}{\leq} \frac{4R+r}{6} \sqrt{\frac{4R+r}{r} \stackrel{t=\frac{R}{r} \geq 2}{\geq} (t+1)^2} \leq \frac{4t+1}{6} \sqrt{4t+1} \Leftrightarrow$$

$$36(t+1)^4 \leq t^2(4t+1)^3 \Leftrightarrow$$

$$64t^5 + 12t^4 - 132t^3 - 215t^2 - 144t - 36 \geq 0 \Leftrightarrow$$

$(t-2)(4t^2+4t+3)(16t^2+19t+6) \geq 0$  true from  $t \geq 2 \Rightarrow (1)$  is true. Proved.

1615. In any  $\triangle ABC$  holds:

$$\left( \frac{h_a}{\sqrt{r_b}} + \frac{h_b}{\sqrt{r_c}} + \frac{h_c}{\sqrt{r_a}} \right)^2 + \sqrt{h_a^2 + h_b^2 + h_c^2} \leq 3(r_a + r_b + r_c) + s$$

Proposed by Nguyen Van Canh-Vietnam

Solution by Bogdan Fuștei-Romania

We know that:  $(x+y+z)^2 \stackrel{C.B.S.}{\leq} 3(x^2+y^2+z^2), \forall x, y, z > 0 \Rightarrow$

$$\left( \frac{h_a}{\sqrt{r_b}} + \frac{h_b}{\sqrt{r_c}} + \frac{h_c}{\sqrt{r_a}} \right)^2 \leq 3 \left( \frac{h_a^2}{r_b} + \frac{h_b^2}{r_c} + \frac{h_c^2}{r_a} \right); (1)$$

$$h_a \leq w_a \leq \sqrt{s(s-a)} = \sqrt{r_b r_c} \text{ and analogs.}$$

$$\frac{h_a^2}{r_b} + \frac{h_b^2}{r_c} + \frac{h_c^2}{r_a} \leq \frac{r_b r_c}{r_b} + \frac{r_c r_a}{r_c} + \frac{r_a r_b}{r_a} = r_a + r_b + r_c$$

$$h_a^2 + h_b^2 + h_c^2 \leq s(s-a) + s(s-b) + s(s-c) = s^2$$

So, we have:  $3 \left( \frac{h_a^2}{r_b} + \frac{h_b^2}{r_c} + \frac{h_c^2}{r_a} \right) + \sqrt{h_a^2 + h_b^2 + h_c^2} \leq 3(r_a + r_b + r_c) + s; (2)$

From (1) and (2) we have:

$$\left( \frac{h_a}{\sqrt{r_b}} + \frac{h_b}{\sqrt{r_c}} + \frac{h_c}{\sqrt{r_a}} \right)^2 + \sqrt{h_a^2 + h_b^2 + h_c^2} \leq 3(r_a + r_b + r_c) + s$$

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1616. In  $\triangle ABC$  the following relationship holds:

$$9 \left( \frac{\tan^2 \frac{A}{2}}{h_a^2} + \frac{\tan^2 \frac{B}{2}}{h_b^2} + \frac{\tan^2 \frac{C}{2}}{h_c^2} \right) \geq \frac{\cot^2 \frac{A}{2}}{h_a^2} + \frac{\cot^2 \frac{B}{2}}{h_b^2} + \frac{\cot^2 \frac{C}{2}}{h_c^2}$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} = \sum_{cyc} \frac{\frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}}{\frac{4S^2}{4R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}} = \frac{R^2}{S^2} \cdot \sum_{cyc} \sin^4 \frac{A}{2}$$

$$= \frac{R^2}{S^2} \left( \frac{8R^2 + r^2 - s^2}{8R^2} \right) = \frac{8R^2 + r^2 - s^2}{8S^2}$$

$$\sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} = \sum_{cyc} \frac{\frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}}{\frac{4S^2}{4R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}} = \frac{R^2}{S^2} \cdot \sum_{cyc} \cos^4 \frac{A}{2}$$

$$= \frac{R^2}{S^2} \left( \frac{(4R + r)^2 - s^2}{8R^2} \right) = \frac{(4R + r)^2 - s^2}{8S^2}$$

We must show that:  $9 \left( \frac{8R^2 + r^2 - s^2}{8S^2} \right) \geq \frac{(4R + r)^2 - s^2}{8S^2} \Leftrightarrow$

$$9(8R^2 + r^2 - s^2) \geq (4R + r)^2 - s^2 \Leftrightarrow 9(8R^2 + r^2) - (4R + r)^2 \geq 8s^2$$

$$72R^2 + 9r^2 - (16R^2 + 8Rr + r^2) \geq 8s^2 \Leftrightarrow 56R^2 - 8Rr + 8r^2 \geq 8s^2$$

$$\Leftrightarrow 7R^2 - Rr + r^2 \geq s^2$$

But:  $s^2 \leq 4R^2 + 4Rr + 3r^2$  (Mitrinovic inequality)

So, we just check:  $7R^2 - Rr + r^2 \geq 4R^2 + 4Rr + 3r^2 \Leftrightarrow$

$3R^2 - 5Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(3R + r) \geq 0$  which is true from  $R \geq 2r$  (Euler).

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1617. In  $\triangle ABC$  the following relationship holds:

$$\frac{4}{9R^2} \leq \frac{\tan^2 \frac{A}{2}}{m_a w_a} + \frac{\tan^2 \frac{B}{2}}{m_b w_b} + \frac{\tan^2 \frac{C}{2}}{m_c w_c} \leq \frac{33R^2}{8S^2} - \frac{1}{2r^2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} &= \sum_{cyc} \frac{\frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}}{4S^2} = \frac{R^2}{S^2} \cdot \sum_{cyc} \sin^4 \frac{A}{2} \\ &= \frac{R^2}{S^2} \cdot \frac{8R^2 + r^2 - s^2}{8R^2} = \frac{8R^2 + r^2 - s^2}{8S^2} \end{aligned}$$

$$\text{Let: } \Omega = 4 \cdot \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{m_a w_a}$$

$$m_a, w_a \leq \frac{R}{2r} \cdot h_a \Rightarrow m_a w_a \leq \left(\frac{R}{2r}\right)^2 h_a^2 \text{ and analogs, then}$$

$$\Omega \geq 4 \left(\frac{2r}{R}\right)^2 \cdot \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} \stackrel{(*)}{\geq} \frac{4}{9R^2}$$

$$(*) \Leftrightarrow 4 \cdot \left(\frac{2r}{R}\right)^2 \cdot \frac{8R^2 + r^2 - s^2}{8S^2} \geq \frac{4}{9R^2} \Leftrightarrow 9(8R^2 + r^2 - s^2) \geq 2s^2$$

$$\Leftrightarrow 9(8R^2 + r^2) \geq 11s^2$$

$$\text{But: } s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Mitrinovic)}$$

$$\Rightarrow 11(4R^2 + 4Rr + 3r^2) \stackrel{(**)}{\leq} 9(8R^2 + r^2)$$

$$(**) \Leftrightarrow 28R^2 - 44Rr - 24r^2 \geq 0 \Leftrightarrow (R - 2r)(7R + 3r) \geq 0 \text{ true, from } R \geq 2r$$

$$\text{(Euler)} \Rightarrow (**) \text{ is true} \Rightarrow (*) \text{ is true.}$$

$$m_a, w_a \geq h_a \Rightarrow \Omega \stackrel{(1)}{\leq} 4 \cdot \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} \stackrel{(2)}{\leq} \frac{33R^2}{8S^2} - \frac{1}{2r^2} = \frac{33R^2 - 4s^2}{8S^2}$$

$$(2) \Leftrightarrow 4 \cdot \frac{8R^2 + r^2 - s^2}{8S^2} \leq \frac{33R^2 - 4s^2}{8S^2} \Leftrightarrow$$

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$$4(8R^2 + r^2 - s^2) \leq 33R^2 - 4s^2 \Leftrightarrow 32R^2 + 4r^2 - 4s^2 \leq 33R^2 - 4s^2$$

$$\Leftrightarrow 4r^2 \leq R^2 \Leftrightarrow 2r \leq R \text{ (Euler)} \Rightarrow (2) \text{ is true} \Rightarrow (1) \text{ is true.}$$

Proved.

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64} (2b^2 + 2c^2 - 2a^2)(2c^2 + 2a^2 - 2b^2)(2a^2 + 2b^2 - 2c^2) \stackrel{(1)}{=} \frac{1}{64} \{-4\sum a^6$$

$$+ 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2\}$$

$$\text{Now, } \sum a^6 = (\sum a^2)^3 - 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

$$= (\sum a^2)^3 - 3(2a^2 b^2 c^2 + \sum a^2 b^2 (\sum a^2 - c^2))$$

$$= (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2 \therefore \sum a^6 \stackrel{(2)}{=} (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2$$

$$\text{Again, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(3)}{=} (\sum a^2 b^2) \sum a^2 - 3a^2 b^2 c^2$$

$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 - 12a^2 b^2 c^2 + 12(\sum a^2 b^2) \sum a^2 + 6(\sum a^2 b^2) \sum a^2 - 18a^2 b^2 c^2$$

$$+ 3a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 + 18(\sum a^2 b^2) \sum a^2 - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 + 18((\sum ab)^2 - 2abc(2s))(\sum a^2) - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2$$

$$- 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\}$$

$$= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\} \leq \frac{R^2 s^4}{4}$$

$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\leq} 0$$

$$\text{Now, LHS of (i)} \stackrel{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4)$$

$$- r^3(4R + r)^3 \stackrel{?}{\leq} 0$$

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$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{(ii)}{\geq} 20rs^4$$

Now, LHS of (ii)  $\stackrel{\text{Gerretsen}}{\underset{(a)}{\geq}} s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$

and RHS of (ii)  $\stackrel{\text{Geretsen}}{\underset{(b)}{\geq}} 20rs^2(4R^2 + 4Rr + 3r^2)$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove :

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2s^2$$

Now, LHS of (iii)  $\stackrel{\text{Gerretsen}}{\underset{(c)}{\geq}} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)$

+  $r(4R + r)^3$  and RHS of (iii)  $\stackrel{\text{Geretsen}}{\underset{(d)}{\geq}} 27r^2(4R^2 + 4Rr + 3r^2)$

(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \stackrel{(4)}{\geq} \frac{Rs^2}{2}$$

$$\text{Now, } r_a w_a = \left( \tan \frac{A}{2} \right) \left( \frac{2bc \cos \frac{A}{2}}{b + c} \right) = \left( \sin \frac{A}{2} \right) \left( \frac{2bc}{b + c} \right) \stackrel{\text{HM} \leq \text{GM}}{\geq} \left( \sin \frac{A}{2} \right) \sqrt{bc}$$

$$= s \sqrt{(s - b)(s - c)} = \sqrt{s(s - b)} \sqrt{s(s - c)} \stackrel{(5)}{\geq} m_b m_c$$

$$\text{Now, } \sum \frac{\tan^2 \frac{A}{2}}{m_a w_a} = \frac{1}{s^2} \sum \frac{r_a^3}{m_a w_a r_a} \stackrel{\text{by (5) and its analogs}}{\geq} \frac{1}{s^2} \sum \frac{r_a^3}{m_a m_b m_c} \stackrel{\text{by (4)}}{\geq} \left( \frac{2}{Rs^4} \right) \sum r_a^3$$

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$$\stackrel{\text{Holder}}{\geq} \frac{2R(4R+r)^3}{9R^2s^4} \stackrel{\text{Trucht}}{\geq} \frac{6\sqrt{3}Rs^3}{9R^2s^4}$$

$$\stackrel{\text{Mitrinovic}}{\geq} \frac{6\sqrt{3}(2s)s^3}{(3\sqrt{3})9R^2s^4} = \frac{4}{9R^2} \therefore \sum \frac{\tan^2 \frac{A}{2} \stackrel{(m)}{}}{m_a w_a} \geq \frac{4}{9R^2}$$

$$\text{Now, } m_a w_a \stackrel{\text{Ioscu}}{\stackrel{(6)}{\geq}} \left( \frac{b+c}{2} \cos \frac{A}{2} \right) \left( \frac{2bc}{b+c} \cos \frac{A}{2} \right) = bc \left( \frac{s(s-a)}{bc} \right) = s(s-a)$$

$$\text{Now, } \sum \frac{\tan^2 \frac{A}{2}}{m_a w_a} \stackrel{\text{by (6) and its analogs}}{\geq} \sum \frac{(s-b)(s-c)}{\{s(s-a)\}^2} \stackrel{\text{GM-AM}}{\geq} \sum \frac{(s-b+s-c)^2}{4s^2(s-a)^2}$$

$$= \frac{1}{4s^2} \sum \frac{(a-s+s)^2}{(s-a)^2}$$

$$= \frac{1}{4s^2} \left[ \sum \frac{(s-a)^2}{(s-a)^2} - \frac{2}{r} \sum \frac{rs}{s-a} + \frac{1}{r^2} \sum \frac{r^2 s^2}{(s-a)^2} \right]$$

$$= \frac{1}{4s^2} \left[ 3 - \frac{2(4R+r)}{r} + \frac{(4R+r)^2 - 2s^2}{r^2} \right] \stackrel{?}{\geq} \frac{33R^2}{8S^2} - \frac{1}{2r^2}$$

$$\Leftrightarrow \frac{(4R+r)^2 - 2s^2 + 3r^2 - 2r(4R+r)}{4r^2s^2} - \frac{33R^2}{8r^2s^2} + \frac{1}{2r^2} \stackrel{?}{\geq} 0$$

$$\Leftrightarrow \frac{2(4R+r)^2 - 4s^2 + 6r^2 - 4r(4R+r) - 33R^2 + 4s^2}{8r^2s^2} \stackrel{?}{\geq} 0$$

$$\Leftrightarrow \frac{4r^2 - R^2}{8r^2s^2} \stackrel{?}{\geq} 0 \rightarrow \text{true, by Euler} \therefore \sum \frac{\tan^2 \frac{A}{2} \stackrel{(n)}{}}{m_a w_a} \geq \frac{33R^2}{8S^2} - \frac{1}{2r^2}$$

$$(m), (n) \Rightarrow \frac{4}{9R^2} \leq \frac{\tan^2 \frac{A}{2}}{m_a w_a} + \frac{\tan^2 \frac{B}{2}}{m_b w_b} + \frac{\tan^2 \frac{C}{2}}{m_c w_c} \leq \frac{33R^2}{8S^2} - \frac{1}{2r^2} \text{ (Proved)}$$

**1618. In  $\triangle ABC$  the following relationship holds:**

$$\frac{h_a^2}{bc} + \frac{h_b^2}{ca} + \frac{h_c^2}{ab} \leq \frac{9}{4}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

**Solution 1 by Bogdan Fuștei-Romania**

$$bc = 2Rh_a \text{ and analogs}$$

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$$\frac{h_a^2}{bc} = \frac{h_a^2}{2Rh_a} = \frac{h_a}{2R} \text{ and analogs}$$

$$\frac{h_a^2}{bc} + \frac{h_b^2}{ca} + \frac{h_c^2}{ab} \leq \frac{9}{4} \Leftrightarrow \frac{h_a + h_b + h_c}{2R} \leq \frac{9}{4} \Leftrightarrow \frac{h_a + h_b + h_c}{R} \leq \frac{9}{2}$$

But:  $h_a + h_b + h_c \leq 2R + 5r$  – (Bankhoff inequality) then

$$2R + 5r \leq \frac{9}{2}R \Leftrightarrow 4R + 10r \leq 9R \Leftrightarrow 10r \leq 5R \Leftrightarrow 2r \leq R \text{ – Euler}$$

**Solution 2 by Adrian Popa-Romania**

$$\begin{aligned} \frac{h_a^2}{bc} + \frac{h_b^2}{ca} + \frac{h_c^2}{ab} &\leq \frac{9}{4} \Leftrightarrow \frac{ah_a^2}{abc} + \frac{bh_b^2}{abc} + \frac{ch_c^2}{abc} \leq \frac{9}{4} \\ \Leftrightarrow \sum_{cyc} \frac{2Sh_a}{2RS} &= \sum_{cyc} \frac{h_a}{2R} = \frac{1}{2R} \sum_{cyc} h_a = \frac{1}{2R} \left( \frac{2S}{a} + \frac{2S}{b} + \frac{2S}{c} \right) = \frac{abc}{4R^2} \cdot \frac{ab + bc + ca}{abc} \\ &= \frac{ab + bc + ca}{4R^2} \leq \frac{a^2 + b^2 + c^2}{4R^2} \stackrel{\text{Leibniz}}{\leq} \frac{9R^2}{4R^2} = \frac{9}{4} \end{aligned}$$

**1619. In  $\triangle ABC$  the following relationship holds:**

$$\sum_{cyc} \frac{r_a + r}{r_a - r} \leq \sum_{cyc} \frac{h_a}{w_a} \sqrt{\frac{h_a}{r_a}} \cdot \sum_{cyc} \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}}$$

**Proposed by Bogdan Fuștei-Romania**

**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \frac{h_a}{w_a} &= \frac{\frac{2S}{a}}{\frac{2bc}{b+c} \cdot \cos \frac{A}{2}} = S \cdot \left( \frac{b+c}{ab \cdot \cos \frac{A}{2}} \right) = \frac{S \cdot (b+c)}{4RS \cdot \cos \frac{A}{2}} = \frac{b+c}{4R \cdot \cos \frac{A}{2}} \\ \frac{h_a}{r_a} &= \frac{\frac{2S}{a}}{s \cdot \tan \frac{A}{2}} = \frac{\frac{2sr}{4R \cdot \sin \frac{A}{2} \cos \frac{A}{2}}}{s \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}} = \frac{1}{\sin^2 \frac{A}{2}} \cdot \frac{r}{2R} \Rightarrow \sqrt{\frac{h_a}{r_a}} = \frac{1}{\sin \frac{A}{2}} \cdot \sqrt{\frac{r}{2R}} \\ \sum_{cyc} \frac{h_a}{w_a} \sqrt{\frac{h_a}{r_a}} &= \sqrt{\frac{r}{2R}} \cdot \sum_{cyc} \frac{b+c}{4R \sin \frac{A}{2} \cos \frac{A}{2}} = \sqrt{\frac{r}{2R}} \cdot \sum_{cyc} \frac{b+c}{a} \end{aligned}$$

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$$r_a - r = 4R \sin^2 \frac{A}{2}$$

$$\Rightarrow \frac{r_a - r}{w_a} \cdot \sqrt{\frac{h_a}{r_a}} = \frac{4R \sin^2 \frac{A}{2}}{\frac{2bc}{b+c} \cdot \sin \frac{A}{2} \cdot \cos \frac{A}{2}} \cdot \sqrt{\frac{r}{2R}} = 2R \sqrt{\frac{r}{2R}} \cdot \sum_{cyc} \frac{b+c}{bc} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}$$

$$= 2R \sqrt{\frac{r}{2R}} \cdot \sum_{cyc} \frac{b+c}{\frac{2S}{2 \sin \frac{A}{2} \cdot \cos \frac{A}{2}}} \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}} = \frac{2R}{S} \cdot \sqrt{\frac{r}{2R}} \cdot \sum_{cyc} \left( (b+c) \sin^2 \frac{A}{2} \right) \Rightarrow$$

$$\sum_{cyc} \frac{h_a}{w_a} \sqrt{\frac{h_a}{r_a}} \cdot \sum_{cyc} \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} = \frac{1}{s} \sum_{cyc} \frac{b+c}{a} \cdot \sum_{cyc} (b+c) \sin^2 \frac{A}{2} = \Omega$$

$$\sum_{cyc} \frac{b+c}{a} = \sum_{cyc} \frac{2s-a}{a} = 2s \sum_{cyc} \frac{1}{a} - 3 = 2s \cdot \frac{s^2 + 4Rr + r^2}{4Rrs} - 3 = \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$\sum_{cyc} (b+c) \sin^2 \frac{A}{2} = \sum_{cyc} (2s-a) \sin^2 \frac{A}{2} = 2s \sum_{cyc} \sin^2 \frac{A}{2} - \sum_{cyc} a \cdot \sin^2 \frac{A}{2}$$

$$= 2s \cdot \frac{2R-r}{2R} - 2R \sum_{cyc} \sin A \left( \frac{1 - \cos A}{2} \right) = \frac{s}{R} (2R-r) - R \left( \sum_{cyc} \sin A - \frac{1}{2} \sum_{cyc} \sin 2A \right)$$

$$= \frac{s}{R} (2R-r) - R \left( \frac{s}{R} - \frac{1}{2} \cdot \frac{2sr}{R^2} \right) = 2s - \frac{sr}{R} - s + \frac{sr}{R} = s$$

So,

$$RHS = \Omega = \frac{1}{s} \cdot \sum_{cyc} \frac{b+c}{a} \cdot \sum_{cyc} (b+c) \sin^2 \frac{A}{2} = \frac{1}{s} \cdot \frac{s^2 - 2Rr + r^2}{2Rr} \cdot s = \frac{s^2 - 2Rr + r^2}{2Rr}$$

$$\frac{r_a + r}{r_a - r} = \frac{s \cdot \tan \frac{A}{2} + (s-a) \cdot \tan \frac{A}{2}}{s \cdot \tan \frac{A}{2} - (s-a) \cdot \tan \frac{A}{2}} = \frac{2s-a}{a} = 2s \cdot \frac{1}{a} - 1$$

$$LHS = \sum_{cyc} \frac{r_a + r}{r_a - r} = 2s \sum_{cyc} \frac{1}{a} - 3 = 2s \cdot \frac{s^2 + 4Rr + r^2}{4Rrs} - 3 = \frac{s^2 - 2Rr + r^2}{2Rr} = RHS$$

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1620. In  $\triangle ABC$  the following relationship holds:

$$\frac{4}{R^4} \leq \frac{\cot^2 \frac{A}{2}}{m_a w_a} + \frac{\cot^2 \frac{B}{2}}{m_b w_b} + \frac{\cot^2 \frac{C}{2}}{m_c w_c} \leq \frac{R}{4r^3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} &= \sum_{cyc} \frac{\frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}}{4S^2} = \frac{R^2}{S^2} \cdot \sum_{cyc} \cos^4 \frac{A}{2} = \frac{R^2}{S^2} \cdot \frac{(4R+r)^2 - s^2}{8R^2} \\ &= \frac{(4R+r)^2 - s^2}{8S^2} \end{aligned}$$

$$\text{Let: } \Omega = 4 \cdot \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{m_a w_a}$$

$$\Omega \stackrel{m_a w_a \geq h_a}{\leq} 4 \cdot \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} = \frac{(4R+r)^2 - s^2}{2S^2}$$

$$\text{We must show that: } \frac{(4R+r)^2 - s^2}{2S^2} \leq \frac{R}{4r^3} \Leftrightarrow \frac{(4R+r)^2 - s^2}{2s^2 r^2} \leq \frac{R}{4r^3}$$

$$\Leftrightarrow r[(4R+r)^2 - s^2] \leq s^2 R \Leftrightarrow r(4R+r)^2 \leq (R+r)s^2$$

$$\text{But: } s^2 \geq 16Rr - 5r^2 \text{ (Mitrinovic)}$$

$$(16Rr - 5r^2)(R+r) \geq r(4R+r)^2 \Leftrightarrow (16R - 5r)(R+r) \geq (4R+r)^2$$

$$\Leftrightarrow 16R^2 + 11Rr - 5r^2 \geq 16R^2 + 8Rr + r^2 \Leftrightarrow 3Rr \geq 6R^2$$

$$\Leftrightarrow R \geq 2r \text{ (Euler)}$$

$$m_a w_a \leq \frac{R}{2r} \cdot h_a \Rightarrow m_a w_a \leq \left(\frac{R}{2r}\right)^2 h_a^2 \text{ and analogs.}$$

$$\Omega = 4 \cdot \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{m_a w_a} \geq 4 \cdot \left(\frac{2r}{R}\right)^2 \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2}$$

$$= 4 \cdot \left(\frac{2r}{R}\right)^2 \cdot \frac{(4R+r)^2 - s^2}{2S^2} \stackrel{(**)}{\geq} \frac{4}{R^2}$$

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$$(**) \Leftrightarrow (4R + r)^2 - s^2 \geq 2s^2 \Leftrightarrow (4R + r)^2 \geq 3s^2 \Leftrightarrow 4R + r \geq s\sqrt{3} \text{ true.}$$

Proved.

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} m_a^2 m_b^2 m_c^2 &= \frac{1}{64} (2b^2 + 2c^2 - 2a^2)(2c^2 + 2a^2 - 2b^2)(2a^2 + 2b^2 - 2c^2) \stackrel{(1)}{=} \\ &= \frac{1}{64} \{-4\sum a^6 + 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2\} \end{aligned}$$

$$\text{Now, } \sum a^6 = (\sum a^2)^3 - 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

$$= (\sum a^2)^3 - 3(2a^2 b^2 c^2 + \sum a^2 b^2 (\sum a^2 - c^2))$$

$$= (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2 \therefore \sum a^6 \stackrel{(2)}{=} (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2$$

$$\text{Again, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(3)}{=} (\sum a^2 b^2) \sum a^2 - 3a^2 b^2 c^2$$

$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2 =$$

$$\frac{1}{64} \{-4(\sum a^2)^3 - 12a^2 b^2 c^2 + 12(\sum a^2 b^2) \sum a^2 + 6(\sum a^2 b^2) \sum a^2 - 18a^2 b^2 c^2 + 3a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 + 18(\sum a^2 b^2) \sum a^2 - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 + 18((\sum ab)^2 - 2abc(2s))(\sum a^2) - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2$$

$$- 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\}$$

$$= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\} \leq \frac{R^2 s^4}{4}$$

$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\leq} 0$$

$$\text{Now, LHS of (i)} \stackrel{\text{Gerretsen}}{\leq} -s^4(8Rr - 36r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4)$$

$$- r^3(4R + r)^3 \stackrel{?}{\leq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{?}{\stackrel{(ii)}{\leq}} 20rs^4$$

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Now, LHS of (ii)  $\underbrace{\sum_{(a)}^{\text{Gerretsen}}}_{(a)} s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$

and RHS of (ii)  $\underbrace{\sum_{(b)}^{\text{Gerretsen}}}_{(b)} 20rs^2(4R^2 + 4Rr + 3r^2)$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove :

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2s^2$$

Now, LHS of (iii)  $\underbrace{\sum_{(c)}^{\text{Gerretsen}}}_{(c)} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)$

+  $r(4R + r)^3$  and RHS of (iii)  $\underbrace{\sum_{(d)}^{\text{Gerretsen}}}_{(d)} 27r^2(4R^2 + 4Rr + 3r^2)$

(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \stackrel{(4)}{\geq} \frac{R s^2}{2}$$

$$\text{Now, } r_a w_a = \left( \tan \frac{A}{2} \right) \left( \frac{2bc \cos \frac{A}{2}}{b + c} \right) = \left( s \sin \frac{A}{2} \right) \left( \frac{2bc}{b + c} \right) \stackrel{\text{HM} \leq \text{GM}}{\geq} \left( s \sin \frac{A}{2} \right) \sqrt{bc}$$

$$= s \sqrt{(s - b)(s - c)} = \sqrt{s(s - b)} \sqrt{s(s - c)} \stackrel{(5)}{\geq} m_b m_c$$

$$\text{Now, } \sum \frac{\cot^2 \frac{A}{2}}{m_a w_a} = \sum \frac{s^2}{r_a^2 m_a w_a} \stackrel{\text{by (5) and its analogs}}{\geq} \sum \frac{s^2}{r_a m_a m_b m_c} \stackrel{\text{by (4)}}{\geq} \left( \frac{2s^2}{R s^2} \right) \sum \frac{1}{r_a}$$

$$= \frac{2R}{R^2 r} \stackrel{\text{Euler}}{\geq} \frac{4}{R^2} \therefore \sum \frac{\cot^2 \frac{A}{2}}{m_a w_a} \stackrel{(m)}{\geq} \frac{4}{R^2}$$

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$$\text{Now, } m_a w_a \stackrel{\text{Ioscu}}{\underset{(6)}{\geq}} \left( \frac{b+c}{2} \cos \frac{A}{2} \right) \left( \frac{2bc}{b+c} \cos \frac{A}{2} \right) = bc \left( \frac{s(s-a)}{bc} \right) = s(s-a)$$

$$\text{Now, } \sum \frac{\cot^2 \frac{A}{2}}{m_a w_a} \stackrel{\text{by (6) and its analogs}}{\geq} \sum \frac{1}{\left\{ \frac{(s-b)(s-c)}{s(s-a)} \right\} s(s-a)} = \frac{\sum (s-a)}{sr^2} = \frac{r}{r^3} \stackrel{\text{Euler}}{\geq} \frac{R}{2r^3}$$

$$\therefore \sum \frac{\cot^2 \frac{A}{2}}{m_a w_a} \stackrel{(n)}{\geq} \frac{R}{2r^3}$$

$$(m), (n) \Rightarrow \frac{4}{R^2} \leq \frac{\cot^2 \frac{A}{2}}{m_a w_a} + \frac{\cot^2 \frac{B}{2}}{m_b w_b} + \frac{\cot^2 \frac{C}{2}}{m_c w_c} \leq \frac{R}{2r^3} \quad (\text{Proved})$$

**1621. In  $\triangle ABC$  the following relationship holds:**

$$\frac{\cos^2 \frac{B-C}{2}}{\cos \frac{A}{2}} + \frac{\cos^2 \frac{C-A}{2}}{\cos \frac{B}{2}} + \frac{\cos^2 \frac{A-B}{2}}{\cos \frac{C}{2}} \geq \frac{4\sqrt{3}r}{R}$$

*Proposed by Marin Chirciu-Romania*

*Solution 1 by Marian Ursărescu-Romania*

$$\frac{\cos^2 \frac{B-C}{2}}{\cos \frac{A}{2}} + \frac{\cos^2 \frac{C-A}{2}}{\cos \frac{B}{2}} + \frac{\cos^2 \frac{A-B}{2}}{\cos \frac{C}{2}} \stackrel{\text{Am-Gm}}{\geq} 3 \sqrt[3]{\frac{\cos^2 \frac{B-C}{2} \cos^2 \frac{C-A}{2} \cos^2 \frac{A-B}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}}$$

We must show:

$$3\sqrt{3} \cdot \frac{\cos^2 \frac{B-C}{2} \cdot \cos^2 \frac{C-A}{2} \cdot \cos^2 \frac{A-B}{2}}{\cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}} \geq \frac{64r^3}{R^3}; \quad (1)$$

$$\text{But: } \prod_{\text{cyc}} \cos \frac{B-C}{2} = \frac{s^2+r^2+2Rr}{8R^2} \text{ and } \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} = \frac{s}{4R}; \quad (2)$$

From (1),(2) we must show:

$$3\sqrt{3} \cdot \frac{(s^2+r^2+2Rr)^2}{64R^4} \cdot \frac{4R}{s} \geq \frac{64r^3}{R^3} \Leftrightarrow 3\sqrt{3}(s^2+r^2+2Rr)^2 \geq 4^5 r^3 s; \quad (3)$$

$$\text{But: } s \leq \frac{3\sqrt{3}}{2} R \text{ (Mitrinovic) and } r \leq \frac{R}{2} \text{ (Euler);} \quad (4)$$

$$\text{From (3),(4) we must show: } (s^2+r^2+2Rr)^2 \geq 4^4 R^2 r^2$$

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$$\Leftrightarrow s^2 + r^2 + 2Rr \geq 16Rr \Leftrightarrow s^2 \geq 14Rr - r^2; \quad (5)$$

From Gerretsen inequality we have:  $s^2 \geq 16Rr - 5r^2$ ; (6)

From (5),(6) we must show:  $16Rr - 5r^2 \geq 14Rr - r^2$

$$\Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r \text{ (Euler). Proved.}$$

### Solution 2 by Bogdan Fuștei-Romania

We known that:  $\cos \frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$  and analogs.

$$\cos^2 \frac{B-C}{2} \geq \frac{2r}{R} \text{ and analogs.}$$

So, we have:  $\sum_{cyc} \frac{\cos^2 \frac{B-C}{2}}{\cos^2 \frac{A}{2}} \geq \frac{2r}{R} \frac{1}{\cos^2 \frac{A}{2}} + \frac{2r}{R} \frac{1}{\cos^2 \frac{B}{2}} + \frac{2r}{R} \frac{1}{\cos^2 \frac{C}{2}}$

We must show that:

$$\frac{2r}{R} \cdot \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2}} \geq \frac{4\sqrt{3}r}{R} = \frac{2r}{R} \cdot 2\sqrt{3} \Rightarrow \sum_{cyc} \frac{1}{\cos^2 \frac{A}{2}} \geq 2\sqrt{3}$$

$$\frac{1}{\cos^2 \frac{A}{2}} + \frac{1}{\cos^2 \frac{B}{2}} + \frac{1}{\cos^2 \frac{C}{2}} \stackrel{Am-Gm}{\geq} 3 \cdot \sqrt[3]{\frac{1}{\cos^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2} \cdot \cos^2 \frac{C}{2}}}; \quad (1)$$

$$\cos \frac{A}{2} = \sqrt{\frac{r_b r_c}{bc}} \text{ and analogs, then:}$$

$$\cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} = \frac{r_a r_b r_c}{abc} = \frac{Ss}{4RS} = \frac{s}{4R}$$

We must show:  $3 \sqrt[3]{\frac{4R}{s}} \geq 2\sqrt{3} \Leftrightarrow 27 \cdot \frac{4R}{s} \geq 8\sqrt{27} \Leftrightarrow 3\sqrt{3}R \geq 2s$

$$\Leftrightarrow \frac{3\sqrt{3}}{2}R \geq s \text{ (Mitrinovic). So, } 3 \cdot \sqrt[3]{\frac{1}{\cos^2 \frac{A}{2} \cdot \cos^2 \frac{B}{2} \cdot \cos^2 \frac{C}{2}}} \geq 2\sqrt{3}; \quad (2)$$

From (1),(2) the inequality is proved.

1622. In  $\triangle ABC$  the following relationship holds:

$$3 \leq \frac{r_b r_c}{w_a^2} + \frac{r_c r_a}{w_b^2} + \frac{r_a r_b}{w_c^2} \leq \frac{3R}{2r}$$

Proposed by Marin Chirciu-Romania

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### Solution 1 by Marian Ursărescu-Romania

$$w_a \leq \sqrt{s(s-a)} \Rightarrow \frac{1}{w_a^2} \geq \frac{1}{s(s-a)} \Rightarrow \sum_{\text{cyc}} \frac{r_b r_c}{w_a^2} = \frac{1}{s} \cdot \sum_{\text{cyc}} \frac{r_b r_c}{s-a}$$

We must show that:  $\sum_{\text{cyc}} \frac{r_b r_c}{s-a} \geq 3s$ ; (1)

But:  $\sum_{\text{cyc}} \frac{r_b r_c}{s-a} \geq 3 \cdot \sqrt[3]{\frac{r_a^2 r_b^2 r_c^2}{(s-a)(s-b)(s-c)}}$ ; (2)

$$\frac{(r_a r_b r_c)^2}{(s-a)(s-b)(s-c)} \geq 3s^3; (3)$$

$\therefore r_a r_b r_c = s^2 r$ ;  $(s-a)(s-b)(s-c) = sr^2$ , then

$$\frac{(r_a r_b r_c)^2}{(s-a)(s-b)(s-c)} = \frac{s^4 r^2}{sr^2} = s^3 \rightarrow (3) \text{ --it's true.}$$

$$w_a \geq h_a \rightarrow \frac{1}{w_a} \leq \frac{1}{h_a} \rightarrow \sum_{\text{cyc}} \frac{r_b r_c}{w_a^2} \leq \sum_{\text{cyc}} \frac{r_b r_c}{h_a^2}; (4)$$

$$\frac{r_b r_c}{h_a^2} = \frac{\frac{s^2}{(s-b)(s-c)}}{\frac{4s^2}{a^2}} = \frac{1}{4} \cdot \frac{a^2}{(s-b)(s-c)}; (5)$$

From (4) and (5) we must show that:

$$\frac{1}{4} \sum_{\text{cyc}} \frac{a^2}{(s-b)(s-c)} \leq \frac{3R}{2r}; (5)$$

But:  $\sum_{\text{cyc}} \frac{a^2}{(s-b)(s-c)} = \frac{4(R+r)}{r}$ ; (6)

From (5) and (6) we must show:  $\frac{R+r}{r} \leq \frac{3R}{2r} \Leftrightarrow 2R + 2r \leq 3R \Leftrightarrow 2r \leq R$  --(Euler)

### Solution 2 by Bogdan Fuștei-Romania

$$w_a^2 = \frac{4bc}{(b+c)^2} s(s-a); r_b r_c = s(s-a) \text{ and analogs.}$$

$$\frac{r_b r_c}{w_a^2} = \frac{b^2 + 2bc + c^2}{4bc} = \frac{1}{2} + \frac{1}{4} \left( \frac{b}{c} + \frac{c}{b} \right) \text{ and analogs.}$$

$$m_a \geq \frac{b^2 + c^2}{4R} \text{ (Teresin inequality) and analogs.}$$

$$bc = 2Rh_a \text{ --and analogs}$$

$$\frac{m_a}{h_a} \geq \frac{b^2 + c^2}{2bc} = \frac{1}{2} \left( \frac{b}{c} + \frac{c}{b} \right); \frac{R}{2r} \geq \frac{m_a}{h_a} \text{ (Panaitopol inequality)}$$

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$$\Rightarrow \frac{R}{2r} \geq \frac{1}{2} \left( \frac{b}{c} + \frac{c}{b} \right) \Rightarrow \frac{R}{r} \geq \frac{b}{c} + \frac{c}{b} \text{ and analogs.}$$

$$\sum_{\text{cyc}} \frac{r_b r_c}{w_a^2} = \frac{3}{2} + \frac{1}{4} \sum_{\text{cyc}} \frac{b+c}{a} \geq 3 \cdot 4 \Rightarrow$$

$$6 + \sum_{\text{cyc}} \frac{b+c}{a} \geq 12 \Rightarrow \sum_{\text{cyc}} \frac{b+c}{a} \geq 6 \text{ true, from Am-Gm.}$$

$$\text{So, } 3 \leq \sum_{\text{cyc}} \frac{r_b r_c}{w_a^2} \quad (1)$$

$$\frac{R}{4r} \geq \frac{1}{4} \left( \frac{b}{c} + \frac{c}{b} \right) \left( + \frac{1}{2} \right) \Rightarrow \frac{1}{2} + \frac{R}{4r} \geq \frac{r_b r_c}{w_a^2} \text{ and analogs.}$$

$$\frac{3}{2} + \frac{3R}{4r} \geq \sum_{\text{cyc}} \frac{r_b r_c}{w_a^2}; \quad \frac{3R}{2r} \geq \frac{3}{2} + \frac{3R}{4r}$$

$$\frac{3R}{2r} = 2 \cdot \frac{3R}{4r} = \frac{3R}{4r} + \frac{3R}{4r} \geq \frac{3}{2} + \frac{3R}{4r} \Rightarrow \frac{3R}{4r} \geq \frac{3}{2} \Rightarrow \frac{R}{2r} \geq 1 \Rightarrow R \geq 2r \text{ --(Euler)}$$

$$\sum_{\text{cyc}} \frac{r_b r_c}{w_a^2} \leq \frac{R}{2r}; \quad (2)$$

$$\text{From (1),(2) we have: } 3 \leq \frac{r_b r_c}{w_a^2} + \frac{r_c r_a}{w_b^2} + \frac{r_a r_b}{w_c^2} \leq \frac{3R}{2r}$$

### Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\text{We have: } w_a = \frac{2\sqrt{bc}}{b+c} \cdot \sqrt{r_b r_c} \text{ and analogs.}$$

$$w_a^2 = \frac{4bc}{(b+c)^2} \cdot r_b r_c \Rightarrow \frac{r_b r_c}{w_a^2} = \frac{(b+c)^2}{4bc} \text{ and analogs.}$$

$$\Rightarrow \Omega = \frac{r_b r_c}{w_a^2} + \frac{r_c r_a}{w_b^2} + \frac{r_a r_b}{w_c^2} = \frac{1}{4} \left[ \frac{(a+b)^2}{ab} + \frac{(b+c)^2}{bc} + \frac{(c+a)^2}{ca} \right]$$

$$= \frac{c(a+b)^2 + a(b+c)^2 + b(c+a)^2}{4abc}$$

$$= \frac{c(a^2 + b^2 + 2ab) + a(b^2 + c^2 + 2bc) + b(a^2 + c^2 + 2ac)}{4abc}$$

$$= \frac{ab(a+b) + bc(b+c) + ca(c+a)}{4abc} = \frac{2s(s^2 + 4Rr + r^2) + 3 \cdot 4Rrs}{4 \cdot 4Rrs}$$

$$= \frac{s^2 + 4Rr + r^2 + 6Rr}{8Rr} = \frac{s^2 + 10Rr + r^2}{8Rr}$$

$$\Omega \geq 3 \Leftrightarrow \frac{s^2 + 10Rr + r^2}{8Rr} \geq 3 \Leftrightarrow s^2 \geq 14Rr - r^2$$

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But:  $s^2 \geq 16Rr - 5r^2$

We must show:  $16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow R \geq 2r$ , true,(Euler)

$$\Omega \leq \frac{3R}{2r} \Leftrightarrow \frac{s^2 + 10Rr + r^2}{8Rr} \Leftrightarrow s^2 + 10Rr + r^2 \leq 12R^2 \Leftrightarrow s^2 \leq 12R^2 - 10Rr - r^2$$

But:  $s^2 \leq 4R^2 + 4Rr + 3r^2$

So, we need to prove:  $4R^2 + 4Rr + 3r^2 \leq 12R^2 - 10Rr - r^2$

$$\Leftrightarrow 8R^2 - 14Rr - 4r^2 \geq 0 \Leftrightarrow 4R^2 - 7Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(4R + r) \geq 0 \text{ true, by}$$

Euler.Proved.

1623. In  $\triangle ABC$  the following relationship holds:

$$2 < \frac{a^2}{w_b^2 + w_c^2} + \frac{b^2}{w_c^2 + w_a^2} + \frac{c^2}{w_a^2 + w_b^2} \leq \frac{R}{r}$$

Proposed by Marin Chirciu-Romania

Solution by Adrian Popa-Romania

$$\begin{aligned} i) w_a \geq h_a &\Rightarrow \sum_{cyc} \frac{a^2}{w_b^2 + w_c^2} \leq \sum_{cyc} \frac{a^2}{h_b^2 + h_c^2} = \sum_{cyc} \frac{a^2}{\frac{4S^2}{b^2} + \frac{4S^2}{c^2}} = \sum_{cyc} \frac{a^2 b^2 c^2}{4S^2(b^2 + c^2)} \\ &= \sum_{cyc} \frac{16R^2 S^2}{4S^2(b^2 + c^2)} = 4R^2 \sum_{cyc} \frac{1}{b^2 + c^2} \leq 4R^2 \sum_{cyc} \frac{1}{2bc} \\ &= 2R^2 \left( \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right) = 2R^2 \frac{a+b+c}{abc} = 2R^2 \cdot \frac{2s}{4RS} = \frac{R^2 s}{Rs} = \frac{R}{r} \end{aligned}$$

$$ii) w_a \leq \sqrt{s(s-a)} \Rightarrow w_a^2 \leq s(s-a) \Rightarrow$$

$$\begin{aligned} \sum_{cyc} \frac{a^2}{w_b^2 + w_c^2} &> \sum_{cyc} \frac{a^2}{s(s-b) + s(s-c)} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum a)^2}{2(3s^2 - sa - sb - sc)} \\ &= \frac{4s^2}{2(3s^2 - s(a+b+c))} = \frac{2s^2}{3s^2 - 2s^2} = 2 \end{aligned}$$

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**1624. In  $\triangle ABC$ ,  $O$  – circumcenter,  $I$  – incenter the following relationship holds:**

$$(b - c)^2 \sin A + (c - a)^2 \sin B + (a - b)^2 \sin C \leq \mu \cdot OI^2, \quad \mu \geq 6\sqrt{3}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\text{We have: } a = 2R \sin A, b = 2R \sin B, c = 2R \sin C$$

$$\sin A = \frac{a}{2R}, \sin B = \frac{b}{2R}, \sin C = \frac{c}{2R}$$

$$\text{LHS} = (b - c)^2 \sin A + (c - a)^2 \sin B + (a - b)^2 \sin C$$

$$= \frac{1}{2R} [a(b - c)^2 + b(c - a)^2 + c(a - b)^2]$$

$$= \frac{1}{2R} [a(b^2 - 2bc + c^2) + b(c^2 - 2ca + a^2) + c(a^2 - 2ab + b^2)]$$

$$= \frac{1}{2R} [ab(a + b) + bc(b + c) + ca(c + a) - 6abc]$$

$$= \frac{1}{2R} [2s(s^2 + 4Rr + r^2) - 3 \cdot 4Rrs - 6 \cdot 4Rrs]$$

$$= \frac{1}{2R} [2s(s^2 + 4Rr + r^2) - 36Rrs] = \frac{s}{R} [s^2 + 4Rr + r^2 - 18Rr] = \frac{s}{R} [s^2 - 14Rr + r^2]$$

$$\text{RHS} = \mu \cdot OI^2 = \mu \cdot (R^2 - 2Rr); \quad \mu \geq 6\sqrt{3}$$

We must show that:

$$\frac{s}{R} [s^2 - 14Rr + r^2] \leq \mu \cdot (R^2 - 2Rr); \quad (*)$$

$$s \leq \frac{3\sqrt{3}}{2} R \Rightarrow \frac{s}{R} \leq \frac{3\sqrt{3}}{2}$$

$$\mu \geq 6\sqrt{3} \Rightarrow \mu(R^2 - 2Rr) \geq 6\sqrt{3}(R^2 - 2Rr)$$

$$\text{So, we need to prove: } \frac{3\sqrt{3}}{2} (s^2 - 14Rr + r^2) \leq 6\sqrt{3}(R^2 - 2Rr)$$

$$\Leftrightarrow s^2 - 14Rr + r^2 \leq 4(R^2 - 2Rr) \Leftrightarrow s^2 \leq 4R^2 + 6Rr - r^2$$

$$\text{But: } s^2 \leq 4R^2 + 4Rr + 3r^2$$

$$\text{We just check: } 4R^2 + 4Rr + 3r^2 \leq 4R^2 + 6Rr - r^2 \Leftrightarrow 4r^2 \leq 2Rr \Leftrightarrow R \geq 2r \text{ (Euler)}$$

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1625. In  $\triangle ABC$  the following relationship holds:

$$\cos A \cos \frac{B}{2} \cos \frac{C}{2} + \cos B \cos \frac{C}{2} \cos \frac{A}{2} + \cos C \cos \frac{A}{2} \cos \frac{B}{2} \leq \frac{9}{8}$$

Proposed by Ionuț Florin Voinea-Romania

Solution by Tran Hong-Dong Thap-Vietnam

We have:

$$\cos A = 2 \cos^2 \frac{A}{2} - 1, \cos B = 2 \cos^2 \frac{B}{2} - 1, \cos C = 2 \cos^2 \frac{C}{2} - 1$$

$$LHS = \sum_{cyc} \cos A \cos \frac{B}{2} \cos \frac{C}{2} = \left( 2 \prod_{cyc} \cos \frac{A}{2} \sum_{cyc} \cos \frac{A}{2} \right) - \sum_{cyc} \left( \cos \frac{A}{2} \cos \frac{B}{2} \right)$$

$$\underbrace{\sum_{cyc} \cos \frac{A}{2}}_{\substack{\leq \frac{3\sqrt{3}}{2} \\ \text{Am-Gm}}} \left( 3\sqrt{3} \prod_{cyc} \cos \frac{A}{2} \right) - 3^3 \sqrt{\prod_{cyc} \cos^2 \frac{A}{2}} = 3\sqrt{3}t^3 - 3t^2 = 3t^2(\sqrt{3}t - 1)$$

$$\leq 3\sqrt{3} \cdot \frac{3\sqrt{3}}{8} - 3 \cdot \frac{3}{4} = \frac{27}{8} - \frac{9}{4} = \frac{9}{8}$$

$$\text{Where } 0 < t = \sqrt[3]{\prod_{cyc} \cos \frac{A}{2}} = \sqrt[3]{\frac{s}{4R}} \stackrel{s \leq \frac{3\sqrt{3}}{2}R}{\leq} \sqrt[3]{\frac{3\sqrt{3}}{8}}. \text{ Proved.}$$

1626. In  $\triangle ABC$  the following relationship holds:

$$\sqrt{\frac{a}{c}} \cdot \cos A + \sqrt{\frac{b}{a}} \cdot \cos B + \sqrt{\frac{c}{b}} \cdot \cos C \leq \frac{3R}{4r}$$

Proposed by Ionuț Florin Voinea

Solution by Tran Hong-Dong Thap-Vietnam

Lemma: If  $x, y, z, \theta_1, \theta_2, \theta_3 \in \mathbb{R}, \theta_1 + \theta_2 + \theta_3 = \pi$  then:

$$x^2 + y^2 + z^2 \stackrel{(*)}{\geq} 2(yz \cos \theta_1 + zx \cos \theta_2 + xy \cos \theta_3)$$

Proof:  $\theta_3 = \pi - (\theta_1 + \theta_2) \rightarrow \cos \theta_3 = -\cos(\theta_1 + \theta_2) = \sin \theta_1 \sin \theta_2 - \cos \theta_1 \cos \theta_2$

$$x^2 + y^2 + z^2 - 2(yz \cos \theta_1 + zx \cos \theta_2 + xy \cos \theta_3)$$

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$$= (z - (x\cos\theta_2 + y\cos\theta_1))^2 + (x\sin\theta_2 - y\sin\theta_1)^2 \geq 0 \text{ true, so (*) true.}$$

$$\text{Now, choose: } x = \sqrt{\frac{qr}{p}}, y = \sqrt{\frac{rp}{q}}, z = \sqrt{\frac{pq}{r}}; p, q, r > 0$$

$$(*) \Leftrightarrow p\cos\theta_1 + q\cos\theta_2 + r\cos\theta_3 \leq \frac{1}{2}\left(\frac{qr}{p} + \frac{rp}{q} + \frac{pq}{r}\right)$$

$$\text{Let: } p = \sqrt{\frac{a}{c}}, q = \sqrt{\frac{b}{a}}, r = \sqrt{\frac{c}{b}}; \theta_1 = A, \theta_2 = B, \theta_3 = C$$

$$\text{We have: } \sqrt{\frac{a}{c}} \cdot \cos A + \sqrt{\frac{b}{a}} \cdot \cos B + \sqrt{\frac{c}{b}} \cdot \cos C \leq \frac{1}{2}\left(\frac{c}{a} + \frac{a}{b} + \frac{b}{c}\right)$$

$$\text{We must show that: } \frac{1}{2}\left(\frac{c}{a} + \frac{a}{b} + \frac{b}{c}\right) \leq \frac{3R}{4r} \Leftrightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{3R}{2r}; \quad (1)$$

$$\text{We have: } \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \leq \frac{R}{r} + \frac{R}{r} + \frac{R}{r} = \frac{3R}{r}$$

$$\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \leq \frac{3R}{r}; \quad (2)$$

Now, wlog, we suppose:  $a \geq b \geq c$ .

$$\text{We must show that: } \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c} \Leftrightarrow$$

$$a^2b + b^2c + c^2a \geq b^2a + c^2b + a^2c \Leftrightarrow (a-b)(a-c)(b-c) \geq 0 \text{ true by } a \geq b \geq c.$$

So,

$$2\left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) \leq \left(\frac{a}{b} + \frac{b}{c} + \frac{c}{a}\right) + \left(\frac{b}{a} + \frac{c}{b} + \frac{a}{c}\right) \stackrel{\text{by (2)}}{\leq} \frac{3R}{r} \Leftrightarrow \frac{a}{b} + \frac{b}{c} + \frac{c}{a} \leq \frac{3R}{r} \rightarrow (1) \text{ is true. Proved.}$$

**1627. In  $\triangle ABC$  the following relationship holds:**

$$w_b \sqrt{\sin \frac{A}{2}} + w_c \sqrt{\sin \frac{B}{2}} + w_a \sqrt{\sin \frac{C}{2}} \leq \frac{s\sqrt{3}}{2} \sqrt{\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}}$$

*Proposed by Mokhtar Khassani-Mostaganem-Algerie*

*Solution by Marian Ursărescu-Romania*

We must show:

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$$\left( w_b \sqrt{\sin \frac{A}{2}} + w_c \sqrt{\sin \frac{B}{2}} + w_a \sqrt{\sin \frac{C}{2}} \right)^2 \leq \frac{3s^2}{2} \left( \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \right); \quad (1)$$

From Cauchy inequality we have:

$$\left( w_b \sqrt{\sin \frac{A}{2}} + w_c \sqrt{\sin \frac{B}{2}} + w_a \sqrt{\sin \frac{C}{2}} \right)^2 \leq (w_a^2 + w_b^2 + w_c^2) \left( \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \right); \quad (2)$$

From (1),(2) we must show:

$$(w_a^2 + w_b^2 + w_c^2) \left( \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \right) \leq \frac{3s^2}{2} \left( \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \right); \quad (3)$$

But in any  $\triangle ABC$  we have:  $w_a \leq \sqrt{s(s-a)}$  then:

$$w_a^2 + w_b^2 + w_c^2 \leq s(s-a) + s(s-b) + s(s-c) = s^2; \quad (4)$$

From (3),(4) we must show:

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \frac{3}{2} \left( \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \right); \quad (5)$$

$$\text{But: } \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}} \leq \frac{s-b+s-c}{2\sqrt{bc}} = \frac{a}{2\sqrt{bc}} \text{ then}$$

$$\sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \leq \frac{1}{2} \left( \frac{a}{\sqrt{bc}} + \frac{b}{\sqrt{ca}} + \frac{c}{\sqrt{ab}} \right); \quad (6)$$

From (5),(6) we must show:

$$\frac{a}{\sqrt{bc}} + \frac{b}{\sqrt{ca}} + \frac{c}{\sqrt{ab}} \leq 3 \left( \frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} \right) \text{ true from Cauchy inequality.}$$

**1628. In  $\triangle ABC$  the following relationship holds:**

$$\frac{\sqrt{3}-1}{2\sqrt{3}r} \leq \frac{\cot A}{s+r_a} + \frac{\cot B}{s+r_b} + \frac{\cot C}{s+r_c} \leq \frac{R\sqrt{3}-2r}{2\sqrt{3}Rr}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Bogdan Fuștei-Romania*

$$\cot x = \frac{1 - \tan^2 \frac{x}{2}}{2 \tan \frac{x}{2}}$$

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$$\tan \frac{A}{2} = \frac{r_a}{s} \text{ and analogs.}$$

$$\cot A = \frac{1 - \frac{r_a^2}{s^2}}{\frac{2r_a}{s}} = \frac{s^2 - r_a^2}{s^2} \cdot \frac{s}{2r_a} = \frac{s^2 - r_a^2}{2sr_a} \text{ and analogs.}$$

$$\frac{\cot A}{s + r_a} = \frac{s^2 - r_a^2}{2sr_a(s + r_a)} = \frac{(s - r_a)(s + r_a)}{2sr_a(s + r_a)} = \frac{s - r_a}{2sr_a} \text{ and analogs.}$$

$$\sum_{cyc} \frac{\cot A}{s + r_a} = \frac{1}{2s} \sum_{cyc} \frac{s - r_a}{r_a} = \frac{1}{2s} \sum_{cyc} \left( \frac{s}{r_a} - 1 \right) = \frac{1}{2s} \left( \frac{s}{r} - 3 \right) = \frac{s - 3r}{2sr}$$

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

$$\frac{\sqrt{3} - 1}{2\sqrt{3}r} \leq \frac{s - 3r}{2sr} \Leftrightarrow \frac{\sqrt{3} - 1}{\sqrt{3}} \leq \frac{s - 3r}{s} \Leftrightarrow s\sqrt{3} - s \leq s\sqrt{3} - 3\sqrt{3}r$$

$$\Leftrightarrow 3\sqrt{3}r \leq s \text{ (Mitrinovic inequality)}$$

$$\frac{s - 3r}{2sr} \leq \frac{R\sqrt{3} - 2r}{2\sqrt{3}Rr} \Leftrightarrow \frac{s - 3r}{s} \leq \frac{R\sqrt{3} - 2r}{R\sqrt{3}} \Leftrightarrow$$

$$Rs\sqrt{3} - 3Rr\sqrt{3} \leq Rs\sqrt{3} - 2rs \Leftrightarrow 2sr \leq 3Rr\sqrt{3}$$

$$\Leftrightarrow s \leq \frac{3\sqrt{3}R}{2} \text{ (Mitrinovic inequality)}$$

1629. In  $\triangle ABC$  the following relationship holds:

$$\frac{\cos^2 \frac{B-C}{2}}{\sin \frac{A}{2}} + \frac{\cos^2 \frac{C-A}{2}}{\sin \frac{B}{2}} + \frac{\cos^2 \frac{A-B}{2}}{\sin \frac{C}{2}} \geq \frac{12r}{R}$$

*Proposed by Marin Chirciu-Romania*

*Solution 1 by Bogdan Fuștei-Romania*

$$\therefore \cos \frac{B-C}{2} \geq \sqrt{\frac{2r}{R}} \text{ and analogs.}$$

$$\sum_{cyc} \frac{\cos^2 \frac{B-C}{2}}{\sin \frac{A}{2}} \geq \frac{2r}{R} \frac{1}{\sin \frac{A}{2}} + \frac{2r}{R} \frac{1}{\sin \frac{B}{2}} + \frac{2r}{R} \frac{1}{\sin \frac{C}{2}}$$

$$\text{We must show that: } \frac{2r}{R} \sum \frac{1}{\sin \frac{A}{2}} \geq \frac{12r}{R} \Rightarrow \sum \frac{1}{\sin \frac{A}{2}} \geq 6$$

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We will show that:  $\sum \frac{1}{\sin \frac{A}{2}} \geq 6$

$\frac{AI}{w_a} = \frac{b+c}{2s}$  and analogs, then

$$AI = \frac{b+c}{2s} \cdot w_a \Rightarrow \frac{AI}{r} = \frac{1}{\sin \frac{A}{2}} = \frac{b+c}{2S} \cdot w_a = \frac{b+c}{a} \cdot \frac{w_a}{h_a} \text{ and analogs.}$$

So,  $\frac{1}{\sin \frac{A}{2}} = \frac{b+c}{a} \cdot \frac{w_a}{h_a} \stackrel{w_a \geq h_a}{\geq} \frac{b+c}{a}$  and analogs.

$$\sum \frac{1}{\sin \frac{A}{2}} \geq \sum \frac{b+c}{a} = \left(\frac{a}{b} + \frac{b}{a}\right) + \left(\frac{b}{c} + \frac{c}{b}\right) + \left(\frac{c}{a} + \frac{a}{c}\right) \stackrel{Am-Gm}{\geq} 6$$

$$\sum \frac{1}{\sin \frac{A}{2}} \geq 6. \text{ Proved.}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\cos\left(\frac{B-C}{2}\right) = \frac{b+c}{a} \cdot \sin \frac{A}{2} \Rightarrow$$

$$\cos^2\left(\frac{B-C}{2}\right) = \left(\frac{b+c}{a}\right)^2 \sin^2 \frac{A}{2} \stackrel{Am-Gm}{\geq} \frac{4bc}{a^2} \cdot \sin^2 \frac{A}{2} \Rightarrow$$

$$\frac{\cos^2\left(\frac{B-C}{2}\right)}{\sin \frac{A}{2}} \geq \frac{4bc \cdot \sin \frac{A}{2}}{a^2}$$

$$LHS = \sum_{cyc} \frac{\cos^2 \frac{B-C}{2}}{\sin \frac{A}{2}} \geq \sum_{cyc} \frac{4bc \cdot \sin \frac{A}{2}}{a^2} \stackrel{Am-Gm}{\geq} 12 \sqrt[3]{\frac{(abc)^2 \prod \sin \frac{A}{2}}{(abc)^2}}$$

$$\stackrel{\prod \sin \frac{A}{2} = \frac{r}{4R}}{=} 12 \sqrt[3]{\frac{r}{4R}}$$

$$\text{We need to prove: } 12 \sqrt[3]{\frac{r}{4R}} \geq \frac{12r}{R} \Leftrightarrow \frac{r}{4R} \geq \left(\frac{r}{R}\right)^3 \Leftrightarrow R^2 \geq 4r^2$$

$$\Leftrightarrow R \geq 2r \text{ (Euler). Proved}$$

**1630. In acute  $\triangle ABC$ ,  $H$  – ortocenter,  $AA_1, BB_1, CC_1$  – altitudes. Prove that:**

$$\frac{1}{HA_1^2} + \frac{1}{HB_1^2} + \frac{1}{HC_1^2} \geq \frac{12}{R^2}, A_1 \in (BC), B_1 \in (CA), C_1 \in (AB)$$

*Proposed by Marian Ursărescu-Romania*

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*Solution by Tran Hong-Dong Thap-Vietnam*

In acute  $\triangle ABC$ :  $AH + BH + CH = 2R(\cos A + \cos B + \cos C) = 2(r + R)$

$$\frac{1}{HA_1^2} + \frac{1}{HB_1^2} + \frac{1}{HC_1^2} \geq \frac{1}{3} \left( \frac{1}{HA_1} + \frac{1}{HB_1} + \frac{1}{HC_1} \right)^2 \stackrel{C.B.S.}{\geq} \frac{1}{3} \left( \frac{9}{HA_1 + HB_1 + HC_1} \right)^2$$

$$\begin{aligned} HA_1 + HB_1 + HC_1 &= h_a + h_b + h_c - (AH + BH + CH) \\ &= \frac{s^2 + r^2 + 4Rr}{2R} - 2(r + R) = \frac{s^2 + r^2 - 4R^2}{2R} \end{aligned}$$

We must show that:

$$\frac{1}{3} \left( \frac{9}{HA_1 + HB_1 + HC_1} \right)^2 \geq \frac{12}{R^2} \Leftrightarrow \frac{9}{\frac{s^2 + r^2 - 4R^2}{2R}} \geq \frac{6}{R}$$

$$\Leftrightarrow 3R^2 \geq s^2 + r^2 - 4R^2 \Leftrightarrow 7R^2 - r^2 \geq s^2$$

$$\text{But: } s^2 \leq 4R^2 + 4Rr + 3r^2$$

$$\text{So, we just check } 4R^2 + 4Rr + 3r^2 \leq 7R^2 - r^2$$

$$\Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow (R - 2r)(3R + 2r) \geq 0 \text{ true by } R \geq 2r \text{ (Euler)}$$

**1631. In  $\triangle ABC$  the following relationship holds:**

$$\left( \frac{R}{r} \right)^3 \geq \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(m_a w_a + r r_a)(m_b w_b + r r_b)(m_c w_c + r r_c)} \geq 8$$

*Proposed by Alex Szoros-Romania*

*Solution by Bogdan Fuștei-Romania*

$$m_a \geq \frac{b+c}{2} \cdot \cos \frac{A}{2} \text{ (and analogs)}$$

$$w_a = \frac{2bc}{b+c} \cdot \cos \frac{A}{2} \text{ (and analogs)}$$

$$\cos^2 \frac{A}{2} = \frac{s(s-a)}{bc} = \frac{r_b r_c}{bc} \text{ (and analogs)}$$

$$\frac{R}{2r} \geq \frac{m_a}{h_a} \text{ (Panaitopol's Inequality)}$$

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$$m_a \geq \frac{b^2 + c^2}{4R} \text{ (Tereshin); } bc = 2Rh_a; \frac{2m_a}{h_a} = \frac{b^2 + c^2}{bc} = \frac{b}{c} + \frac{c}{b}; \text{ (2)}$$

From (1),(2) we have:  $\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b}$  (Bădilă's Inequality); (3)

$$m_a w_a \geq \left( \frac{b+c}{2} \cdot \cos \frac{A}{2} \right) \cdot \left( \frac{2bc}{b+c} \cdot \cos \frac{A}{2} \right) = bc \cdot \frac{s(s-a)}{bc} = s(s-a) = r_b r_c$$

$$r_b r_c + ar_a = s(s-a) + (s-b)(s-c) = 2s^2 - s(a+b+c) + bc = bc$$

$$\frac{b^2 + c^2}{bc} = \frac{b}{c} + \frac{c}{b} \geq \frac{b^2 + c^2}{m_a w_a + r r_a}; \text{ (4)}$$

From (3),(4) we have:  $\frac{R}{r} \geq \frac{b^2 + c^2}{m_a w_a + r r_a}$  (and analogs)

$$\left( \frac{R}{r} \right)^3 \geq \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(m_a w_a + r r_a)(m_b w_b + r r_b)(m_c w_c + r r_c)}; \text{ (5)}$$

$$b^2 + c^2 = n_a^2 + g_a^2 + 2r r_a \text{ (and analogs) (B.Fuștei)}$$

$$n_a g_a \geq m_a w_a \text{ (and analogs) (B.Fuștei)}$$

$$b^2 + c^2 = n_a^2 + g_a^2 + 2r r_a \stackrel{AM-GM}{\geq} 2m_a g_a + 2r r_a \geq 2m_a w_a + 2r r_a \Rightarrow$$

$$\frac{b^2 + c^2}{m_a w_a + r r_a} \geq 2 \Rightarrow \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(m_a w_a + r r_a)(m_b w_b + r r_b)(m_c w_c + r r_c)} \geq 8; \text{ (6)}$$

From (5),(6) we get

$$\left( \frac{R}{r} \right)^3 \geq \frac{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}{(m_a w_a + r r_a)(m_b w_b + r r_b)(m_c w_c + r r_c)} \geq 8$$

**1632. In any  $\triangle ABC$ ,  $\frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 2 \sum \frac{m_b + m_c - m_a}{\sqrt{s_b s_c}}$**

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

Let  $x = m_b + m_c - m_a$ ,  $y = m_c + m_a - m_b$  and  $z = m_a + m_b - m_c$

$$\therefore \sum x = \sum m_a \Rightarrow m_a = \frac{y+z}{2}, m_b = \frac{z+x}{2}, m_c = \frac{x+y}{2}$$

$\therefore$  sum of any 2 medians  $>$  third

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$\therefore x, y, z > 0$  and also, let  $\frac{1}{\sqrt{s_a}} = u, \frac{1}{\sqrt{s_b}} = v, \frac{1}{\sqrt{s_c}} = w$

$$\begin{aligned} \text{Now, } \frac{m_a}{s_a} + \frac{m_b}{s_b} + \frac{m_c}{s_c} &= \sum \frac{m_a}{\frac{2bcm_a}{b^2+c^2}} = \sum \frac{b^2+c^2}{2bc} = \frac{1}{2} \sum \left( \frac{b}{c} + \frac{c}{b} \right) = \frac{1}{2} \sum \frac{b+c}{a} \\ &\Rightarrow \sum \frac{b+c}{a} = 2 \left( \frac{m_a}{s_a} + \frac{m_b}{s_b} + \frac{m_c}{s_c} \right) \\ &= (y+z)u^2 + (z+x)v^2 + (x+y)w^2 \geq 2 \sum \frac{m_b+m_c-m_a}{\sqrt{s_b s_c}} = 2xvw + 2ywu + 2zuv \\ &\Leftrightarrow x \left( \frac{v^2+w^2}{2} \right) + y \left( \frac{w^2+u^2}{2} \right) + z \left( \frac{u^2+v^2}{2} \right) \geq xvw + ywu + zuv \rightarrow \text{true} \\ &\quad \because x \left( \frac{v^2+w^2}{2} \right) \stackrel{\text{A-G}}{\geq} xvw \text{ and analogs} \\ &\therefore \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c} \geq 2 \sum \frac{m_b+m_c-m_a}{\sqrt{s_b s_c}} \text{ (Proved)} \end{aligned}$$

**1633. In any  $\triangle ABC$  holds:**

$$\sum \frac{\cos B + \cos C}{r_a} \leq \frac{s^2}{m_a m_b m_c}$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Now, } m_a^2 m_b^2 m_c^2 = \frac{1}{64} (2b^2 + 2c^2 - a^2)(2c^2 + 2a^2 - b^2)(2a^2 + 2b^2 - c^2)$$

$$\stackrel{(1)}{\cong} \frac{1}{64} \{-4\sum a^6 + 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2\}$$

$$\begin{aligned} \text{Now, } \sum a^6 &= (\sum a^2)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2) \\ &= (\sum a^2)^3 - 3(2a^2 b^2 c^2 + \sum a^2 b^2 (\sum a^2 - c^2)) \\ &= (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2 \end{aligned}$$

$$\therefore \sum a^6 \stackrel{(2)}{\cong} (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2$$

$$\text{Again, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(3)}{\cong} (\sum a^2 b^2) \sum a^2 - 3a^2 b^2 c^2$$

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$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2 =$$

$$\frac{1}{64} \{-4(\sum a^2)^3 - 12a^2 b^2 c^2 + 12(\sum a^2 b^2) \sum a^2 + 6(\sum a^2 b^2) \sum a^2 - 18a^2 b^2 c^2 + 3a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 + 18(\sum a^2 b^2) \sum a^2 - 27a^2 b^2 c^2\} =$$

$$\frac{1}{64} \{-4(\sum a^2)^3 + 18((\sum ab)^2 - 2abc(2s))(\sum a^2) - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2$$

$$- 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\}$$

$$= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\} \leq \frac{R^2 s^4}{4}$$

$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\geq} 0$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} -s^4(8Rr - 36r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4)$

$$- r^3(4R + r)^3 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{?}{\geq} 20rs^4 \quad (ii)$$

Now, LHS of (ii)  $\stackrel{\text{Gerretsen}}{\geq} \underset{(a)}{s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3)}$

$$+ r^2(4R + r)^3 \text{ and RHS of (ii) } \stackrel{\text{Gerretsen}}{\geq} \underset{(b)}{20rs^2(4R^2 + 4Rr + 3r^2)}$$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove :

$$s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$$

$$\geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2 s^2$$

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Now, LHS of (iii)  $\underbrace{\sum_{(c)}^{\text{Gerretsen}}}_{(c)} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)$

+  $r(4R + r)^3$  and RHS of (iii)  $\underbrace{\sum_{(d)}^{\text{Gerretsen}}}_{(d)} 27r^2(4R^2 + 4Rr + 3r^2)$

(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \quad \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \stackrel{(4)}{\geq} \frac{R s^2}{2}$$

$$\begin{aligned} \text{Now, } \cos B + \cos C &= \frac{c^2 + a^2 - b^2}{2ca} + \frac{a^2 + b^2 - c^2}{2ca} \\ &= \frac{bc(b + c) - (b + c)(b^2 - bc + c^2) + a^2(b + c)}{2abc} \\ &= \frac{(b + c)\{a^2 - (b - c)^2\}}{2abc} \\ &= \frac{2(b + c)(s - b)(s - c)}{abc} \Rightarrow \frac{\cos B + \cos C}{r_a} = \frac{2(b + c)(s - b)(s - c)(s - a)}{4Rrs \cdot rs} \\ &= \frac{2(b + c)sr^2}{4Rr^2s^2} = \frac{b + c}{2Rs} \text{ and analogs} \end{aligned}$$

$$\therefore \sum \frac{\cos B + \cos C}{r_a} = \frac{\sum(b + c)}{2Rs} = \frac{4s}{2Rs} = \frac{2}{R} = \frac{s^2}{Rs^2} \stackrel{\text{by (4)}}{\geq} \frac{s^2}{m_a m_b m_c} \quad (\text{Proved})$$

1634. In  $\triangle ABC$ ,  $I$  – incenter,  $R_a, R_b, R_c$  – circumradii of  $\triangle BIC, \triangle CIA, \triangle AIB$ .

Prove that:

$$R_a + R_b + R_c \geq \frac{5R}{2} + r$$

Proposed by Marian Ursărescu-Romania

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*Solution by Bogdan Fuștei-Romania*

$$R_a = 2R \sin \frac{A}{2} \text{ (and analogs)}$$

$$R_a + R_b + R_c \geq \frac{5R}{2} + r \Leftrightarrow \sin \frac{A}{2} + \sin \frac{B}{2} + \sin \frac{C}{2} \geq \frac{5}{4} + \frac{r}{2R}$$

Let be the function:  $f: \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, f(x) = \cos x, f'(x) = \sin x,$

$$f''(x) = -\cos x < 0, \forall x \in \left(0, \frac{\pi}{2}\right) \Rightarrow f \text{ -concave on } \left(0, \frac{\pi}{2}\right).$$

Applying T.Popoviciu's Inequality, we have:

$$f(x) + f(y) + f(z) + 3f\left(\frac{x+y+z}{3}\right) \leq 2 \left[ f\left(\frac{x+y}{2}\right) + f\left(\frac{y+z}{2}\right) + f\left(\frac{z+x}{2}\right) \right],$$

$$\forall x, y, z \in \left(0, \frac{\pi}{2}\right)$$

Let  $x = A, y = B, z = C; A, B, C \in \left(0, \frac{\pi}{2}\right)$  and from  $\cos A + \cos B + \cos C = 1 + \frac{r}{R}$

We get:

$$\cos A + \cos B + \cos C + 3 \cos \left( \frac{A+B+C}{3} \right) \leq 2 \cdot \sum_{cyc} \cos \left( \frac{B+C}{2} \right) \Leftrightarrow$$

$$1 + \frac{r}{R} + \frac{3}{2} \leq 2 \cdot \sum_{cyc} \sin \frac{A}{2} \Leftrightarrow \sum_{cyc} \sin \frac{A}{2} \geq \frac{5}{4} + \frac{r}{2R}$$

**1635. In any  $\Delta ABC$  holds:**

$$(m_a + m_b + m_c)^2 \geq 3\sqrt{3}S \left( \frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a} \right)$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

Let  $s - a = x, s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$  and

$$c = x + y$$

$$\text{Now, } \frac{s^2}{r^2} = \frac{s^4}{\Delta^2} = \frac{s^4}{s(s-a)(s-b)(s-c)} \stackrel{(1)}{=} \frac{(\sum x)^3}{xyz} \text{ and } 1 + \frac{4R}{r}$$

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$$= 1 + \frac{4sabc}{4s(s-a)(s-b)(s-c)} = 1 + \frac{\prod(y+z) \stackrel{(2)}{\cong} xyz + \prod(y+z)}{xyz}$$

$$\text{Also, } \sum \frac{a}{b} = \sum \frac{y+z}{z+x} \stackrel{(3)}{\cong} \frac{\sum(x+y)(y+z)^2}{\prod(y+z)} \therefore (2), (3), (4) \Rightarrow \frac{s^2}{r^2} \geq \left(1 + \frac{4R}{r}\right) \left(\sum \frac{a}{b}\right)$$

$$\Leftrightarrow \frac{(\sum x)^3}{xyz} \geq \left[ \frac{xyz + \prod(y+z)}{xyz} \right] \left[ \frac{\sum(x+y)(y+z)^2}{\prod(y+z)} \right]$$

$$\Leftrightarrow \{\prod(y+z)\}(\sum x)^3 \geq \{xyz + \prod(y+z)\}\sum(x+y)(y+z)^2$$

$$\Leftrightarrow \sum x^2 y^4 + \sum x^3 y^3 \stackrel{(i)}{\cong} xyz(\sum x^2 y) + 3x^2 y^2 z^2$$

Now, if  $u, v, w > 0$ , then

$$: v^3 + v^3 + u^3 \stackrel{A-G}{\cong} 3v^2 u, w^3 + w^3 + v^3 \stackrel{A-G}{\cong} 3w^2 v \text{ and } u^3 + u^3$$

$$+ w^3 \stackrel{A-G}{\cong} 3u^2 w \text{ and adding these three :}$$

$\sum u^3 \geq \sum uv^2$  and choosing  $u = xy, v = yz$  and  $w = zx$ , we get

$$: \sum x^3 y^3 \stackrel{(a)}{\cong} xyz(\sum x^2 y) \text{ and } \sum x^2 y^4 \stackrel{A-G}{\cong} \sum_{(b)} 3x^2 y^2 z^2$$

$$\therefore (a) + (b) \Rightarrow (i) \text{ is true } \Rightarrow \frac{s^2}{r^2} \geq \left(\frac{4R+r}{r}\right) \left(\sum \frac{a}{b}\right) \stackrel{\text{Trucht}}{\cong} \frac{s\sqrt{3}}{r} \left(\sum \frac{a}{b}\right) \Rightarrow s^2$$

$$\geq \sqrt{3}S \left(\sum \frac{a}{b}\right) \Rightarrow \left(\sum a\right)^2 \geq 4\sqrt{3}S \left(\sum \frac{a}{b}\right) \text{ applying which}$$

on a triangle with sides  $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$  whose area of course  $= \frac{S}{3}$ , we get

$$: \left(\sum \frac{2m_a}{3}\right)^2 \geq 4\sqrt{3} \left(\frac{S}{3}\right) \left(\sum \frac{\left(\frac{2m_a}{3}\right)}{\left(\frac{2m_b}{3}\right)}\right)$$

$$\Rightarrow (m_a + m_b + m_c)^2 \geq 3\sqrt{3}S \left(\frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a}\right)$$

**1636. In acute  $\triangle ABC$  the following relationship holds:**

$$\cos \frac{\mu^2(A)}{4} + \cot \frac{\mu^2(B)}{4} + \sec \frac{\mu^2(C)}{4} > 2 + \frac{r}{2R}$$

*Proposed by Radu Diaconu-Romania*

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**Solution by George Florin Șerban-Romania**

Let be the function  $f: \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, f(x) = \cos x; f'(x) = -\sin x < 0 \Rightarrow f$  –decreasing

$$x < \frac{\pi}{2} \Rightarrow x^2 < \frac{\pi^2}{4} \Rightarrow \frac{1}{4} < \frac{\pi^2}{16} < \frac{\pi}{3} \Rightarrow 3\pi < 16 (\text{true}) \Rightarrow \frac{\mu^2(A)}{4} < \frac{\pi}{3} \Rightarrow$$

$$\cos \frac{\mu^2(A)}{4} > \cos \frac{\pi}{3} = \frac{1}{2}$$

Let be the function  $g: \left(0, \frac{\pi}{2}\right) \rightarrow \mathbb{R}, g(x) = \cot x, g'(x) = -\frac{1}{\sin^2 x} < 0 \Rightarrow g$  –decreasing

$$x < \frac{\pi}{2} \Rightarrow x^2 < \frac{\pi^2}{4} \Rightarrow \frac{x^2}{4} < \frac{\pi^2}{16} < \frac{\pi}{4}; (\pi < 4 \text{ true}) \Rightarrow \frac{\mu^2(B)}{4} < \frac{\pi}{4} \Rightarrow$$

$$\cot \frac{\mu^2(B)}{4} > \cot \frac{\pi}{4} = 1$$

$$\sec \frac{\mu^2(C)}{4} = \frac{1}{\cos \frac{\mu^2(C)}{4}} > \frac{1}{\frac{3}{4}} = \frac{4}{3}, \left(\cos x < 1 < \frac{4}{3}\right) \Rightarrow$$

$$\cos \frac{\mu^2(A)}{4} + \cot \frac{\mu^2(B)}{4} + \sec \frac{\mu^2(C)}{4} > \frac{1}{2} + 1 + \frac{3}{4} = \frac{9}{4}$$

We must show that:

$$\frac{9}{4} > 2 + \frac{r}{2R} \Leftrightarrow \frac{r}{2R} < \frac{1}{4} \Leftrightarrow R \geq 2r \text{ (Euler)}$$

$$\text{If } R = 2r \Rightarrow a = b = c \Rightarrow \mu(A) + \mu(B) + \mu(C) = \frac{\pi}{3}$$

$$\frac{x^2}{4} = \frac{\pi^2}{36}, \cos \frac{\pi^2}{36} + \cot \frac{\pi^2}{36} + \sec \frac{\pi^2}{36} \stackrel{(1)}{=} 2 + \frac{1}{4} = \frac{9}{4}$$

On the other hand, we have:

$$\frac{\pi^2}{36} < \frac{\pi}{3} \Leftrightarrow 3\pi < 36 \Leftrightarrow \pi < 12 (\text{true}) \Rightarrow \cos \frac{\pi^2}{36} > \cos \frac{\pi}{3} = \frac{1}{2}$$

$$\frac{\pi^2}{36} < \frac{\pi}{6} \Leftrightarrow \pi < 6 (\text{true}) \Rightarrow \cot \frac{\pi^2}{36} > \cot \frac{\pi}{6} = \sqrt{3}$$

$$\sec \frac{\pi^2}{36} = \frac{1}{\cos \frac{\pi^2}{36}} > 1$$

Adding up relations, we get

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$$\cos \frac{\pi^2}{36} + \cot \frac{\pi^2}{36} + \sec \frac{\pi^2}{36} = \frac{1}{2} + \sqrt{3} + 1 = \frac{3}{2} + \sqrt{3} > \frac{9}{4} \Rightarrow (1) \text{true.}$$

1637. In any  $\Delta ABC$  holds:

$$\sum m_a \sqrt{\frac{r_a}{h_a}} \geq \frac{1}{2} \sqrt{\frac{R}{r}} \sum (h_a + |b - c| \sin^2 \frac{A}{2})$$

Proposed by Bogdan Fuștei – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} 8m_a^2 &\geq (b+c)^2 \left(\cos \frac{A}{2}\right)^2 + (b-c)^2 \left(\sin \frac{A}{2}\right)^2 + 2(b+c)|b-c| \cos \frac{A}{2} \sin \frac{A}{2} \\ \Leftrightarrow 8m_a^2 &\geq (b-c)^2 \left( \left(\cos \frac{A}{2}\right)^2 + \left(\sin \frac{A}{2}\right)^2 \right) + 4bc \left(\cos \frac{A}{2}\right)^2 + \left(\frac{a}{2R}\right)(b+c)|b-c| \\ &\Leftrightarrow 8m_a^2 \geq (b-c)^2 + \frac{4bcs(s-a)}{bc} + \left(\frac{a}{2R}\right)(b+c)|b-c| \\ &\Leftrightarrow 8m_a^2 \geq (b-c)^2 + (b+c+a)(b+c-a) + \left(\frac{a}{2R}\right)(b+c)|b-c| \Leftrightarrow 8m_a^2 \\ &\geq (b-c)^2 + (b+c)^2 - a^2 + \left(\frac{a}{2R}\right)(b+c)|b-c| \\ \Leftrightarrow 8m_a^2 &\geq 4m_a^2 + \left(\frac{a}{2R}\right)(b+c)|b-c| \Leftrightarrow 8Rm_a^2 \geq a(b+c)|b-c| \Leftrightarrow \left(\frac{2abc}{4\Delta}\right) 4m_a^2 \geq \\ &a(b+c)|b-c| \\ &\Leftrightarrow 4a^2b^2c^2(2b^2 + 2c^2 - a^2)^2 \\ &\geq (a+b+c)(b+c-a)(c+a-b)(a+b-c)a^2(b^2 - c^2)^2 \\ &\Leftrightarrow 4b^2c^2(2b^2 + 2c^2 - a^2)^2 \geq (2\sum a^2b^2 - \sum a^4)(b^2 - c^2)^2 \\ \text{(expanding and re-arranging)} \\ &\Leftrightarrow a^4(b^2 + c^2)^2 - 2a^2(b^6 + c^6) - 14a^2b^2c^2(b^2 + c^2) + \\ &(b^2 + c^2)^4 + 8b^2c^2(b^2 + c^2)^2 + 16b^4c^4 \geq 0 \\ \Leftrightarrow \{a^4(b^2 + c^2)^2 + 16b^4c^4 - 8a^2b^2c^2(b^2 + c^2)\} - 6a^2b^2c^2(b^2 + c^2) + (b^2 + c^2)^4 \\ &+ 8b^2c^2(b^2 + c^2)^2 \\ &- 2a^2(b^2 + c^2)(b^4 + c^4 - b^2c^2) \geq 0 \end{aligned}$$

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$$\begin{aligned} &\Leftrightarrow \{a^2(b^2 + c^2) - 4b^2c^2\}^2 - 6a^2b^2c^2(b^2 + c^2) + (b^2 + c^2)^4 \\ &\quad + 8b^2c^2(b^2 + c^2)^2 - 2a^2(b^2 + c^2)\{(b^2 + c^2)^2 - 3b^2c^2\} \geq 0 \\ \Leftrightarrow &\{a^2(b^2 + c^2) - 4b^2c^2\}^2 + (b^2 + c^2)^4 + 8b^2c^2(b^2 + c^2)^2 - 2a^2(b^2 + c^2)^3 \geq 0 \\ \Leftrightarrow &\{a^2(b^2 + c^2) - 4b^2c^2\}^2 + (b^2 + c^2)^4 - 2(b^2 + c^2)^2\{a^2(b^2 + c^2) - 4b^2c^2\} \geq 0 \\ \Leftrightarrow &[\{a^2(b^2 + c^2) - 4b^2c^2\} - (b^2 + c^2)^2]^2 \geq 0 \rightarrow \text{true} \end{aligned}$$

$$\begin{aligned} \therefore (2\sqrt{2}m_a)^2 &\geq \left( (b+c)\cos\frac{A}{2} + |b-c|\sin\frac{A}{2} \right)^2 \\ \Rightarrow m_a \sqrt{\frac{r_a}{h_a}} &= m_a \sqrt{\frac{4R\sin\frac{A}{2}\cos\frac{A}{2}\tan\frac{A}{2}}{2rs}} = m_a \sqrt{\frac{2R}{r}} \sin\frac{A}{2} \\ &\geq \frac{1}{2\sqrt{2}} \sqrt{\frac{2R}{r}} \sin\frac{A}{2} \left( (b+c)\cos\frac{A}{2} + |b-c|\sin\frac{A}{2} \right) \\ &= \frac{1}{2} \sqrt{\frac{R}{r}} \left\{ (b+c)\frac{\sin A}{2} + |b-c|\sin^2\frac{A}{2} \right\} \Rightarrow m_a \sqrt{\frac{r_a}{h_a}} \\ &\geq \frac{1}{2} \sqrt{\frac{R}{r}} \left\{ \frac{a(b+c)}{4R} + |b-c|\sin^2\frac{A}{2} \right\} \text{ and analogs} \\ \Rightarrow \sum m_a \sqrt{\frac{r_a}{h_a}} &\geq \frac{1}{2} \sqrt{\frac{R}{r}} \left\{ \frac{\sum ab}{2R} + \sum (|b-c|\sin^2\frac{A}{2}) \right\} \\ &= \frac{1}{2} \sqrt{\frac{R}{r}} \left\{ \sum h_a + \sum (|b-c|\sin^2\frac{A}{2}) \right\} \\ &= \frac{1}{2} \sqrt{\frac{R}{r}} \sum (h_a + |b-c|\sin^2\frac{A}{2}) \text{ (QED)} \end{aligned}$$

**1638. In any  $\triangle ABC$  holds:**

$$\left\{ \sum \left( \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 \geq \frac{2}{3} \sum \frac{m_b + m_c}{r_a}$$

*Proposed by Bogdan Fuștei – Romania*

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*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} b + c - a &= 4R\cos\frac{A}{2}\cos\frac{B-C}{2} - 4R\sin\frac{A}{2}\cos\frac{A}{2} = 4R\cos\frac{A}{2}\left(\cos\frac{B-C}{2} - \cos\frac{B+C}{2}\right) \\ &= 8R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \end{aligned}$$

$$\Rightarrow s - a \stackrel{(1)}{\cong} 4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\begin{aligned} \text{Also, } a\cos A + b\cos B + c\cos C &= R(\sin 2A + \sin 2B + \sin 2C) \\ &= R\{2\sin(A+B)\cos(A-B) + 2\sin C\cos C\} \end{aligned}$$

$$= 2R\sin C\{\cos(A-B) - \cos(A+B)\} = 4R[\sin A] = 4R\left(\frac{abc}{8R^3}\right) = \frac{4Rrs}{2R^2} = \frac{2rs}{R}$$

$$\Rightarrow \sum a\cos A \stackrel{(2)}{\cong} \frac{2rs}{R}$$

$$\text{Now, } \sum \left( \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) = \sum \left\{ \frac{a \left( \frac{rs}{s-a} - \frac{rs}{s} \right) (b+c)}{2abcc\cos\frac{A}{2}} \sqrt{\frac{2rs}{4Rst\sin\frac{A}{2}\cos\frac{A}{2}}} \right\}$$

$$= \sqrt{\frac{r}{2R}} \sum \left[ \frac{4R\sin\frac{A}{2}\cos\frac{A}{2} \left\{ \frac{ars}{s(s-a)} \right\} (b+c)}{8Rrsc\cos\frac{A}{2}\sin\frac{A}{2}} \right]$$

$$\stackrel{\text{by (1)}}{\cong} \sqrt{\frac{r}{2R}} \sum \left[ \frac{4R\sin\frac{A}{2}\cos\frac{A}{2} \left\{ \frac{4Rr\sin\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \right\} (b+c)}{8Rrsc\cos\frac{A}{2}\sin\frac{A}{2}} \right]$$

$$= \left(\frac{1}{2s}\right) \sqrt{\frac{r}{2R}} \sum \left[ \left\{ \frac{4R\sin^2\frac{A}{2}\cos\frac{A}{2}}{4R\cos\frac{A}{2}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}} \right\} (b+c) \right]$$

$$= \left(\frac{1}{2s}\right) \sqrt{\frac{r}{2R}} \sum \left[ \left\{ \frac{4R\sin^2\frac{A}{2}}{4R\left(\frac{r}{4R}\right)} \right\} (b+c) \right] = \left(\frac{R}{rs}\right) \sqrt{\frac{r}{2R}} \sum \left\{ 2\sin^2\frac{A}{2} (b+c) \right\}$$

$$= \left(\frac{R}{rs}\right) \sqrt{\frac{r}{2R}} \sum \{1 - \cos A\} (b+c)$$

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$$\begin{aligned}
 &= \left(\frac{R}{rs}\right) \sqrt{\frac{r}{2R}} \left[4s - \sum \{(2s - a)\cos A\}\right] \\
 &= \left(\frac{R}{rs}\right) \sqrt{\frac{r}{2R}} \left\{4s - 2s\left(1 + \frac{r}{R}\right) + \sum a\cos A\right\} \stackrel{\text{by (2)}}{\cong} \left(\frac{R}{rs}\right) \sqrt{\frac{r}{2R}} \left(2s - \frac{2rs}{R}\right. \\
 &\quad \left. + \frac{2rs}{R}\right) = \left(\frac{2R}{r}\right) \sqrt{\frac{r}{2R}} = \sqrt{\frac{2R}{r}} \\
 &\Rightarrow \left\{ \sum \left( \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 \stackrel{(a)}{\cong} \frac{2R}{r}
 \end{aligned}$$

Now, WLOG, we may assume  $a \geq b \geq c$  and then,  $m_b + m_c \geq m_c + m_a$

$\geq m_a + m_b$  and  $\frac{1}{r_a} \leq \frac{1}{r_b} \leq \frac{1}{r_c} \therefore$  by Chebyshev's inequality,

$$\begin{aligned}
 \frac{2}{3} \sum \frac{m_b + m_c}{r_a} &\leq \frac{2}{9} \left\{ \sum (m_b + m_c) \right\} \left( \sum \frac{1}{r_a} \right) = \frac{4 \sum m_a}{9r} \stackrel{\text{Bager}}{\leq} \frac{4(4R + r)}{9r} \\
 &\Rightarrow \frac{2}{3} \sum \frac{m_b + m_c}{r_a} \stackrel{(b)}{\geq} \frac{4(4R + r)}{9r} \therefore (a), (b) \Rightarrow \text{suffices to prove :}
 \end{aligned}$$

$$\begin{aligned}
 \frac{2R}{r} \geq \frac{4(4R + r)}{9r} &\Leftrightarrow 9R \geq 8R + 2r \Leftrightarrow R \geq 2r \rightarrow \text{true (Euler)} \therefore \left\{ \sum \left( \frac{r_a - r}{w_a} \sqrt{\frac{h_a}{r_a}} \right) \right\}^2 \\
 &\geq \frac{2}{3} \sum \frac{m_b + m_c}{r_a} \quad (\text{Proved})
 \end{aligned}$$

**1639. In  $\triangle ABC$  the following relationship holds:**

$$\frac{h_a^2}{a(s-a)} + \frac{h_b^2}{b(s-b)} + \frac{h_c^2}{c(s-c)} \leq \frac{r_a^2}{a(s-a)} + \frac{r_b^2}{b(s-b)} + \frac{r_c^2}{c(s-c)}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Marian Ursărescu-Romania*

$$\begin{aligned}
 h_a^2 \leq w_a^2 \leq s(s-a) &\Rightarrow \frac{h_a^2}{a(s-a)} \leq \frac{s}{a} \Rightarrow \\
 \sum_{cyc} \frac{h_a^2}{a(s-a)} &\leq s \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) = \frac{s(ab + bc + ca)}{abc}
 \end{aligned}$$

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But  $ab + bc + ca = s^2 + r^2 + 4Rr$  and  $abc = 4Rrs$  then, we get

$$\sum_{cyc} \frac{h_a^2}{a(s-a)} \leq \frac{s^2 + r^2 + 4Rr}{4Rr}; \quad (1)$$

Now, from Bergstrom inequality we have:

$$\sum_{cyc} \frac{r_a^2}{a(s-a)} \geq \frac{(r_a + r_b + r_c)^2}{2s^2 - (a^2 + b^2 + c^2)}$$

But  $r_a + r_b + r_c = (4R + r)^2$  and  $a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr)$  then, we get

$$\sum_{cyc} \frac{r_a^2}{a(s-a)} \geq \frac{(4R + r)^2}{2r^2 + 8Rr} = \frac{(4R + r)^2}{2r(4R + r)} = \frac{4R + r}{2r}; \quad (2)$$

From (1),(2) we must show that:

$$\frac{s^2 + r^2 + 4Rr}{4Rr} \leq \frac{4R + r}{2r} \Leftrightarrow s^2 + r^2 + 4Rr \leq 8R^2 + 2Rr \Leftrightarrow$$

$$s^2 \leq 8R^2 - 2Rr - r^2; \quad (3)$$

$$\text{From Gerretsen: } s^2 \leq 4R^2 + 4Rr + 3r^2; \quad (4)$$

From (3),(4) we get:

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2 \Leftrightarrow 6Rr + 4r^2 \leq 4R^2 \Leftrightarrow 3Rr + 2r^2 \leq 2R^2 \text{ true.}$$

**1640. In  $\triangle ABC$  the following relationship holds:**

$$648\sqrt{3}r^3 \leq a(a - 3b - 3c)^2 + b(3a - b - c)^2 + c(3a - b - c)^2 \leq 81\sqrt{3}R^3$$

*Proposed by Daniel Sitaru-Romania*

*Solution by George Florin Şerban-Romania*

$$\begin{aligned} X &= a(a - 3b - 3c)^2 + b(3a - b - c)^2 + c(3a - b - c)^2 = \\ &= a^3 + 9ab^2 + 9ac^2 - 6a^2b + 18abc - 6a^2c + 9a^2b + b^3 + bc^2 - 6ab^2 + 2b^2c - \\ &\quad - 6abc + 9a^2c + b^2c + c^3 - 6abc + 2bc^2 - 6ac^2 = \\ &= a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3ac^2 + 3a^2c + 3b^2c + 3bc^2 = \\ &= (a + b + c)^3 = 8s^3 \end{aligned}$$

Applying Mitrinovic Inequality:  $3\sqrt{3}r \leq s \leq \frac{3\sqrt{3}R}{2}$  we get:

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$$(3\sqrt{3}r)^3 \cdot 8 \leq 8s^3 \leq 8 \cdot \left(\frac{3\sqrt{3}R}{2}\right)^3 \Leftrightarrow$$

$$27 \cdot 3\sqrt{3} \cdot 8r^3 \leq X \leq 8 \cdot \frac{27 \cdot 3\sqrt{3}R^3}{8} \Leftrightarrow$$

$$648\sqrt{3}r^3 \leq a(a-3b-3c)^2 + b(3a-b-c)^2 + c(3a-b-c)^2 \leq 81\sqrt{3}R^3$$

1641. In  $\triangle ABC$  the following relationship holds:

$$\frac{4}{9R^2} \leq \frac{\tan^2 \frac{A}{2}}{m_a h_a} + \frac{\tan^2 \frac{B}{2}}{m_b h_b} + \frac{\tan^2 \frac{C}{2}}{m_c h_c} \leq \frac{33R^2}{8S^2} - \frac{1}{2r^2}$$

Proposed by Marin Chirciu-Romania

**Solution 1 by Bogdan Fuștei-Romania**

$$\tan \frac{A}{2} = \frac{r_a - r}{a}; m_a \geq h_a; a \cdot h_a = 2S \text{ and analogs}$$

$$\begin{aligned} \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{m_a h_a} &\leq \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} = \sum_{cyc} \frac{\left(\frac{r_a - r}{a}\right)^2}{h_a^2} = \sum_{cyc} \frac{(r_a - r)^2}{4S^2} \\ &= \frac{r_a^2 + r_b^2 + r_c^2 + 3r^2 + 2r(r_a + r_b + r_c)}{4S^2} \end{aligned}$$

$$r_a + r_b + r_c = 4R + r \Rightarrow (r_a + r_b + r_c)^2 = (4R + r)^2$$

$$r_a^2 + r_b^2 + r_c^2 + 2 \left( \underbrace{r_a r_b + r_b r_c + r_c r_a}_{=s^2} \right) = (4R + r)^2$$

$$r_a^2 + r_b^2 + r_c^2 = (4R + r)^2 - s^2 \text{ then}$$

$$\begin{aligned} \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} &= \frac{(4R + r)^2 - 2s^2 + 3r^2 - 2r(4R + r)}{4S^2} \\ &= \frac{(4R + r)(4R + r - 2r) + 3r^2 - 2s^2}{4S^2} = \frac{16R^2 + 2r^2 - 2s^2}{4S^2} \end{aligned}$$

$$\text{We show that: } \frac{16R^2 + 2r^2 - 2s^2}{4S^2} \leq \frac{33R^2}{8S^2} - \frac{1}{2r^2} \Leftrightarrow$$

$$32R^2 + 4r^2 - 4s^2 \leq 33R^2 - 4s^2 \Leftrightarrow 4r^2 \leq R^2 \Leftrightarrow 2r \leq R \text{ (Euler)}$$

$$\text{So, we have: } \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} \leq \frac{33R^2}{8S^2} - \frac{1}{2r^2} \quad (1)$$

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From  $\frac{R}{2r} \geq \frac{m_a}{h_a}$  (Panaitopol Inequality) we have:  $\frac{R}{2r} \cdot h_a^2 \geq m_a \cdot h_a \Rightarrow$

$$\frac{1}{m_a \cdot h_a} \geq \frac{2r}{R} \cdot \frac{1}{h_a^2} \Rightarrow \frac{\tan^2 \frac{A}{2}}{m_a \cdot h_a} \geq \frac{2r}{R} \cdot \frac{(r_a - r)^2}{h_a^2} \text{ and analogs.}$$

$$\text{So, } \frac{2r}{R} \sum_{cyc} \frac{(r_a - r)^2}{4S^2} \leq \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{m_a \cdot h_a} \quad (2)$$

$$\text{We show that: } \frac{4}{9R^2} \leq \frac{2r}{R} \sum_{cyc} \frac{(r_a - r)^2}{4S^2} \Leftrightarrow$$

$$\frac{4}{9R^2} \leq \frac{2r}{R} \cdot \frac{16R^2 + 2r^2 - 2s^2}{4S^2} = \frac{2r}{R} \cdot \frac{8R^2 + r^2 - s^2}{2S^2} \Leftrightarrow$$

$$\frac{4}{9R^2} \leq \frac{8R^2 + r^2 - 2s^2}{s^2 R r} \Leftrightarrow 4s^2 r \leq 9R(8R^2 + r^2 - s^2)$$

$$\Leftrightarrow 4s^2 R \leq 9R(8R^2 + r^2) - 9Rs^2 \Leftrightarrow s^2(9R + 4r) \leq 9R(8R^2 + r^2)$$

$$\Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen Inequality)}$$

$$\text{We show that: } (9R + 4r)(4R^2 + 4Rr + 3r^2) \leq 9R(8R^2 + r^2)$$

$$\Leftrightarrow \left(\frac{9R}{r} + 4\right) \left(\frac{4R^2}{r^2} + \frac{4R}{r} + 3\right) \leq \frac{9R}{r} \left(\frac{8R^2}{r^2} + 1\right)$$

Denote:  $t = \frac{R}{r} \geq 2$  (Euler) we have:

$$(9t + 4)(4t^2 + 4t + 3) \leq 9t(8t^2 + 1) \Leftrightarrow$$

$$36t^3 - 52t^2 - 34t - 12 \geq 0 \Leftrightarrow 18t^3 - 26t^2 - 17t - 6 \geq 0$$

$$\Leftrightarrow (t - 2)(18t^2 + 10t + 3) \geq 0 \text{ true, from } t = \frac{R}{r} \geq 2 \text{ (Euler) } (3)$$

From (1),(2),(3) the inequality is proved.

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2} &= \sum_{cyc} \frac{\frac{\sin^2 \frac{A}{2}}{\cos^2 \frac{A}{2}}}{4S^2} = \frac{R^2}{S^2} \cdot \sum_{cyc} \sin^4 \frac{A}{2} \\ &= \frac{R^2}{S^2} \left( \frac{8R^2 + r^2 - s^2}{8R^2} \right) = \frac{8R^2 + r^2 - s^2}{8S^2} \end{aligned}$$

$$\text{Let: } \Omega = 4 \cdot \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{h_a^2}$$

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$$\begin{aligned} \Omega \stackrel{(*)}{\geq} \frac{4}{9R^2} &\Leftrightarrow 4 \cdot \frac{8R^2 + r^2 - s^2}{8S^2} \geq \frac{4}{9R^2} \Leftrightarrow 9R^2(8R^2 + r^2 - s^2) \geq 8s^2r^2 \\ &\Leftrightarrow 9R^2(8R^2 + r^2) \geq (9R^2 + 8r^2)s^2 \end{aligned}$$

But:  $s^2 \leq 4R^2 + 4Rr + 3r^2$  (Mitrinovic inequality), then

$$(9R^2 + 8r^2)(4R^2 + 4Rr + 3r^2) \stackrel{(**)}{\leq} 9R^2(8R^2 + r^2)$$

$$(**) \stackrel{t = \frac{R}{r} \geq 2}{\Leftrightarrow} (9t^2 + 8)(4t^2 + 4t + 3) \leq 9t^2(8t^2 + 1) \Leftrightarrow$$

$$2(18t^4 - 18t^3 - 25t^2 - 16t - 12) \geq 0 \Leftrightarrow$$

$$2(t - 2)(18t^3 + 18t^2 + 11t + 6) \geq 0 \text{ true, from } t = \frac{R}{r} \geq 2 \text{ (Euler)}$$

$\Rightarrow (**)$  – is true  $\Rightarrow (*)$  – is true.

$$\Omega \stackrel{(1)}{\leq} \frac{33R^2}{8S^2} - \frac{1}{2r^2} = \frac{33R^2 - 4s^2}{8S^2} \Leftrightarrow 4 \cdot \frac{8R^2 + r^2 - s^2}{8S^2} \leq \frac{33R^2 - 4s^2}{8S^2}$$

$$\Leftrightarrow 4(8R^2 + r^2 - s^2) \leq 33R^2 - 4s^2 \Leftrightarrow 32R^2 + 4r^2 - 4s^2 \leq 33R^2 - 4s^2$$

$$\Leftrightarrow 4r^2 \leq R^2 \Leftrightarrow 2r \leq R \text{ (Euler)} \Rightarrow (1) \text{ proved.}$$

### Solution 3 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} m_a^2 m_b^2 m_c^2 &= \frac{1}{64} (2b^2 + 2c^2 - 2a^2)(2c^2 + 2a^2 - 2b^2)(2a^2 + 2b^2 - 2c^2) \stackrel{(1)}{=} \frac{1}{64} \{-4\sum a^6 \\ &\quad + 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2\} \end{aligned}$$

$$\text{Now, } \sum a^6 = (\sum a^2)^3 - 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)q$$

$$= (\sum a^2)^3 - 3(2a^2 b^2 c^2 + \sum a^2 b^2 (\sum a^2 - c^2))$$

$$= (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2 \therefore \sum a^6 \stackrel{(2)}{=} (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2$$

$$\text{Again, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(3)}{=} (\sum a^2 b^2) \sum a^2 - 3a^2 b^2 c^2$$

$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$\begin{aligned} &= \frac{1}{64} \{-4(\sum a^2)^3 - 12a^2 b^2 c^2 + 12(\sum a^2 b^2) \sum a^2 + 6(\sum a^2 b^2) \sum a^2 \\ &\quad - 18a^2 b^2 c^2 + 3a^2 b^2 c^2\} \end{aligned}$$

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$$\begin{aligned}
 &= \frac{1}{64} \{-4(\sum a^2)^3 + 18(\sum a^2 b^2) \sum a^2 - 27a^2 b^2 c^2\} \\
 &= \frac{1}{64} \{-4(\sum a^2)^3 + 18((\sum ab)^2 - 2abc(2s))(\sum a^2) - 27a^2 b^2 c^2\} \\
 &= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2 \\
 &\quad - 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\} \\
 &= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\} \leq \frac{R^2 s^4}{4}
 \end{aligned}$$

$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\leq} 0$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} -s^4(8Rr - 36r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{?}{\geq} 0$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{?}{\geq} 20rs^4 \quad (ii)$$

Now, LHS of (ii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3)}_{(a)} + r^2(4R + r)^3$

and RHS of (ii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{20rs^2(4R^2 + 4Rr + 3r^2)}_{(b)}$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove

$$\begin{aligned}
 &: s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) \\
 &\quad + r^2(4R + r)^3 \\
 &\geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0
 \end{aligned}$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2 s^2$$

Now, LHS of (iii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)}_{(c)}$

$$+ r(4R + r)^3 \text{ and RHS of (iii) } \stackrel{\text{Gerretsen}}{\geq} \underbrace{27r^2(4R^2 + 4Rr + 3r^2)}_{(d)}$$

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(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \quad \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \stackrel{(4)}{\leq} \frac{Rs^2}{2}$$

$$\begin{aligned} \text{Now, } r_a h_a &\leq r_a w_a = \left(\tan \frac{A}{2}\right) \left(\frac{2bc \cos \frac{A}{2}}{b+c}\right) = \left(\sin \frac{A}{2}\right) \left(\frac{2bc}{b+c}\right) \stackrel{\text{HM} \leq \text{GM}}{\leq} \left(\sin \frac{A}{2}\right) \sqrt{bc} \\ &= s \sqrt{(s-b)(s-c)} \end{aligned}$$

$$= \sqrt{s(s-b)} \sqrt{s(s-c)} \stackrel{(5)}{\leq} m_b m_c$$

$$\text{Now, } \sum \frac{\tan^2 \frac{A}{2}}{m_a h_a} = \frac{1}{s^2} \sum \frac{r_a^3}{m_a h_a r_a} \stackrel{\text{by (5) and its analogs}}{\leq} \frac{1}{s^2} \sum \frac{r_a^3}{m_a m_b m_c} \stackrel{\text{by (4)}}{\leq} \left(\frac{2}{Rs^4}\right) \sum r_a^3$$

$$\stackrel{\text{Holder}}{\leq} \frac{2R(4R+r)^3}{9R^2 s^4} \stackrel{\text{Trucht}}{\leq} \frac{6\sqrt{3}Rs^3}{9R^2 s^4}$$

$$\stackrel{\text{Mitrinovic}}{\leq} \frac{6\sqrt{3}(2s)s^3}{(3\sqrt{3})9R^2 s^4} = \frac{4}{9R^2} \therefore \sum \frac{\tan^2 \frac{A}{2}}{m_a w_a} \stackrel{(m)}{\leq} \frac{4}{9R^2}$$

$$\text{Again, } \sum \frac{\tan^2 \frac{A}{2}}{m_a h_a} \stackrel{m_a \geq h_a \text{ and analogs}}{\leq} \sum \frac{\tan^2 \frac{A}{2}}{h_a^2} = \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2} \tan^2 \frac{A}{2}}{4s^2 r^2}$$

$$= \frac{4R^2}{s^2 r^2} \sum \sin^4 \frac{A}{2} = \frac{R^2}{s^2 r^2} \sum (1 - \cos A)^2$$

$$= \frac{R^2}{s^2 r^2} \sum (1 + 1 - \sin^2 A - 2\cos A) = \frac{R^2}{s^2 r^2} \left\{ 6 - \frac{s^2 - 4Rr - r^2}{2R^2} - 2\left(\frac{R+r}{R}\right) \right\}$$

$$= \frac{12R^2 - s^2 + 4Rr + r^2 - 4R^2 - 4Rr}{2s^2 r^2} \stackrel{?}{\leq} \frac{33R^2}{8s^2} - \frac{1}{2r^2}$$

$$\Leftrightarrow \frac{8R^2 - s^2 + r^2}{2s^2 r^2} - \frac{33R^2}{8s^2 r^2} + \frac{1}{2r^2} \stackrel{?}{\leq} 0 \Leftrightarrow \frac{32R^2 - 4s^2 + 4r^2 - 33R^2 + 4s^2}{8s^2 r^2} \stackrel{?}{\leq} 0$$

$$\Leftrightarrow \frac{4r^2 - R^2}{8s^2 r^2} \stackrel{?}{\leq} 0 \rightarrow \text{true, by Euler}$$

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$$\begin{aligned} \therefore \sum \frac{\tan^2 \frac{A}{2}}{m_a h_a} &\stackrel{(n)}{\leq} \frac{33R^2}{8S^2} - \frac{1}{2r^2} \therefore (m), (n) \Rightarrow \frac{4}{9R^2} \leq \frac{\tan^2 \frac{A}{2}}{m_a h_a} + \frac{\tan^2 \frac{B}{2}}{m_b h_b} + \frac{\tan^2 \frac{C}{2}}{m_c h_c} \\ &\leq \frac{33R^2}{8S^2} - \frac{1}{2r^2} \end{aligned}$$

1642. In  $\triangle ABC$  the following relationship holds:

$$\frac{4}{R^2} \leq \frac{\cot^2 \frac{A}{2}}{m_a h_a} + \frac{\cot^2 \frac{B}{2}}{m_b h_b} + \frac{\cot^2 \frac{C}{2}}{m_c h_c} \leq \frac{R}{2r^3}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$\begin{aligned} \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} &= \sum_{cyc} \frac{\frac{\cos^2 \frac{A}{2}}{\sin^2 \frac{A}{2}}}{4S^2} = \frac{R^2}{S^2} \cdot \sum_{cyc} \cos^4 \frac{A}{2} \\ &= \frac{R^2}{S^2} \left( \frac{(4R+r)^2 - s^2}{8R^2} \right) = \frac{(4R+r)^2 - s^2}{8S^2} \end{aligned}$$

$$\text{Let: } \Omega = \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{m_a h_a}$$

$$\Omega \stackrel{m_a \geq h_a}{\leq} 4 \cdot \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} = \frac{(4R+r)^2 - s^2}{2S^2}$$

$$\text{We must show that: } \frac{(4R+r)^2 - s^2}{2S^2} \leq \frac{R}{2r^3} \Leftrightarrow \frac{(4R+r)^2 - s^2}{2s^2 r^2} \leq \frac{R}{2r^3}$$

$$\Leftrightarrow r[(4R+r)^2 - s^2] \leq s^2 R \Leftrightarrow r(4R+r)^2 \leq (R+r)s^2$$

$$\text{But: } s^2 \geq 16Rr - 5r^2 \text{ (Mitrinovic inequality)}$$

$$\text{We need to prove: } (16Rr - 5r^2)(R+r) \geq r(4R+r)^2 \Leftrightarrow$$

$$(16R - 5r)(R+r) \geq (4R+r)^2$$

$$\Leftrightarrow 16R^2 + 11Rr - 5r^2 \geq 16R^2 + 8Rr + r^2$$

$$\Leftrightarrow 3Rr \geq 6r^2 \Leftrightarrow R \geq 2r \text{ true from Euler.}$$

$$m_a \leq \frac{R}{2r} \cdot h_a \Rightarrow m_a h_a \leq \frac{R}{2r} \cdot h_a^2 \text{ and analogs.}$$

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$$\Rightarrow \Omega = \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{m_a h_a} \geq \frac{2r}{R} \cdot \Omega = \sum_{cyc} \frac{\cot^2 \frac{A}{2}}{h_a^2} = 4 \cdot \frac{2r}{R} \cdot \frac{(4R+r)^2 - s^2}{8S^2} \stackrel{(**)}{\geq} \frac{4}{R^2}$$

$$(**) \Leftrightarrow R[(4R+r)^2 - s^2] \geq 4rs^2 \Leftrightarrow R(4R+r)^2 \geq (R+4r)s^2$$

But:  $s^2 \leq 4R^2 + 4Rr + 3r^2$  (Mitrinovic inequality), then

$$(R+4r)(4R^2 + 4Rr + 3r^2) \leq R(4R+r)^2 \stackrel{t=\frac{R}{r} \geq 2}{\Leftrightarrow}$$

$$(t+4)(4t^2 + 4t + 3) \leq t(4t+1)^2 \Leftrightarrow 6(t-2)(2t^2 + 2t + 1) \geq 0$$

true from  $t = \frac{R}{r} \geq 2$  (Euler). Proved.

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64} (2b^2 + 2c^2 - 2a^2)(2c^2 + 2a^2 - 2b^2)(2a^2 + 2b^2 - 2c^2) \stackrel{(1)}{\cong} \frac{1}{64} \{-4\Sigma a^6 + 6(\Sigma a^4 b^2 + \Sigma a^2 b^4) + 3a^2 b^2 c^2\}$$

$$\text{Now, } \Sigma a^6 = (\Sigma a^2)^3 - 3(a^2+b^2)(b^2+c^2)(c^2+a^2)$$

$$= (\Sigma a^2)^3 - 3(2a^2 b^2 c^2 + \Sigma a^2 b^2 (\Sigma a^2 - c^2))$$

$$= (\Sigma a^2)^3 + 3a^2 b^2 c^2 - 3(\Sigma a^2 b^2) \Sigma a^2 \therefore \Sigma a^6 \cong (\Sigma a^2)^3 + 3a^2 b^2 c^2 - 3(\Sigma a^2 b^2) \Sigma a^2 \stackrel{(2)}$$

$$\text{Again, } \Sigma a^4 b^2 + \Sigma a^2 b^4 = \Sigma a^2 b^2 (\Sigma a^2 - c^2) \stackrel{(3)}{\cong} (\Sigma a^2 b^2) \Sigma a^2 - 3a^2 b^2 c^2$$

$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64} \{-4(\Sigma a^2)^3 - 12a^2 b^2 c^2 + 12(\Sigma a^2 b^2) \Sigma a^2 + 6(\Sigma a^2 b^2) \Sigma a^2 - 18a^2 b^2 c^2 + 3a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\Sigma a^2)^3 + 18(\Sigma a^2 b^2) \Sigma a^2 - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-4(\Sigma a^2)^3 + 18((\Sigma ab)^2 - 2abc(2s))(\Sigma a^2) - 27a^2 b^2 c^2\}$$

$$= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2$$

$$- 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\}$$

$$= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R+r)^3\} \leq \frac{R^2 s^4}{4}$$

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$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\geq} 0$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} -s^4(8Rr - 36r^2) - s^2(60R^2r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{?}{\geq} 0$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{?}{\geq} 20rs^4 \quad (ii)$$

Now, LHS of (ii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3)}_{(a)} + r^2(4R + r)^3$

and RHS of (ii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{20rs^2(4R^2 + 4Rr + 3r^2)}_{(b)}$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove

$$: s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$$

$$\geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2s^2$$

Now, LHS of (iii)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)}_{(c)}$

$$+ r(4R + r)^3 \text{ and RHS of (iii) } \stackrel{\text{Gerretsen}}{\geq} \underbrace{27r^2(4R^2 + 4Rr + 3r^2)}_{(d)}$$

(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \quad \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (iii) \Rightarrow (ii)$$

$$\Rightarrow (i) \text{ is true } \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow m_a m_b m_c \stackrel{(4)}{\geq} \frac{R s^2}{2}$$

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$$\text{Now, } r_a h_a \leq r_a w_a = \left( s \tan \frac{A}{2} \right) \left( \frac{2bc \cos \frac{A}{2}}{b+c} \right) = \left( s \sin \frac{A}{2} \right) \left( \frac{2bc}{b+c} \right) \stackrel{HM \leq GM}{\leq} \left( s \sin \frac{A}{2} \right) \sqrt{bc}$$

$$= s \sqrt{(s-b)(s-c)}$$

$$= \sqrt{s(s-b)} \sqrt{s(s-c)} \stackrel{(5)}{\geq} m_b m_c$$

$$\text{Now, } \sum \frac{\cot^2 \frac{A}{2}}{m_a h_a} = \sum \frac{s^2}{r_a^2 m_a h_a} \stackrel{\text{by (5) and its analogs}}{\geq} \sum \frac{s^2}{r_a m_a m_b m_c} \stackrel{\text{by (4)}}{\geq} \left( \frac{2s^2}{R^2} \right) \sum \frac{1}{r_a}$$

$$= \frac{2R}{R^2 r} \stackrel{\text{Euler}}{\geq} \frac{4}{R^2} \therefore \sum \frac{\cot^2 \frac{A}{2}}{m_a w_a} \stackrel{(m)}{\geq} \frac{4}{R^2}$$

$$\text{Again, } \sum \frac{\cot^2 \frac{A}{2}}{m_a h_a} \stackrel{\because m_a \geq h_a \text{ and analogs}}{\geq} \sum \frac{\cot^2 \frac{A}{2}}{h_a^2} = \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2} \cot^2 \frac{A}{2}}{4s^2 r^2} =$$

$$= \frac{4R^2}{s^2 r^2} \sum \cos^4 \frac{A}{2} = \frac{R^2}{s^2 r^2} \sum (1 + \cos A)^2$$

$$= \frac{R^2}{s^2 r^2} \sum (1 + 1 - \sin^2 A + 2 \cos A) = \frac{R^2}{s^2 r^2} \left\{ 6 - \frac{s^2 - 4Rr - r^2}{2R^2} + 2 \left( \frac{R+r}{R} \right) \right\}$$

$$= \frac{12R^2 - s^2 + 4Rr + r^2 + 4R^2 + 4Rr}{2s^2 r^2} \stackrel{?}{\geq} \frac{R}{2r^3}$$

$$\Leftrightarrow \frac{16R^2 + 8Rr + r^2 - s^2}{2s^2 r^2} - \frac{R}{2r^3} \stackrel{?}{\geq} 0 \Leftrightarrow \frac{r(16R^2 + 8Rr + r^2) - (R+r)s^2}{2s^2 r^3} \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (R+r)s^2 \stackrel{?}{\geq} r(16R^2 + 8Rr + r^2) \quad (iv)$$

$$\text{Now, LHS of (iv)} \stackrel{\text{Gerretsen}}{\geq} (R+r)(16Rr - 5r^2) \stackrel{?}{\geq} r(16R^2 + 8Rr + r^2) \Leftrightarrow 3Rr \stackrel{?}{\geq} 6r^2$$

$$\Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true, by Euler} \Rightarrow (iv) \text{ is true}$$

$$\therefore \sum \frac{\cot^2 \frac{A}{2}}{m_a h_a} \stackrel{(n)}{\geq} \frac{R}{2r^3} \therefore (m), (n) \Rightarrow \frac{4}{R^2} \leq \frac{\cot^2 \frac{A}{2}}{m_a w_a} + \frac{\cot^2 \frac{B}{2}}{m_b w_b} + \frac{\cot^2 \frac{C}{2}}{m_c w_c} \leq \frac{R}{2r^3} \text{ (Proved)}$$

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1643. In  $\triangle ABC$  the following relationship holds:

$$\frac{\cos^2 \frac{B-C}{2}}{\cot \frac{A}{2}} + \frac{\cos^2 \frac{C-A}{2}}{\cot \frac{B}{2}} + \frac{\cos^2 \frac{A-B}{2}}{\cot \frac{C}{2}} \geq \frac{2\sqrt{3}r}{R}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Marian Ursărescu-Romania

$$\sum_{cyc} \frac{\cos^2 \frac{B-C}{2}}{\cot \frac{A}{2}} \stackrel{Am-Gm}{\geq} 3 \cdot \sqrt[3]{\frac{\cos^2 \frac{B-C}{2} \cos^2 \frac{C-A}{2} \cos^2 \frac{A-B}{2}}{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}}}$$

$$\text{We must show: } \sqrt{3} \cdot \sqrt[3]{\frac{\cos^2 \frac{B-C}{2} \cos^2 \frac{C-A}{2} \cos^2 \frac{A-B}{2}}{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}}} \geq \frac{2r}{R} \Leftrightarrow$$

$$3\sqrt{3} \cdot \frac{\cos^2 \frac{B-C}{2} \cos^2 \frac{C-A}{2} \cos^2 \frac{A-B}{2}}{\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}} \geq \frac{8r^3}{R^3}; \quad (1)$$

$$\text{But: } \prod_{cyc} \cos \frac{B-C}{2} = \frac{s^2+r^2+2Rr}{8R^2} \text{ and } \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = \frac{s}{r}; \quad (2)$$

$$\text{From (1),(2) we must show: } 3\sqrt{3} \cdot \frac{(s^2+r^2+2Rr)^2}{64R^4} \cdot \frac{r}{s} \geq \frac{8r^3}{R^3} \Leftrightarrow$$

$$3\sqrt{3}(s^2+r^2+2Rr)^2 \geq 64 \cdot 8sr^2R; \quad (3)$$

From Mitrinovic inequality:  $s \leq \frac{3\sqrt{3}}{2}R$ ; (4) and with (3), we must show:

$$(s^2+r^2+2Rr)^2 \geq 64 \cdot 4R^2r^2 \Leftrightarrow s^2+r^2+2Rr \geq 16Rr$$

$$\Leftrightarrow s^2 \geq 14Rr - r^2, \text{ true because from Gerretsen}$$

$$s^2 \geq 16Rr - 5r^2 \geq 14Rr - r^2 \Leftrightarrow 2Rr \geq 4r^2 \Leftrightarrow R \geq 2r \text{ (Euler).}$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\cos \frac{A-B}{2} = \frac{(2s-c)\sin \frac{C}{2}}{4R\sin \frac{C}{2} \cos \frac{C}{2}} = \frac{s}{2R} \cdot \frac{1}{\cos \frac{C}{2}} - \sin \frac{C}{2}$$

$$\text{Simiary: } \cos \frac{B-C}{2} = \frac{s}{2R} \cdot \frac{1}{\cos \frac{A}{2}} - \sin \frac{A}{2}; \quad \cos \frac{A-C}{2} = \frac{s}{2R} \cdot \frac{1}{\cos \frac{B}{2}} - \sin \frac{B}{2}$$

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$$\begin{aligned} \Rightarrow \cos^2\left(\frac{A-B}{2}\right) &= \frac{s^2}{4R^2} \cdot \frac{1}{\cos^2\frac{C}{2}} - \frac{s}{R} \tan\frac{C}{2} + \sin^2\frac{C}{2} \\ \Rightarrow \frac{\cos^2\left(\frac{A-B}{2}\right)}{\cot\frac{C}{2}} &= \frac{s^2}{4R^2} \cdot \left(1 + \tan^2\frac{C}{2}\right) \tan\frac{C}{2} - \frac{s}{R} \cdot \tan^2\frac{C}{2} + \tan\frac{C}{2} - \frac{\sin C}{2} \\ &= \left(\frac{s^2}{4R^2} + 1\right) \tan\frac{C}{2} + \frac{s^2}{4R^2} \cdot \tan^3\frac{C}{2} - \frac{s}{R} \cdot \tan^2\frac{C}{2} - \frac{\sin C}{2} \\ \Rightarrow LHS &= \sum_{cyc} \frac{\cos^2\left(\frac{A-B}{2}\right)}{\cot\frac{C}{2}} = \\ &= \left(\frac{s^2}{4R^2} + 1\right) \sum_{cyc} \tan\frac{A}{2} + \frac{s^2}{4R^2} \sum_{cyc} \tan^3\frac{A}{2} - \frac{s}{R} \sum_{cyc} \tan^2\frac{A}{2} - \frac{1}{2} \sum_{cyc} \sin A \\ &= \frac{s^2 + 4R^2}{4R^2} \cdot \frac{4R+r}{s} + \frac{s^2}{4R^2} \cdot \frac{(4R+r)^3 - 12s^2R}{s^3} - \frac{s}{R} \cdot \frac{(4R+r)^2 - 2s^2}{s^2} - \frac{s}{2R} = \\ &= \frac{(s^2 + 4R^2)(4R+r)}{4R^2s} + \frac{(4R+r)^3 - 12s^2R}{4R^2s} - \left(\frac{(4R+r)^2 - 2s^2}{Rs}\right) - \frac{s}{2R} \stackrel{(*)}{\geq} \frac{2\sqrt{3}r}{R} \\ (*) \Leftrightarrow &\frac{(s^2 + 4R^2)(4R+r) + (4R+r)^3 - 12s^2R}{4Rs} - \left(\frac{(4R+r)^2 - 2s^2}{s}\right) - \frac{s}{2} \geq 2\sqrt{3}r \\ \Leftrightarrow &(s^2 + 4R^2)(4R+r) + (4R+r)^3 - 12s^2R - 4R((4R+r)^2 - 2s^2) - 2Rs^2 \\ &\geq 8Rrs\sqrt{3} \\ \Leftrightarrow &[(4R+r) \cdot 4R^2 - 4R(4R+r)^2 + (4R+r)^3] - 2Rs^2 + rs^2 \geq 8Rrs\sqrt{3} \\ \Leftrightarrow &(4R+r)[4R^2 - 4R(4R+r) + (4R+r)^2] + rs^2 \geq 2Rs^2 + 8Rrs\sqrt{3} \\ \Leftrightarrow &(4R+r)(2R+r)^2 + rs^2 \geq 2R(s^2 + 4rs\sqrt{3}) \end{aligned}$$

But:  $16Rr - 5r^2 \leq s^2 \leq 4R^2 + 4Rr + 3r^2$ ;  $s \leq \frac{3\sqrt{3}}{2}R$  - (Mitrinovic)

We need to prove:

$$\begin{aligned} (4R+r)(2R+r)^2 + r(16Rr - 5r^2) &\stackrel{(**)}{\geq} 2R(4R^2 + 4Rr + 3r^2 + 18Rr) \\ \Leftrightarrow (4R+r)(2R+r)^2 + r(16Rr - 5r^2) &\geq 2R(4R^2 + 22Rr + 3r^2) \\ \Leftrightarrow &\stackrel{\substack{R \\ t = \frac{r}{R} \geq 2}}{\Leftrightarrow} (4t+1)(2t+1)^2 + (16t-5) \geq 2t(4t^2 + 22t + 3) \end{aligned}$$

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$$\Leftrightarrow 8t^3 - 24t^2 + 18t - 4 \geq 0 \Leftrightarrow 2(t-2)(2t-1)^2 \geq 0 \text{ which is true from } t = \frac{R}{r} \geq 2$$

(Euler).

### Solution 3 by Bogdan Fuștei-Romania

We know that:  $\cos \frac{B-C}{2} \geq \sqrt{\frac{2r}{R}}$  and analogs.

$$\frac{1}{\cot \frac{A}{2}} = \tan \frac{A}{2} = \frac{r_a}{s} \text{ and analogs.}$$

$$r_a + r_b + r_c \geq s\sqrt{3}$$

So, we have:

$$\sum_{cyc} \frac{\cos^2 \left( \frac{A-B}{2} \right)}{\cot \frac{C}{2}} \geq \frac{2r}{R} \cdot \left( \tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right) = \frac{2r}{R} \cdot \frac{r_a + r_b + r_c}{s}; \quad (1)$$

$$\frac{2r}{R} \cdot \frac{r_a + r_b + r_c}{s} \geq \frac{2\sqrt{3}r}{R} \Rightarrow \frac{r_a + r_b + r_c}{s} \geq \sqrt{3} \Rightarrow r_a + r_b + r_c \geq s\sqrt{3}; \quad (2)$$

From (1),(2) the inequality is proved.

**1644.** If  $x, y, z > 0$  then in  $\triangle ABC$  the following relationship holds:

$$\frac{(y+z)\sin A}{x\sin B\sin C} + \frac{(z+x)\sin B}{y\sin C\sin A} + \frac{(x+y)\sin C}{z\sin A\sin B} \geq 4\sqrt{3}$$

(generalization for Daniel Sitaru RMM problem)

Proposed by Marin Chirciu-Romania

### Solution 1 by Avishek Mitra-West Bengal-India

$$\frac{(y+z)\sin A}{x\sin B\sin C} + \frac{(z+x)\sin B}{y\sin C\sin A} + \frac{(x+y)\sin C}{z\sin A\sin B} \geq 4\sqrt{3}$$

$$\Leftrightarrow \sum_{cyc} \frac{y+z}{x} \cdot \frac{\sin A}{\sin B\sin C} = \sum_{cyc} \frac{y+z}{x} \cdot \frac{\frac{a}{2R}}{\frac{bc}{4R^2}} = 2R \sum_{cyc} \frac{y+z}{x} \cdot \frac{a}{bc}$$

$$\stackrel{Am-Gm}{\geq} 2R \cdot 3 \left[ \frac{\prod (y+z)}{\prod x} \cdot \frac{abc}{a^2 b^2 c^2} \right]^{\frac{1}{3}} \stackrel{Am-Gm}{\geq} 3 \cdot 2R \left[ \frac{\prod 2\sqrt{yz}}{\prod x} \cdot \frac{1}{abc} \right]^{\frac{1}{3}}$$

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$$= 6R \cdot 2 \left( \frac{\prod x}{\prod x} \cdot \frac{1}{abc} \right)^{\frac{1}{3}}$$

We need to show:

$$\frac{12R}{\sqrt[3]{abc}} \geq 4\sqrt{3} \Leftrightarrow 12^3 R^3 \geq 64 \cdot 3\sqrt{3} \cdot 4Rrs \Leftrightarrow \frac{9}{\sqrt{3}} R^2 \geq 4rs$$

$\Leftrightarrow (3\sqrt{3}R)R \geq (2s)(2s)$  true from  $R \geq 2r$  (Euler) and  $3\sqrt{3}R \geq 2s$  (Mitrinovic). Proved.

### Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For  $x, y, z > 0$  we have:

$$\begin{aligned} & \frac{x \sin B}{y \sin C \sin A} + \frac{y \sin C}{z \sin A \sin B} + \frac{z \sin A}{x \sin B \sin C} + \frac{x \sin C}{z \sin A \sin B} + \frac{z \sin B}{y \sin C \sin A} + \frac{y \sin A}{x \sin B \sin C} = \\ & = \frac{x \sin^2 B}{y \sin A \sin B \sin C} + \frac{y \sin^2 C}{z \sin A \sin B \sin C} + \frac{z \sin^2 A}{x \sin A \sin B \sin C} \\ & \geq \frac{3 \sqrt[3]{\sin^2 A \sin^2 B \sin^2 C \cdot \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}}}{\sin A \sin B \sin C} + \frac{3 \sqrt[3]{\sin^2 A \sin^2 B \sin^2 C \cdot \frac{x}{y} \cdot \frac{y}{z} \cdot \frac{z}{x}}}{\sin A \sin B \sin C} \\ & = 6 \sqrt[3]{\frac{\sin^2 A \sin^2 B \sin^2 C}{\sin^3 A \sin^3 B \sin^3 C}} = 6 \sqrt[3]{\frac{1}{\sin A \sin B \sin C}} \geq \frac{6 \cdot 2}{\sqrt{3}} = 4\sqrt{3} \end{aligned}$$

$$\text{Because: } \sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$$

$$\text{Hence: } 3 \sqrt[3]{\sin A \sin B \sin C} \leq \frac{3\sqrt{3}}{2}$$

$$\text{Hence: } \sqrt[3]{\frac{1}{\sin A \sin B \sin C}} \geq \frac{2}{\sqrt{3}}$$

### Solution 3 by Tran Hong-Dong Thap-Vietnam

With  $\sin A \sin B \sin C \neq 0$  we have:

$$\begin{aligned} \Omega &= \frac{(y+z)\sin A}{x \sin B \sin C} + \frac{(z+x)\sin B}{y \sin C \sin A} + \frac{(x+y)\sin C}{z \sin A \sin B} \\ &= \left(\frac{y}{x} + \frac{z}{x}\right) \frac{\sin A}{\sin B \sin C} + \left(\frac{z}{y} + \frac{x}{y}\right) \frac{\sin B}{\sin C \sin A} + \left(\frac{x}{z} + \frac{y}{z}\right) \frac{\sin C}{\sin A \sin B} \\ &= \sum_{\text{cyc}} \left( \frac{y \sin A}{x \sin B \sin C} + \frac{x \sin B}{y \sin C \sin A} \right) \stackrel{Am-Gm}{\geq} 2 \sum_{\text{cyc}} \sqrt{\frac{y \sin A \sin B}{x y \sin A \sin B \sin^2 C}} \end{aligned}$$

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$$= 2 \left( \frac{1}{\sin A} + \frac{1}{\sin B} + \frac{1}{\sin C} \right)^{c-B-S} \geq 2 \cdot \frac{9}{\sin A + \sin B + \sin C} \geq \frac{2 \cdot 9}{\frac{3\sqrt{3}}{2}} = 4\sqrt{3}$$

Because:  $\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}$

**Solution 4 by Marian Dincă-Romania**

$$\begin{aligned} & \frac{(y+z)\sin A}{x\sin B\sin C} + \frac{(z+x)\sin B}{y\sin C\sin A} + \frac{(x+y)\sin C}{z\sin A\sin B} \geq \\ & \geq 3 \sqrt[3]{\frac{(y+z)\sin A}{x\sin B\sin C} \cdot \frac{(z+x)\sin B}{y\sin C\sin A} \cdot \frac{(x+y)\sin C}{z\sin A\sin B}} \\ & = 3 \sqrt[3]{\frac{(y+z)(z+x)(x+y)}{xyz} \cdot \frac{1}{\sin A\sin B\sin C}} \geq 3 \sqrt[3]{8 \cdot \frac{1}{\left(\frac{\sqrt{3}}{2}\right)^3}} = 3 \cdot 2 \cdot \frac{2}{\sqrt{3}} = 4\sqrt{3} \end{aligned}$$

$$\frac{(y+z)(z+x)(x+y)}{xyz} \geq 8 \text{ (Cesaro inequality)}$$

$$\sin A\sin B\sin C \leq \sin^3 \frac{\pi}{3} = \left(\frac{\sqrt{3}}{2}\right)^3 \text{ Jensen inequality for the concave function}$$

$$f(x) = \log(\sin x), x \in (0, \pi)$$

**1645. In  $\triangle ABC$ ,  $K$  –Lemoine's point, the following relationship holds:**

$$\frac{aAK + bBK + cCK}{m_a + m_b + m_c} \leq \frac{2R\sqrt{3}}{3}$$

*Proposed by Daniel Sitaru-Romania*

**Solution by Marian Ursărescu-Romania**

From Van Aubel theorem we have:  $\frac{AK}{KA'} = \frac{AC'}{C'B} + \frac{AB'}{B'C}$ ; (1)

From Steiner theorem we have:  $\frac{AC'}{C'B} = \frac{b^2}{a^2}$ ;  $\frac{AB'}{B'C} = \frac{c^2}{a^2}$ ; (2)

From (1),(2) we have:

$$\begin{aligned} \frac{AK}{KA'} &= \frac{b^2 + c^2}{a^2} \Rightarrow \frac{AK}{S_a} = \frac{b^2 + c^2}{a^2 + b^2 + c^2} \Rightarrow AK = \frac{b^2 + c^2}{a^2 + b^2 + c^2} \cdot S_a \\ &= \frac{b^2 + c^2}{a^2 + b^2 + c^2} \cdot \frac{2bc}{b^2 + c^2} \cdot m_a = \frac{2bcm_a}{a^2 + b^2 + c^2} \end{aligned}$$

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$$\Rightarrow aKA = \frac{2abc}{a^2+b^2+c^2} \cdot m_a \text{ and simillary.}$$

$$\frac{aAK + bBK + cCK}{m_a + m_b + m_c} = \frac{2abc}{a^2 + b^2 + c^2}$$

We must show:

$$\frac{2abc}{a^2 + b^2 + c^2} \leq \frac{2R\sqrt{3}}{3} \Leftrightarrow \frac{4Rrs}{a^2 + b^2 + c^2} \leq \frac{R\sqrt{3}}{3} \Leftrightarrow$$

$$12sr \leq \sqrt{3}(a^2 + b^2 + c^2); \quad (3)$$

$$\text{From Mitrinovic: } s \geq 3\sqrt{3}r \Rightarrow r \leq \frac{s}{3\sqrt{3}}; \quad (4)$$

From (3),(4) we must show:

$$4s^2 \leq 3(a^2 + b^2 + c^2) \Leftrightarrow (a + b + c)^2 \leq 3(a^2 + b^2 + c^2) \text{ true because it's Cauchy inequality. Proved.}$$

**1646. In any  $\triangle ABC$  holds:**

$$\sum \left( \tan \frac{A}{4} + \cot \frac{A}{4} \right) \geq \sum \left( \frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_a}} \right)$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} b + c - a &= 4R \cos \frac{A}{2} \cos \frac{B-C}{2} - 4R \cos \frac{A}{2} \sin \frac{A}{2} = 4R \cos \frac{A}{2} \left( \cos \frac{B-C}{2} - \cos \frac{B+C}{2} \right) \\ &= 8R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \end{aligned}$$

$$\Rightarrow s - a \stackrel{(1)}{=} 4R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\text{Now, } AI = \frac{r}{\sin \frac{A}{2}} = \frac{4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}} = 4R \sin \frac{B}{2} \sin \frac{C}{2} \stackrel{\text{by (1)}}{=} \frac{s-a}{\cos \frac{A}{2}} \Rightarrow \cos \frac{A}{2} \stackrel{(2)}{=} \frac{s-a}{AI}$$

$$\text{Again, } \tan \frac{A}{4} = \frac{1 - \cos \frac{A}{2}}{\sin \frac{A}{2}} \stackrel{\text{by (2)}}{=} \frac{1 - \frac{s-a}{AI}}{\frac{r}{AI}} = \frac{AI - (s-a)}{r} \Rightarrow \tan \frac{A}{4} \stackrel{(a)}{=} \frac{AI - (s-a)}{r}$$

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$$\text{Also, } \cot \frac{A}{4} = \frac{\sin \frac{A}{2}}{1 - \cos \frac{A}{2}} = \frac{(1 + \cos \frac{A}{2}) \sin \frac{A}{2}}{1 - \cos^2 \frac{A}{2}} = \frac{(1 + \cos \frac{A}{2}) \sin \frac{A}{2}}{\sin^2 \frac{A}{2}}$$

$$= \frac{1 + \cos \frac{A}{2}}{\sin \frac{A}{2}} \stackrel{\text{by (2)}}{\cong} \frac{1 + \frac{s-a}{AI}}{\frac{r}{AI}} = \frac{AI + (s-a)}{r} \Rightarrow \cot \frac{A}{4} \stackrel{(b)}{\cong} \frac{AI + (s-a)}{r}$$

$$\therefore \tan \frac{A}{4} + \cot \frac{A}{4} = \frac{AI - (s-a) + AI + (s-a)}{r} = \frac{2AI}{r} \text{ and analogs}$$

$$\Rightarrow \sum \left( \tan \frac{A}{4} + \cot \frac{A}{4} \right) \stackrel{(i)}{\cong} \frac{2}{r} \sum AI$$

$$\sum \frac{m_a}{w_a} \stackrel{\text{Tsintsifas}}{\cong} \sum \left( \frac{b^2 + c^2}{2bc} \right) = \frac{1}{2} \sum \left( \frac{b}{c} + \frac{c}{b} \right) = \frac{1}{2} \sum \left( \frac{c}{a} + \frac{b}{a} \right) = \frac{1}{2} \sum \frac{b+c}{a}$$

$$= \frac{1}{2} \sum \frac{4R \cos \frac{A}{2} \cos \frac{B-C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}} \leq \frac{1}{2} \sum \frac{1}{\sin \frac{A}{2}}$$

$$\left( \because 0 \leq \cos \frac{B-C}{2} \leq 1 \text{ and analogs} \right) = \frac{1}{2r} \sum AI \Rightarrow \sum \frac{m_a}{w_a} \stackrel{(ii)}{\cong} \frac{1}{2r} \sum AI$$

$$\sum \sqrt{\frac{m_a}{r_a}} \stackrel{\text{Panaitopol}}{\cong} \sum \sqrt{\frac{Rh_a}{2rr_a}} = \sum \sqrt{\frac{2Rrs(s-a)}{2r^2as}}$$

$$= \sum \sqrt{\frac{R(s-a)}{ra}} \stackrel{\text{by (1)}}{\cong} \sum \sqrt{\frac{4R^2 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{4R \cos \frac{A}{2} \sin \frac{A}{2} \cdot 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}} = \frac{1}{2} \sum \frac{1}{\sin \frac{A}{2}} = \frac{1}{2r} \sum AI$$

$$\Rightarrow \sum \sqrt{\frac{m_a}{r_a}} \stackrel{(iii)}{\cong} \frac{1}{2r} \sum AI$$

$$\text{Now, } x^4 + 1 \geq \frac{1}{2} (x^2 + 1)^2 \stackrel{\text{A-G}}{\cong} \frac{1}{2} (2x)(x^2 + 1) \Rightarrow x^4 + 1 \geq x(x^2 + 1) \Rightarrow x^2 + \frac{1}{x^2} \stackrel{(3)}{\cong} x + \frac{1}{x}$$

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Choosing  $x = \sqrt{\frac{a}{b}}$  in (3), we get :  $\sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \leq \frac{a}{b} + \frac{b}{a}$  and analogs

$$\Rightarrow \sum \left( \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_a}} \right) = \sum \left( \sqrt{\frac{bc}{ca}} + \sqrt{\frac{ca}{bc}} \right) = \sum \left( \sqrt{\frac{a}{b}} + \sqrt{\frac{b}{a}} \right)$$

$$\leq \sum \left( \frac{a}{b} + \frac{b}{a} \right) = \sum \left( \frac{c}{a} + \frac{b}{a} \right) = \sum \frac{b+c}{a} = \sum \frac{4R \cos \frac{A}{2} \cos \frac{B-C}{2}}{4R \sin \frac{A}{2} \cos \frac{A}{2}}$$

$$\leq \sum \frac{1}{\sin \frac{A}{2}} \left( \because 0 \leq \cos \frac{B-C}{2} \leq 1 \text{ and analogs} \right) = \frac{1}{r} \Sigma AI$$

$$\Rightarrow \sum \left( \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_a}} \right) \stackrel{(iv)}{\geq} \frac{1}{r} \Sigma AI$$

$$\therefore \sum \frac{m_a}{w_a} + \sum \sqrt{\frac{m_a}{r_a}} + \sum \left( \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_a}} \right) \stackrel{\text{by (ii)+(iii)+(iv)}}{\geq} \frac{1}{2r} \Sigma AI + \frac{1}{2r} \Sigma AI + \frac{1}{r} \Sigma AI$$

$$= \frac{2}{r} \Sigma AI \stackrel{\text{by (i)}}{\geq} \sum \left( \tan \frac{A}{4} + \cot \frac{A}{4} \right)$$

$$\therefore \sum \left( \tan \frac{A}{4} + \cot \frac{A}{4} \right) \geq \sum \left( \frac{m_a}{w_a} + \sqrt{\frac{m_a}{r_a}} + \sqrt{\frac{h_a}{h_b}} + \sqrt{\frac{h_b}{h_a}} \right) \text{ (Proved)}$$

**1647. In  $\triangle ABC, \triangle A'B'C'$  the following relationship holds:**

$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \leq \frac{1}{r^3} + \frac{1}{r'^3}$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \leq \frac{1}{r^3} + \frac{1}{r'^3} + \frac{1}{r^3} \stackrel{Am-Gm}{\geq} 2 \sqrt{\frac{1}{(rr')^3}}$$

We need to prove:

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$$\frac{192\sqrt{3}}{(a+a')(b+b')(c+c')} \leq 2 \sqrt{\frac{1}{(rr')^3}} \Leftrightarrow (a+a')(b+b')(c+c') \stackrel{Am-Gm}{\geq} 96\sqrt{3}(rr')^3$$

$$\text{But: } (a+a')(b+b')(c+c') \stackrel{Am-Gm}{\geq} 8\sqrt{abc \cdot a'b'c'}$$

$$\text{So, we just check: } 8\sqrt{abc \cdot a'b'c'} \geq 96\sqrt{(rr')^3}$$

$$\Leftrightarrow \sqrt{abc \cdot a'b'c'} \geq 12\sqrt{3}(rr')^3 \Leftrightarrow abc \cdot a'b'c' \geq 432(rr')^3$$

$$\Leftrightarrow (4Rrs) \cdot (4R'r's') \geq 432(rr')^3 \Leftrightarrow RsR's' \geq 27 \cdot r^2r'^2 \quad (*)$$

Which is clearly true because :

$$R \geq \frac{2}{3\sqrt{3}}s \Rightarrow Rs \geq \frac{2}{3\sqrt{3}}s^2 \geq \frac{2}{3\sqrt{3}} \cdot 27r^2 = 2 \cdot 3\sqrt{3}r^2 \quad (1)$$

$$R' \geq \frac{2}{3\sqrt{3}}s' \Rightarrow R's' \geq \frac{2}{3\sqrt{3}}s'^2 \geq \frac{2}{3\sqrt{3}} \cdot 27r'^2 = 2 \cdot 3\sqrt{3}r'^2 \quad (2)$$

$$\stackrel{(1),(2)}{\implies} Rs \cdot R's' \geq 4 \cdot 27 \cdot r^2r'^2 \geq 27r^2r'^2 \Rightarrow (*) \text{ is true. Proved.}$$

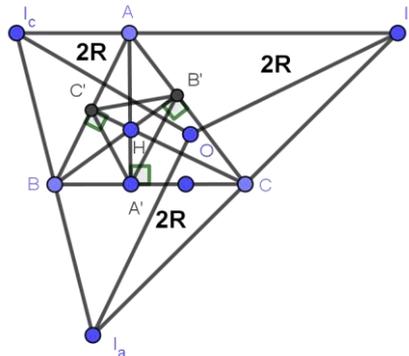
1648. In acute  $\triangle ABC$ ,  $O$  – circumcentre,  $I$  – incentre,  $I_a, I_b, I_c$  - excenters,

$\triangle A'B'C'$  -orthic triangle. Prove that:

$$8r < \sum_{cyc} \frac{B'C'}{OI_a} \cdot \sum_{cyc} II_a < \frac{9\sqrt{3}R^2}{2r}$$

Proposed by Radu Diaconu-Romania

Solution by Tran Hong-Dong Thap-Vietnam



$$OI_a = OI_b = OI_c = 2R;$$

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$$II_a = 4R \sin \frac{A}{2}; II_b = 4R \sin \frac{B}{2}; II_c = 4R \sin \frac{C}{2}$$

$$\therefore B'C^2 = AC^2 + AB'^2 - 2AB' \cdot AC' \cdot \cos A$$

$$= (b \cos A)^2 + (c \cos A)^2 - 2b \cos^2 A \cdot \cos A$$

$$= \cos^2(b^2 + c^2 - 2bc \cos A) = (a \cos A)^2;$$

$$\overset{\Delta ABC\text{-acute}}{\Rightarrow} B'C' = a \cos A;$$

$$\text{Similarly: } A'C' = b \cos B; A'B' = c \cos C$$

$$\Rightarrow \Omega = \sum_{cyc} \frac{B'C'}{OI_a} \cdot \sum_{cyc} II_a = \sum_{cyc} \left( \frac{a \cos A}{2R} \right) \cdot \sum_{cyc} \left( 4R \sin \frac{A}{2} \right) = 2R \cdot \sum_{cyc} \sin 2A \cdot \sum_{cyc} \sin \frac{A}{2} \stackrel{(*)}{>} 8r$$

$$(*) \Leftrightarrow R \sum_{cyc} \sin 2A \cdot \sum \frac{A}{2} > 4r$$

$$\Leftrightarrow R \cdot \frac{2sr}{R^2} \cdot \sum \sin \frac{A}{2} > 4r \Leftrightarrow s \sum \sin \frac{A}{2} > 2R. \text{ Which is true because: } \Delta ABC - \text{acute:}$$

$$s^2 > (2R + r)^2 > (2R)^2 \Rightarrow s > 2R$$

$$\sum_{cyc} \sin \frac{A}{2} = 1 + 4 \sin \left( \frac{\pi - A}{4} \right) \sin \left( \frac{\pi - B}{4} \right) \sin \left( \frac{\pi - C}{4} \right) > 1 + 4 \cdot 0 = 1$$

$$\Rightarrow (*) \text{ is true. Lastly: } \Omega < \frac{9\sqrt{3}R^2}{2r} \Leftrightarrow 2R \sum_{cyc} \sin 2A \cdot \sum_{cyc} \sin \frac{A}{2} < \frac{9\sqrt{3}R^2}{2r}$$

$$\Leftrightarrow 2R \cdot \frac{2sr}{R^2} \cdot \sum_{cyc} \sin \frac{A}{2} < \frac{9\sqrt{3}R^2}{2r} \Leftrightarrow 8sr^2 \cdot \sum_{cyc} \sin \frac{A}{2} \leq 9\sqrt{3}R^3$$

$$\text{But: } \sum_{cyc} \frac{A}{2} \leq \frac{3}{2} \Rightarrow 8sr^2 \cdot \frac{3}{2} = 12sr^2 \stackrel{(**)}{<} 9\sqrt{3}R^3$$

$$(**) \Leftrightarrow 4sr^2 < 3\sqrt{3}R^3 \Leftrightarrow (2r)^2 \cdot s < R^2 \cdot 3\sqrt{3}R$$

$$\text{Which is clearly true because: } R \geq 2r \Rightarrow R^2 \geq (2r)^2$$

$$s \leq \frac{3\sqrt{3}}{2}R < 3\sqrt{3}R \Rightarrow (**) \text{ is true. Proved.}$$

1649. In  $\Delta ABC$  the following relationship holds:

$$\left( \frac{a^4 m_a^2}{m_b m_c} \right)^5 + \left( \frac{b^4 m_b^2}{m_c m_a} \right)^5 + \left( \frac{c^4 m_c^2}{m_a m_b} \right)^5 \geq \frac{(4S)^{10}}{81}$$

Proposed by Daniel Sitaru-Romania

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**Solution 1 by Avishek Mitra-West Bengal-India**

$$\sum_{cyc} \left( \frac{a^4 m_a^2}{m_b m_c} \right)^5 = \sum_{cyc} \frac{a^{20} m_a^{10}}{m_b^5 m_c^5} \stackrel{Am-Gm}{\geq} 3^3 \sqrt[3]{\frac{a^{20} b^{20} c^{20} m_a^{10} m_b^{10} m_c^{10}}{m_a^{10} m_b^{10} m_c^{10}}} = 3^3 \sqrt[3]{(abc)^{20}}$$

$$= 3^3 \sqrt[3]{(4RS)^{20}}$$

Need to show:  $3^3 \sqrt[3]{(4RS)^{20}} \geq \frac{(4S)^{10}}{81} \Leftrightarrow 3 \cdot 81^3 \sqrt[3]{R^{20}} \geq \sqrt[3]{4^{10}} \cdot \sqrt[3]{r^{10}} \cdot \sqrt[3]{s^{10}}$

$\Leftrightarrow 27 \cdot 81^3 \cdot R^{20} \geq 4^{10} r^{10} s^{10}$  true from  $R \geq 2r$  (Euler) and  $3\sqrt{3}R \geq 2s$  (Mitrinovic), we have:

$$3\sqrt{3}R^2 \geq 4rs \Rightarrow (\sqrt{3^3})^{10} R^{20} \geq 4^{10} (rs)^{10} \Rightarrow 3^{15} R^{20} \geq 4^{10} (rs)^{10}$$

$$\Rightarrow 27 \cdot 81^3 \cdot R^{20} \geq 4^{10} r^{10} s^{10} \text{ .Proved.}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\sum_{cyc} \left( \frac{a^4 m_a^2}{m_b m_c} \right)^5 \stackrel{Am-Gm}{\geq} 3^3 \sqrt[3]{\frac{(a^4 b^4 c^4)^5 m_a^2 m_b^2 m_c^2}{(m_a m_b m_c)^2}} = 3^3 \sqrt[3]{(abc)^{20}} \stackrel{(*)}{\geq} \frac{(4S)^{10}}{81}$$

$$(*) \Leftrightarrow 27(4RS)^{20} \geq \frac{(4S)^{30}}{81^3} \Leftrightarrow 27R^{20} \geq \frac{(4S)^{20}}{81^3} = \frac{(4sr)^{10}}{81^3}$$

$$\Leftrightarrow 27R^{10} R^{10} \geq \frac{(4sr)^{10}}{81^3}$$

From  $R \geq 2r$  (Euler) and  $3\sqrt{3}R \geq 2s$  (Mitrinovic), we have:

$$27R^{10} R^{10} \geq 27(2r)^{10} \left( \frac{2s}{3\sqrt{3}} \right)^{10} = \frac{(4sr)^{10}}{81^3} \Rightarrow (*) \text{ is true. Proved.}$$

**1650. In  $\triangle ABC$  the following relationship holds:**

$$\sqrt[3]{\left(\frac{h_a}{r_a}\right)^2} + \sqrt[3]{\left(\frac{h_b}{r_b}\right)^2} + \sqrt[3]{\left(\frac{h_c}{r_c}\right)^2} \geq 3$$

*Proposed by Rahim Shahbazov*

**Solution by Tran Hong-Dong Thap-Vietnam**

- $\frac{r_a}{h_a} = \frac{\frac{s}{p-a}}{\frac{2s}{a}} = \frac{a}{2(p-a)}; \frac{r_b}{h_b} = \frac{\frac{s}{p-b}}{\frac{2s}{b}} = \frac{b}{2(p-b)}; \frac{r_c}{h_c} = \frac{\frac{s}{p-c}}{\frac{2s}{c}} = \frac{c}{2(p-c)}$

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Now,

$$\sqrt[3]{\left(\frac{h_a}{r_a}\right)^2} = \sqrt[3]{1 \cdot \frac{h_a}{r_a} \cdot \frac{h_a}{r_a}} \stackrel{AM-HM}{\geq} \frac{3}{1 + \frac{r_a}{h_a} + \frac{r_a}{h_a}} = \frac{3}{1 + 2 \cdot \frac{a}{2(p-a)}} = \frac{3}{1 + \frac{a}{p-a}} = \frac{3(p-a)}{p};$$

$$\text{Similary: } \sqrt[3]{\left(\frac{h_b}{r_b}\right)^2} \geq \frac{3(p-b)}{p}; \quad \sqrt[3]{\left(\frac{h_c}{r_c}\right)^2} \geq \frac{3(p-c)}{p};$$

$$\rightarrow LHS = \sum \sqrt[3]{\left(\frac{h_a}{r_a}\right)^2} \geq \frac{3(p-a) + 3(p-b) + 3(p-c)}{p} = \frac{3(3p-2p)}{p} = 3;$$

**1651. In  $\triangle ABC$  the following relationship holds:**

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{36r^2}{s^2} \geq \frac{13}{3}$$

*Proposed by Rahim Shahbazov*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\bullet \quad S = pr \rightarrow r = \frac{S}{p} \rightarrow r^2 = \frac{S^2}{p^2} = \frac{p(p-a)(p-b)(p-c)}{p^2} = \frac{(p-a)(p-b)(p-c)}{p}$$

$$\bullet \quad r_a = \frac{S}{p-a}; \quad r_b = \frac{S}{p-b}; \quad r_c = \frac{S}{p-c}$$

$$\text{Inequality} \Leftrightarrow \frac{p-b}{p-a} + \frac{p-c}{p-b} + \frac{p-a}{p-c} + 36 \cdot \frac{(p-a)(p-b)(p-c)}{p^3} \stackrel{(*)}{\geq} \frac{13}{3}$$

$$\text{Let } x = p-a; \quad y = p-b; \quad z = p-c \rightarrow x+y+z = p$$

$$(*) \Leftrightarrow \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 36 \cdot \frac{xyz}{(x+y+z)^3} \geq \frac{13}{3}$$

$$\text{But: } \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \geq \frac{9(x^2+y^2+z^2)}{(x+y+z)^2}$$

$$\text{We need to prove: } \frac{9(x^2+y^2+z^2)}{(x+y+z)^2} + 36 \cdot \frac{xyz}{(x+y+z)^3} \geq \frac{13}{3}$$

$$\Leftrightarrow 3 \cdot [9(x^2 + y^2 + z^2)(x + y + z) + 36 \cdot xyz] \geq 13(x + y + z)^3$$

$$\Leftrightarrow 14(x^3 + y^3 + z^3) + 30xyz \stackrel{(**)}{\geq} 12[xy(x+y) + yz(y+z) + zx(z+x)]$$

Which is true because:

$$(x^3 + y^3 + z^3) + 3xyz \stackrel{\text{Schur}}{\geq} [xy(x+y) + yz(y+z) + zx(z+x)]$$

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$$\rightarrow 10(x^3 + y^3 + z^3) + 30xyz \geq 10[xy(x + y) + yz(y + z) + zx(z + x)]; \quad (1)$$

And:

$$2(x^3 + y^3 + z^3) \geq 2(xy^2 + yz^2 + zx^2); \quad (2)$$

$$2(x^3 + y^3 + z^3) \geq 2(yx^2 + zy^2 + xz^2); \quad (3)$$

(1)+(2)+(3)

$$\begin{aligned} \Rightarrow & 10(x^3 + y^3 + z^3) + 30xyz + 4(x^3 + y^3 + z^3) \\ & \geq 10[xy(x + y) + yz(y + z) + zx(z + x)] \\ & + 2[xy(x + y) + yz(y + z) + zx(z + x)]; \\ & \rightarrow (**) \text{ is true } \rightarrow (*) \text{ is true .} \end{aligned}$$

1652. In  $\triangle ABC$  the following relationship holds:

$$\frac{3}{2} < \frac{1}{1 + \sin^3 \frac{A}{2}} + \frac{1}{1 + \sin^3 \frac{B}{2}} + \frac{1}{1 + \sin^3 \frac{C}{2}} \leq \frac{12R}{4R + r}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \sum \frac{1}{1 + \sin^3 \frac{A}{2}} &= \sum \frac{1 + \sin^3 \frac{A}{2} - \sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} = \sum \frac{1 + \sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} - \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} \\ &= 3 - \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} \leq \frac{12R}{4R + r} \end{aligned}$$

$$\Leftrightarrow \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} \stackrel{(i)}{\geq} \frac{3r}{4R + r}$$

$$\text{Now, } \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} = \sum \frac{\sin^4 \frac{A}{2}}{\sin \frac{A}{2} + \sin^4 \frac{A}{2}} \stackrel{\text{Bergstrom}}{\stackrel{(1)}{\geq}} \frac{(\sum \sin^2 \frac{A}{2})^2}{\sum \sin \frac{A}{2} + \sum \sin^4 \frac{A}{2}}$$

$$\sum \sin^2 \frac{A}{2} = \frac{1}{2} \sum (1 - \cos A) = \frac{1}{2} \left( 3 - 1 - \frac{r}{R} \right) \stackrel{(2)}{=} \frac{2R - r}{2R} \text{ and } \sum \sin^4 \frac{A}{2}$$

$$= \left( \sum \sin^2 \frac{A}{2} \right)^2 - 2 \sum \sin^2 \frac{B}{2} \sin^2 \frac{C}{2}$$

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$$\begin{aligned}
 & \stackrel{\text{by (2)}}{=} \left(\frac{2R-r}{2R}\right)^2 - \left(2 \prod \sin^2 \frac{A}{2}\right) \sum \operatorname{cosec}^2 \frac{A}{2} \\
 & = \left(\frac{2R-r}{2R}\right)^2 - \left(\frac{2r^2}{16R^2}\right) \frac{\sum bc(s-a)}{(s-a)(s-b)(s-c)} \\
 & = \left(\frac{2R-r}{2R}\right)^2 - \frac{r^2\{s(s^2+4Rr+r^2)-12Rrs\}}{8R^2sr^2} = \left(\frac{2R-r}{2R}\right)^2 - \frac{s^2-8Rr+r^2}{8R^2} \\
 & = \frac{2(2R-r)^2 - s^2 + 8Rr - r^2}{8R^2} \stackrel{(3)}{=} \frac{8R^2 + r^2 - s^2}{8R^2}
 \end{aligned}$$

Again,  $f(x) = \sin \frac{x}{2} \forall x \in (0, \pi)$  is concave as  $f''(x) = \frac{-\sin \frac{x}{2}}{4} < 0$

$$\therefore \sum \sin \frac{A}{2} \stackrel{\text{Jensen}}{\geq} 3 \sin \left(\frac{A+B+C}{6}\right) = \frac{3}{2} \Rightarrow \sum \sin \frac{A}{2} \stackrel{(4)}{\geq} \frac{3}{2}$$

$$\begin{aligned}
 \therefore (1), (2), (3), (4) & \Rightarrow \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} \geq \frac{\left(\frac{2R-r}{2R}\right)^2}{\frac{3}{2} + \frac{8R^2 + r^2 - s^2}{8R^2}} = \frac{\frac{(2R-r)^2}{4R^2}}{\frac{20R^2 + r^2 - s^2}{8R^2}} \\
 & = \frac{2(2R-r)^2}{20R^2 + r^2 - s^2} \Rightarrow \sum \frac{\sin^3 \frac{A}{2}}{1 + \sin^3 \frac{A}{2}} \stackrel{(5)}{\geq} \frac{2(2R-r)^2}{20R^2 + r^2 - s^2}
 \end{aligned}$$

(5)  $\Rightarrow$  in order to prove (i) and consequently (b), it suffices to prove

$$\frac{2(2R-r)^2}{20R^2 + r^2 - s^2} \stackrel{(ii)}{\geq} \frac{3r}{4R+r}$$

$$\text{Now, } s^2 - r^2 \stackrel{\text{Gerretsen}}{\geq} 4R^2 + 4Rr + 2r^2 \stackrel{\text{Euler}}{\geq} 4R^2 + 2R^2 + \frac{R^2}{2} < 20R^2 \Rightarrow 20R^2 + r^2 - s^2$$

$> 0$  and  $\therefore (ii) \Leftrightarrow$

$$2(4R+r)(2R-r)^2 \geq 3r(20R^2 + r^2 - s^2)$$

$$\Leftrightarrow 2(4R+r)(2R-r)^2 + 3rs^2 \stackrel{(iii)}{\geq} 3r(20R^2 + r^2)$$

$$\text{Now, LHS of (iii)} \stackrel{\text{Gerretsen}}{\geq} 2(4R+r)(2R-r)^2 + 3r(16Rr - 5r^2) \stackrel{?}{\geq} 3r(20R^2 + r^2)$$

$$\Leftrightarrow 8t^3 - 21t^2 + 12t - 4 \stackrel{?}{\geq} 0 \left(\text{where } t = \frac{R}{r}\right)$$

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$$\Leftrightarrow (t-2)\{(t-2)(8t+11)+24\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)} \Rightarrow \text{(i)}$$

$$\Rightarrow \boxed{\text{(b) is true}}$$

$$\because 0 < \sin \frac{A}{2} < 1 \Rightarrow 0 < \sin^3 \frac{A}{2} < 1 \Rightarrow 1 < 1 + \sin^3 \frac{A}{2} < 2 \Rightarrow \frac{1}{1 + \sin^3 \frac{A}{2}} > \frac{1}{2} \text{ \& analogs}$$

$$\Rightarrow \sum \frac{1}{1 + \sin^3 \frac{A}{2}} > \frac{3}{2} \Rightarrow \boxed{\text{(a) is true}} \text{ (QED)}$$

### Solution 2 by Tran Hong-Dong Thap-Vietnam

$$\text{Lemma: If } x, y, z > 0, xyz < \frac{1}{8} \text{ then } \frac{1}{x+1} + \frac{1}{y+1} + \frac{1}{z+1} \stackrel{(1)}{<} \frac{3}{1+\sqrt[3]{xyz}}$$

$$\text{Proof: (1) } \stackrel{u=\sqrt[3]{xyz}}{\Leftrightarrow} ((1+x)(1+y) + (1+y)(1+z) + (1+z)(1+x)) < 3(1+x)(1+y)(1+z)$$

$$\Leftrightarrow (x+y+z)(1-2u) + (xy+yz+zx)(2-u) + 3u^3 - 3u \stackrel{(*)}{>} 0$$

$$\text{Because: } u = \sqrt[3]{xyz} \Rightarrow 0 < u < \sqrt[3]{\frac{1}{8}} = \frac{1}{2} < 2 \Rightarrow 1-2u > 0; 2-u > 0$$

$$\text{LHS}_{(*)} \stackrel{\text{Am-Gm}}{>} 3\sqrt[3]{xyz}(1-2u) + 3\sqrt[3]{xyz}(2-u) + 3u^3 - 3u = 3u(1-2u) + 3u^2(2-u) + 3u^3 - 3u = 0 \Rightarrow (*) \text{ is true.}$$

$$\text{Now, choose: } x = \sin^3 \frac{A}{2}, y = \sin^3 \frac{B}{2}, z = \sin^3 \frac{C}{2}$$

$$\Rightarrow x, y, z > 0 \text{ and } xyz = \left(\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\right)^3 \leq \frac{1}{8^3} < \frac{1}{8}$$

$$\begin{aligned} & \frac{1}{1 + \sin^3 \frac{A}{2}} + \frac{1}{1 + \sin^3 \frac{B}{2}} + \frac{1}{1 + \sin^3 \frac{C}{2}} < \\ & < \frac{3}{1 + \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}} \stackrel{\sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{4R}}{=} \frac{3}{1 + \frac{r}{4R}} = \frac{12R}{4R+r} \end{aligned}$$

$$\text{Lastly, } 0 < \sin \frac{A}{2}; \sin \frac{B}{2}; \sin \frac{C}{2} < 1$$

$$\Rightarrow \frac{1}{1 + \sin^3 \frac{A}{2}} + \frac{1}{1 + \sin^3 \frac{B}{2}} + \frac{1}{1 + \sin^3 \frac{C}{2}} > \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = \frac{3}{2}. \text{ Proved.}$$

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1653. In  $\triangle ABC$  the following relationship holds:

$$(m_a + m_b + m_c) \left( \frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a} \right) \geq \frac{9}{2} \sqrt{a^2 + b^2 + c^2}$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\left( \sum a \right)^2 \left( \sum \frac{a}{b} \right)^2 \geq \left( \sum a \right)^2 \left\{ \frac{9 \sum a^2}{(\sum a)^2} \right\}^2 = \frac{81 (\sum a^2)^2}{(\sum a)^2} \geq \frac{27 (\sum a^2) (\sum a)^2}{(\sum a)^2} = 27 \sum a^2$$

$$\therefore \left( \sum a \right) \sum \frac{a}{b} \stackrel{(1)}{\geq} 3\sqrt{3} \sqrt{\sum a^2}$$

Applying (1) on a triangle with sides  $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ , we get :

$$\frac{2}{3} \left( \sum m_a \right) \sum \frac{m_a}{m_b} \geq 3\sqrt{3} \sqrt{\frac{4 \sum m_a^2}{9}}$$

$$\begin{aligned} \therefore \left( \sum m_a \right) \sum \frac{m_a}{m_b} &\geq 3\sqrt{3} \sqrt{\sum m_a^2} = 3\sqrt{3} \sqrt{\frac{3 \sum a^2}{4}} \Rightarrow (m_a + m_b + m_c) \left( \frac{m_a}{m_b} + \frac{m_b}{m_c} + \frac{m_c}{m_a} \right) \\ &\geq \frac{9}{2} \sqrt{a^2 + b^2 + c^2} \text{ (Proved)} \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\text{Because: } m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2)$$

$$\text{Let: } x = m_a, y = m_b, z = m_c, (x, y, z > 0)$$

$$\text{So, we need to prove: } (x + y + z) \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) \stackrel{(*)}{\geq} 3\sqrt{3} \sqrt{x^2 + y^2 + z^2}$$

$$\text{Proof: } \left( \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \right) (xy + yz + zx) \stackrel{C-B-S}{\geq} (x + y + z)^2; \quad (1)$$

$$3(xy + yz + zx) \sqrt{3(x^2 + y^2 + z^2)} = \sqrt{27(x^2 + y^2 + z^2)(xy + yz + zx)^2}$$

$$\stackrel{Am-Gm}{\leq} \sqrt{(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)^3} = \sqrt{(x + y + z)^6} = (x + y + z)^3; \quad (2)$$

From (1),(2) (\*) is true.Proved.

$$\text{Equality} \Leftrightarrow x = y = z \Leftrightarrow m_a = m_b = m_c \Leftrightarrow a = b = c$$

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1654. In  $\triangle ABC$  the following relationship holds:

$$\sqrt{(8R^2 - a^2)(8R^2 - b^2)(8R^2 - c^2)} \geq 2\sqrt{5}R^2(2R + r)$$

Proposed by Marian Ursărescu-Romania

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\sqrt{(8R^2 - a^2)(8R^2 - b^2)(8R^2 - c^2)} \stackrel{(1)}{\geq} 2\sqrt{5}R^2(2R + r)$$

$$(1) \Leftrightarrow (8R^2 - a^2)(8R^2 - b^2)(8R^2 - c^2) \geq 20R^4(2R + r)^2$$

$$\Leftrightarrow 512R^6 - 64R^4 \left( \sum a^2 \right) + 8R^2 \sum a^2 b^2 - a^2 b^2 c^2 \geq 20R^4(2R + r)^2$$

$$\Leftrightarrow 512R^6 - 128R^4(s^2 - 4Rr - r^2) + 8R^2\{(s^2 + 4Rr + r^2)^2 - 16Rrs^2\} - 16R^2r^2s^2 - 20R^4(2R + r)^2 \geq 0$$

$$\Leftrightarrow 108R^4 + 108R^3r + 59R^2r^2 + 16Rr^3 + 2r^4 + 2s^4 - s^2(32R^2 + 16Rr) \stackrel{(2)}{\geq} 0$$

$$\text{Now, Gerretsen} \Rightarrow 4R^2 + 4Rr + 3r^2 - s^2 \geq 0 \Rightarrow 2(4R^2 + 4Rr + 3r^2 - s^2)^2 \stackrel{(i)}{\geq} 0 \therefore (i)$$

$\Rightarrow$  in order to prove (2), it suffices to prove :

$$108R^4 + 108R^3r + 59R^2r^2 + 16Rr^3 + 2r^4 + 2s^4 - s^2(32R^2 + 16Rr) \geq 2(4R^2 + 4Rr + 3r^2 - s^2)^2$$

$$\Leftrightarrow 76R^4 + 44R^3r - 21R^2r^2 - 32Rr^3 - 16r^4 \stackrel{(3)}{\geq} s^2(16R^2 - 12r^2)$$

$$\text{Now, RHS of (3)} \stackrel{\text{Gerretsen}}{\geq} (16R^2 - 12r^2)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 76R^4 + 44R^3r - 21R^2r^2 - 32Rr^3 - 16r^4$$

$$\Leftrightarrow 12t^4 - 20t^3 - 21t^2 + 16t + 20 \stackrel{?}{\geq} 0 \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(12t^2 + 28t + 43) + 76\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore t \stackrel{\text{Euler}}{\geq} 2$$

$\Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$  is true (Proved)

**Solution 2 by Rahim Shahbazov-Baku-Azerbaijan**

$$\sqrt{(8R^2 - a^2)(8R^2 - b^2)(8R^2 - c^2)} \geq 2\sqrt{5}R^2(2R + r); \quad (1)$$

$$a = 2R\sin A; b = 2R\sin B; c = 2R\sin C \stackrel{(1)}{\Rightarrow}$$

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$$16R^2(2 - \sin^2 A)(2 - \sin^2 B)(2 - \sin^2 C) \geq 5(2R + r)^2$$

$$16R^2(1 + \cos^2 A)(1 + \cos^2 B)(1 + \cos^2 C) \geq 5R^2 \left(2 + \frac{r}{R}\right)^2$$

$$16(1 + \cos^2 A)(1 + \cos^2 B)(1 + \cos^2 C) \geq 5(1 + \cos A + \cos B + \cos C)^2$$

Let:  $\cos A = \frac{x}{2}$ ;  $\cos B = \frac{y}{2}$ ;  $\cos C = \frac{z}{2}$  then

$$(x^2 + 4)(y^2 + 4)(z^2 + 4) \geq 5(x + y + z + 2)^2; \quad (2)$$

$$(x + y + z + 2)^2 \stackrel{Am-Gm}{\leq} (x^2 + 4) \left(1 + \left(\frac{y + z + 2}{2}\right)^2\right) \Rightarrow$$

$$5 \left(1 + \left(\frac{y + z + 2}{2}\right)^2\right) \leq (y^2 + 4)(z^2 + 4) \Leftrightarrow 5(y + z - 2)^2 \geq 0$$

**1655. In  $\triangle ABC$  the following relationship holds:**

$$\frac{r_a - h_a}{r_a + h_a} + \frac{r_b - h_b}{r_b + h_b} + \frac{r_c - h_c}{r_c + h_c} + \frac{2r}{R} \leq 1$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\frac{r_a - h_a}{r_a + h_a} + \frac{r_b - h_b}{r_b + h_b} + \frac{r_c - h_c}{r_c + h_c} = \sum_{cyc} \frac{\frac{r_a}{h_a} - 1}{\frac{r_a}{h_a} + 1} = \sum_{cyc} \frac{\frac{a}{2(s-a)} - 1}{\frac{a}{2(s-a)} + 1} = \sum_{cyc} \frac{\frac{a}{b+c-a} - 1}{\frac{a}{b+c-a} + 1}$$

$$\frac{2r}{R} = \frac{(a+b-c)(b+c-a)(a+c-b)}{abc}$$

$$\text{Let: } x = \frac{b+c-a}{a}, y = \frac{a+c-b}{b}, z = \frac{a+b-c}{c} \Rightarrow 0 < xyx \leq 1$$

$$\Rightarrow \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} = \frac{1}{\frac{b+c-a}{a}+2} + \frac{1}{\frac{a+c-b}{b}+2} + \frac{1}{\frac{a+b-c}{c}+2}$$

$$= \frac{a}{b+c+a} + \frac{b}{a+c+b} + \frac{c}{a+b+c} = 1 \Leftrightarrow$$

$$xyz + xy + yz + zx = 4$$

$$\text{More, } (xy + yz + zx)^2 \geq 3xyz(x + y + z) \Rightarrow$$

$$x + y + z \leq \frac{(xy + yz + zx)^2}{3xyz} = \frac{(4 - xyz)^2}{3xyz}$$

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Inequality become as:

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} + \frac{1-z}{1+z} + xyz \leq 1 \Leftrightarrow$$

$$(xyz)^2 + xyz(xy + yz + zx + x + y + z - 3) - 2(xy + yz + zx) + 2 \leq 0$$

$$\Leftrightarrow (xyz)^2 + xyz(4 - xyz + x + y + z - 3) - 2(4 - xyz) + 2 \leq 0$$

$$\Leftrightarrow (xyz)^2 + xyz(1 - xyz + x + y + z) + 2xyz - 6 \leq 0$$

$$\Leftrightarrow xyz(x + y + z) + 3xyz - 6 \leq 0$$

Let:  $t = xyz$ , ( $0 < t \leq 1$ ). We need to prove:

$$t \cdot \frac{(4-t)^2}{3t} + 3t - 6 \leq 0 \Leftrightarrow \frac{1}{3}(4-t)^2 + 3t - 6 \leq 0 \Leftrightarrow$$

$$(4-t)^2 + 9t - 18 \leq 0 \Leftrightarrow t^2 + t - 2 \leq 0 \Leftrightarrow (t-1)(t+2) \leq 0 \text{ true, because}$$

$$0 < t \leq 1 \Rightarrow t-1 \leq 0, t+2 > 2 > 0.$$

**1656. In any  $\triangle ABC$ ,  $n_a$  –Nagel’s cevian,  $g_a$  –Gergonne’s cevian, holds:**

$$\prod (n_a - m_a) \geq \sqrt{g_a g_b g_c} \prod (\sqrt{m_a} - \sqrt{g_a})$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\text{Triangle inequality} \Rightarrow g_a \leq AI + r \stackrel{?}{\leq} w_a \Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{2abccos \frac{A}{2}}{a(b+c)}$$

$$\Leftrightarrow \frac{r}{\sin \frac{A}{2}} + r \stackrel{?}{\leq} \frac{8Rrscos \frac{A}{2}}{4R(b+c)\sin \frac{A}{2} \cos \frac{A}{2}}$$

$$\Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a+b+c}{(b+c)\sin \frac{A}{2}} \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + 1 \stackrel{?}{\leq} \frac{a}{(b+c)\sin \frac{A}{2}} + \frac{1}{\sin \frac{A}{2}} \Leftrightarrow (b+c)\sin \frac{A}{2} \stackrel{?}{\leq} a$$

$$\Leftrightarrow 4R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leq} 4R \sin \frac{A}{2} \cos \frac{A}{2}$$

$$\Leftrightarrow \cos \frac{B-C}{2} \stackrel{?}{\leq} 1 \rightarrow \text{true} \therefore g_a \leq w_a \leq m_a \Rightarrow g_a \stackrel{(1)}{\leq} m_a$$



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$$\begin{aligned}
 &\Rightarrow 2a(b^2 + c^2) = 2a(n_a^2 + g_a^2) + a(a + b - c)(c + a - b) \Rightarrow 2(b^2 + c^2) \\
 &\quad = 2(n_a^2 + g_a^2) + a^2 - (b - c)^2 \\
 &\Rightarrow 2(b^2 + c^2) - a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \Rightarrow 4m_a^2 + (b - c)^2 = 2(n_a^2 + g_a^2) \\
 &\Rightarrow 4m_a^2 + (b - c)^2 + 4r_b r_c = 2(n_a^2 + g_a^2) + 4r_b r_c \Rightarrow 4m_a^2 + (b - c)^2 + 4s(s - a) \\
 &\quad = 2(n_a^2 + g_a^2) + 4s(s - a) \\
 &\Rightarrow 4m_a^2 + 4m_a^2 = 2(n_a^2 + g_a^2) + 4s(s - a) \Rightarrow n_a^2 + g_a^2 = 4m_a^2 - 2s(s - a) \\
 &\quad \text{by (3)} \\
 &\Rightarrow n_a^2 + g_a^2 + 2n_a g_a \stackrel{\geq}{\sim} 4m_a^2 - 2s(s - a) + 2s(s - a) \\
 &\quad \text{by (1)} \\
 &= 4m_a^2 \Rightarrow n_a + g_a \geq 2m_a \stackrel{\geq}{\sim} m_a + \sqrt{m_a g_a} \Rightarrow n_a - m_a \geq \sqrt{m_a g_a} - g_a \\
 &\quad = \sqrt{g_a}(\sqrt{m_a} - \sqrt{g_a}) \\
 &\quad \therefore n_a - m_a \geq \sqrt{g_a}(\sqrt{m_a} - \sqrt{g_a}) \text{ and analogs and} \\
 &\quad \text{by (2)} \quad \text{by (1)} \\
 &\therefore n_a - m_a \stackrel{\geq}{\sim} 0 \text{ and analogs and } \sqrt{m_a} - \sqrt{g_a} \stackrel{\geq}{\sim} 0 \text{ and analogs} \\
 &\therefore \prod (n_a - m_a) \geq \prod \{\sqrt{g_a}(\sqrt{m_a} - \sqrt{g_a})\} \Rightarrow \prod (n_a - m_a) \\
 &\geq \sqrt{g_a g_b g_c} \prod (\sqrt{m_a} - \sqrt{g_a}) \text{ (Proved)}
 \end{aligned}$$

**1657. In  $\triangle ABC$  the following relationship holds:**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}} \geq 2$$

*Proposed by Nguyen Van Canh-Ben Tre-Vietnam*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned}
 2 - \sum \frac{a}{b+c} &= 2 - \frac{\sum a(c+a)(a+b)}{\prod (b+c)} = 2 - \frac{\sum \{a(\sum ab + a^2)\}}{2s(s^2 + 2Rr + r^2)} \\
 &= 2 - \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)}
 \end{aligned}$$

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$$= \frac{2(s^2 + 2Rr + r^2) - (2s^2 - 2Rr - 2r^2)}{s^2 + 2Rr + r^2} = \frac{6Rr + 4r^2}{s^2 + 2Rr + r^2} \leq \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

$$\Leftrightarrow \frac{(6Rr + 4r^2)^2}{(s^2 + 2Rr + r^2)^2} \leq \frac{8Rrs}{2s(s^2 + 2Rr + r^2)}$$

$$\Leftrightarrow R(s^2 + 2Rr + r^2) \stackrel{(1)}{\geq} r(3R + 2r)^2$$

Now, LHS of (1)  $\stackrel{\text{Gerretsen}}{\geq} R(18Rr - 4r^2) \stackrel{?}{\geq} r(3R + 2r)^2 \Leftrightarrow 9R^2 - 16Rr - 4r^2 \stackrel{?}{\geq} 0$

$$\Leftrightarrow (R - 2r)(9R + 2r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \stackrel{\text{Euler}}{\geq} 2r$$

$$\Rightarrow (1) \text{ is true} \Rightarrow 2 - \sum \frac{a}{b+c} \leq \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}}$$

$$\Rightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \sqrt{\frac{2abc}{(a+b)(b+c)(c+a)}} \geq 2 \text{ (Proved)}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}} \geq 2$$

$$\Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}; (*)$$

By Schur's inequality:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2$$

$$\Leftrightarrow \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq 2 - \frac{4abc}{(a+b)(b+c)(c+a)}; (**)$$

From (\*), (\*\*) we need to prove:

$$2 - \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2 - 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}}$$

$$\Leftrightarrow 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}} \geq \frac{4abc}{(a+b)(b+c)(c+a)}$$

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$$\Leftrightarrow 2 \cdot \sqrt{\frac{abc}{(a+b)(b+c)(c+a)}} \leq 1 \Leftrightarrow \frac{abc}{(a+b)(b+c)(c+a)} \leq \frac{1}{4}$$

Which is true by Am-Gm:

$$(a+b)(b+c)(c+a) \geq (2\sqrt{ab})(2\sqrt{bc})(2\sqrt{ca}) = 2\sqrt{(abc)^2} = 2abc$$

$$\Rightarrow \frac{abc}{(a+b)(b+c)(c+a)} \leq \frac{1}{8} \leq \frac{1}{4}$$

1658. In  $\triangle ABC$ ,  $K$  –Lemoine's point, the following relationship holds:

$$\left( \sum_{cyc} \frac{AK}{w_a} \right) \left( \sum_{cyc} \frac{1}{\varphi + \sin \frac{A}{2}} \right) > \frac{12r(a+b+c)}{a^2 + b^2 + c^2}, \varphi \leq 1$$

Proposed by Radu Diaconu-Romania

Solution by George Florin Şerban-Romania

$$\text{Let: } f: (0, \pi) \rightarrow \mathbb{R}, f(x) = \sin \frac{x}{2}, f'(x) = \frac{1}{2} \cos \frac{x}{2}, f''(x) = -\frac{1}{4} \sin \frac{x}{2} < 0,$$

$$\forall x \in (0, \pi) \Rightarrow f \text{ -concave.}$$

$$f\left(\frac{A+B+C}{3}\right) \geq \frac{f(A)+f(B)+f(C)}{3} \text{ (by Jensen inequality)}$$

$$\sum_{cyc} \sin \frac{A}{2} \leq 3f\left(\frac{\pi}{3}\right) = 3\sin \frac{\pi}{6} = \frac{3}{2}$$

$$\sum_{cyc} \frac{1}{\varphi + \sin \frac{A}{2}} \stackrel{\text{Bergstrom}}{\geq} \frac{9}{\sum_{cyc} \varphi + \sin \frac{A}{2}} \geq \frac{9}{3\varphi + \sum_{cyc} \sin \frac{A}{2}} \geq \frac{9}{3 \cdot 1 + \frac{3}{2}} = 2$$

$$\text{We must show that: } \sum_{cyc} \frac{AK}{w_a} \geq \frac{6r(a+b+c)}{a^2+b^2+c^2}$$

$$\sum_{cyc} \frac{AK}{w_a} = \sum_{cyc} \frac{2bcm_a}{(a^2+b^2+c^2)w_a} = \frac{2}{a^2+b^2+c^2} \sum_{cyc} \frac{bcm_a}{w_a} \stackrel{?}{\geq} \frac{6r(a+b+c)}{a^2+b^2+c^2}$$

$$\Leftrightarrow \sum_{cyc} \frac{bcm_a}{w_a} \geq 3r(a+b+c) = 6rs \stackrel{m_a \geq w_a}{\Leftrightarrow}$$

$$\sum_{cyc} \frac{bcm_a}{w_a} \geq \sum_{cyc} \frac{bcw_a}{w_a} = \sum_{cyc} bc \geq 4S\sqrt{3} \stackrel{?}{\geq} 6rs$$

$$4S\sqrt{3} = 4\sqrt{3}rs \geq 6rs. \text{ Proved.}$$

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1659. In  $\triangle ABC$ ,  $G$  –centroid,  $O$  –circumcentre,  $I$  –incenter,  $H$  –orthocenter,  $N_a$  –Nagel's point, the following relationship holds:

$$GO \perp GI \Leftrightarrow OH^2 + IN_a^2 = 9OI^2$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Marian Ursărescu-Romania*

We have:  $OH^2 = 9R^2 - (a^2 + b^2 + c^2)$ ,  $OI^2 = R^2 - 2Rr$  and in any  $\triangle ABC$ ,  $I, G, N_a$  –it's collinear and  $IN_a = 3GI$ .

$$\text{But: } IG^2 = \frac{s^2 - 16Rr + 5r^2}{9} \Rightarrow OH^2 + IN_a^2 = 9OI^2 \Leftrightarrow$$

$$9R^2 - (a^2 + b^2 + c^2) + s^2 - 16Rr + 5r^2 = 9R^2 - 18Rr \Leftrightarrow$$

$$-(a^2 + b^2 + c^2) + s^2 + 2Rr + 5r^2 = 0 \text{ and}$$

$$a^2 + b^2 + c^2 = 2(s^2 - r^2 - 4Rr) \text{ then } OH^2 + IN_a^2 = 9OI^2 \Leftrightarrow$$

$$-2s^2 + 2r^2 + 8Rr + s^2 + 2Rr + 5r^2 = 0 \Leftrightarrow s^2 = 10Rr + 7r^2$$

$$OH^2 + IN_a^2 = 9OI^2 \Leftrightarrow s^2 = 10Rr + 7r^2, \quad (1)$$

$$GO \perp GI \Leftrightarrow GO^2 + GI^2 = OI^2$$

$$\Leftrightarrow R^2 - \frac{1}{9}(a^2 + b^2 + c^2) + \frac{s^2 - 16Rr + 5r^2}{9} = R^2 - 2Rr$$

$$\Leftrightarrow -(a^2 + b^2 + c^2) + s^2 - 16Rr + 5r^2 = -18Rr$$

$$\Leftrightarrow -2(s^2 - r^2 - 4Rr) + s^2 + 2Rr + 5r^2 = 0$$

$$\Leftrightarrow s^2 = 10Rr + 7r^2, \text{ means } GO \perp GI \Leftrightarrow s^2 = 10Rr + 7r^2, \quad (2)$$

$$\text{From (1), (2) we have: } GO \perp GI \Leftrightarrow OH^2 + IN_a^2 = 9OI^2$$

1660. In  $\triangle ABC$  the following relationship holds:

$$\sqrt{\frac{2r_a r_b}{r_a^2 + r_b^2}} + \sqrt{\frac{2r_b r_c}{r_b^2 + r_c^2}} + \sqrt{\frac{2r_c r_a}{r_c^2 + r_a^2}} \geq \frac{3h_a h_b h_c}{m_a m_b m_c}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution 1 by George Florin Şerban-Romania*

$$\text{We show that: } \frac{h_a h_b h_c}{m_a m_b m_c} \leq \frac{2r}{R}$$

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$$\frac{3h_a h_b h_c}{m_a m_b m_c} \leq \frac{\frac{2s^2 r^2}{R}}{\sqrt{s(s-a)} \cdot \sqrt{s(s-b)} \cdot \sqrt{s(s-c)}} = \frac{2s^2 r^2}{R \cdot rs \cdot s} = \frac{2r}{R}$$

We show that:  $\frac{1}{3} \sum_{cyc} \sqrt{\frac{2r_a r_b}{r_a^2 + r_b^2}} \geq \frac{2r}{R} \geq \frac{h_a h_b h_c}{m_a m_b m_c}$

$$\frac{r_a^2 + r_b^2}{2r_a r_b} \geq 1 \Leftrightarrow (r_a - r_b)^2 \geq 0$$

Let:  $f: (0, \infty) \rightarrow \mathbb{R}, f(x) = x^{-\frac{1}{2}}, f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}, f''(x) = \frac{3}{4}x^{-\frac{5}{2}} > 0 \Rightarrow f$  -convex.

$$f\left(\frac{x+y+z}{3}\right) \leq \frac{f(x)+f(y)+f(z)}{3} \text{ (by Jensen inequality)}$$

$$\frac{1}{3} \sum_{cyc} \sqrt{\frac{2r_a r_b}{r_a^2 + r_b^2}} = \frac{1}{3} \sum_{cyc} \left(\frac{r_a^2 + r_b^2}{2r_a r_b}\right)^{-\frac{1}{2}} \geq \left(\frac{\sum_{cyc} \frac{r_a^2 + r_b^2}{2r_a r_b}}{3}\right)^{-\frac{1}{2}} \geq 1 \geq \frac{2r}{R}$$

### Solution 2 by Rahim Shahbazov-Baku-Azerbaijan

Lemma:  $\sqrt{\frac{2xy}{x^2+y^2}} + \sqrt{\frac{2yz}{y^2+z^2}} + \sqrt{\frac{2zx}{z^2+x^2}} \geq \frac{24xyz}{(x+y)(y+z)(z+x)}$

Proof:  $\sqrt{\frac{2xy}{x^2+y^2}} \geq \frac{2xy}{\sqrt{2xy(x^2+y^2)}} \geq \frac{4xy}{2xy+x^2+y^2} = \frac{4xy}{(x+y)^2}$

$$\sum_{cyc} \sqrt{\frac{2xy}{x^2+y^2}} \geq \sum_{cyc} \frac{4xy}{(x+y)^2} \geq 3 \sqrt[3]{\frac{64x^2y^2z^2}{(x+y)^2(y+z)^2(z+x)^2}}$$

If we take  $k = \frac{8xyz}{(x+y)(y+z)(z+x)} \leq 1 \Rightarrow 3\sqrt[3]{k^2} \geq 3k \Leftrightarrow k \leq 1$  true.

$$\sum_{cyc} \sqrt{\frac{2r_a r_b}{r_a^2 + r_b^2}} \geq 3 \cdot \frac{2r}{R} \geq 3 \cdot \frac{h_a h_b h_c}{m_a m_b m_c} \Rightarrow \frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r} \text{ (Well known)}$$

$$m_a \geq \sqrt{r_b r_c}$$

1661. In  $\triangle ABC$  the following relationship holds:

$$3 \sqrt[3]{\frac{r_a}{h_a}} + 3 \sqrt[3]{\frac{r_b}{h_b}} + 3 \sqrt[3]{\frac{r_c}{h_c}} \leq \frac{3m_a m_b m_c}{h_a h_b h_c}$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

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*Solution by George Florin Şerban-Romania*

We show that:  $\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{R}{2r}$

$$\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{\sqrt{s(s-a)} \cdot \sqrt{s(s-b)} \cdot \sqrt{s(s-c)}}{\frac{2s^2 r^2}{R}} = \frac{s \cdot rs \cdot R}{2s^2 r^2} = \frac{R}{2r}$$

Let be the function:  $f: (0, \infty) \rightarrow \mathbb{R}, f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}, f'(x) = \frac{1}{3} x^{-\frac{2}{3}},$

$f''(x) = -\frac{2}{9} x^{-\frac{5}{3}} < 0 \Rightarrow f$  –convave, by Jensen  $f\left(\frac{x+y+z}{3}\right) \geq \frac{f(x)+f(y)+f(z)}{3}$  we get:

$$\frac{1}{3} \sum_{cyc} \sqrt[3]{\frac{r_a}{h_a}} \leq \sqrt[3]{\frac{1}{3} \sum_{cyc} \frac{r_a}{h_a}} \stackrel{?}{\leq} \frac{R}{2r} \leq \frac{m_a m_b m_c}{h_a h_b h_c} \Leftrightarrow$$

$$\frac{1}{3} \sum_{cyc} \frac{r_a}{h_a} = \frac{1}{3} \sum_{cyc} \frac{S}{s-a} \cdot \frac{a}{2S} \leq \frac{R^3}{8r^3} \Leftrightarrow$$

$$\frac{1}{6} \sum_{cyc} \frac{a}{s-a} = \frac{1}{6} \cdot \frac{4R-2r}{r} \leq \frac{R^3}{8r^3} \Leftrightarrow \frac{2}{3} \cdot \frac{R}{r} - \frac{1}{3} \leq \frac{1}{8} \left(\frac{R}{r}\right)^3 \stackrel{t=\frac{R}{r} \geq 2}{\Leftrightarrow}$$

$$3t^3 \geq 16t - 8 \Leftrightarrow (t-2)(t^2 + 6t - 4) \geq 0 \text{ true by } t \geq 2 \text{ and}$$

$$t^2 + 6t - 4 \geq 2^2 + 6 \cdot 2 - 4 = 20 > 0. \text{ Proved.}$$

**1662. In  $\triangle ABC$  the following relationship holds:**

$$\frac{1}{\sin^2 \frac{A}{2} \sin^2 \frac{B}{2}} + \frac{1}{\sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} + \frac{1}{\sin^2 \frac{C}{2} \sin^2 \frac{A}{2}} \geq \frac{48(a^2 + b^2 + c^2)}{ab + bc + ca}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution 1 by Bogdan Fuştei-Romania*

We know that:  $AI = 4R \sin \frac{B}{2} \sin \frac{C}{2} = \frac{r}{\sin \frac{A}{2}}$  and analogs.

$$\sin \frac{A}{2} = \sqrt{\frac{r_a - r}{4R}} \text{ and analogs.}$$

$$\sqrt{\frac{R}{2r}} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

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$\sum_{cyc} \frac{1}{\sin^2 \frac{A}{2} \sin^2 \frac{B}{2}} = \sum_{cyc} \frac{16R^2}{A^2}$  and  $\frac{1}{A^2} = \frac{r_a - r}{4Rr^2}$  and analogs, then

$$\sum_{cyc} \frac{1}{\sin^2 \frac{A}{2} \sin^2 \frac{B}{2}} = \frac{16R^2(r_a - r + r_b - r + r_c - r)}{4Rr^2}$$

So, we have:  $\sum_{cyc} \frac{1}{\sin^2 \frac{A}{2} \sin^2 \frac{B}{2}} = \frac{4R}{r^2}(4R + r - 3r) = \frac{8R}{r^2}(2R - r)$

We must show that:  $\frac{8R}{r^2}(2R - r) \geq 48 \cdot \sqrt{\frac{R}{2r}}$

$$\sqrt{\frac{R}{r}} \cdot \sqrt{\frac{R}{r}} \cdot \frac{2R - r}{r} \geq 6 \sqrt{\frac{R}{2r}} \Rightarrow \sqrt{\frac{R}{r}} \cdot \frac{2R - r}{r} \geq 3\sqrt{2}$$

$$\frac{R}{r} \geq 2 \text{ (Euler)} \Rightarrow \frac{2R - r}{r} \geq 3 \Rightarrow 2R - r \geq 3r \Rightarrow 2R \geq 4r \Rightarrow$$

$R \geq 2r$  (Euler). Proved.

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \frac{1}{\sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} = \frac{8R^2}{r^2} \sum (1 - \cos A) = \frac{8R(2R - r)}{r^2} \geq \frac{48 \sum a^2}{\sum ab} = \frac{96(s^2 - 4Rr - r^2)}{s^2 + 4Rr + r^2}$$

$$\Leftrightarrow R(2R - r)(s^2 + 4Rr + r^2) \geq 12r^2(s^2 - 4Rr - r^2)$$

$$\Leftrightarrow (2R^2 - Rr - 6r^2)s^2 + \{R(2R - r) + 12r^2\}(4Rr + r^2) \stackrel{(1)}{\geq} 6r^2s^2$$

Now, LHS of (1)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{(2R^2 - Rr - 6r^2)(16Rr - 5r^2)}_{(a)}$

$$+ \{R(2R - r) + 12r^2\}(4Rr + r^2)$$

and RHS of (1)  $\stackrel{\text{Gerretsen}}{\geq} \underbrace{6r^2(4R^2 + 4Rr + 3r^2)}_{(b)} \therefore (a), (b)$

$\Rightarrow$  in order to prove(1), it suffices to prove:

$$(2R^2 - Rr - 6r^2)(16Rr - 5r^2) + \{R(2R - r) + 12r^2\}(4Rr + r^2) \geq 6r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 10t^3 - 13t^2 - 17t + 6 \geq 0 \left( \text{where } t = \frac{R}{r} \right) \Leftrightarrow (t - 2)(10t^2 + 7t - 3) \geq 0 \rightarrow \text{true}$$

$\stackrel{\text{Euler}}{\therefore} t \geq 2 \Rightarrow (1) \text{ is true}$

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$$\therefore \sum \frac{1}{\sin^2 \frac{B}{2} \sin^2 \frac{C}{2}} \geq \frac{48 \sum a^2}{\sum ab} \quad (\text{Proved})$$

**1663. In any  $\Delta ABC$  holds:**

$$9r^3 \leq \frac{1}{16} \left( \frac{a^4}{r_a} + \frac{b^4}{r_b} + \frac{c^4}{r_c} \right) \leq R^3 + r^3$$

*Proposed by Marin Chirciu – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\sum \frac{a^4}{r_a} = \sum \frac{a^4(s-a)}{rs} \leq 16(R^3 + r^3) \Leftrightarrow \sum a^4(s-a) \leq 16rs(R^3 + r^3)$$

$$\Leftrightarrow 2s \sum a^4 - 2 \sum a^5 \stackrel{(1)}{\leq} 32rs(R^3 + r^3)$$

$$\text{Now, } (\sum a)(\sum a^4) = \sum a^5 + \sum ab(a^3 + b^3) \Rightarrow 2s \sum a^4 - \sum a^5$$

$$= \sum \{ab(\sum a^3 - c^3)\} \stackrel{(i)}{\cong} (\sum a^3)(\sum ab) - abc \sum a^2$$

$$\text{Now, } (\sum a^2)(\sum a^3) = \sum a^5 + \sum \{a^2b^2(2s-c)\} = 2s \sum a^2b^2 + \sum a^5 - abc \sum ab$$

$$\Rightarrow - \sum a^5 \stackrel{(ii)}{\cong} 2s \left( (\sum ab)^2 - 2abc(\sum a) \right) - abc \sum ab - (\sum a^2)(\sum a^3)$$

$$(i) + (ii) \Rightarrow 2s \sum a^4 - 2 \sum a^5$$

$$= (\sum a^3)(\sum ab) - abc \sum a^2 + 2s \left( (\sum ab)^2 - 2abc(\sum a) \right)$$

$$- abc \sum ab - (\sum a^2)(\sum a^3)$$

$$= 2s(s^2 + 4Rr + r^2)(s^2 - 6Rr - 3r^2) - 8Rrs(s^2 - 4Rr - r^2)$$

$$+ 2s((s^2 + 4Rr + r^2)^2 - 16Rrs^2) - 4Rrs(s^2 + 4Rr + r^2)$$

$$- 4s(s^2 - 4Rr - r^2)(s^2 - 6Rr - 3r^2) \leq 32rs(R^3 + r^3)$$

$$\Leftrightarrow 4R^3 + 12R^2r + 11Rr^2 + 6r^3 \stackrel{(2)}{\leq} (R + 2r)s^2$$

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Now, RHS of (2)  $\stackrel{\text{Gerretsen}}{\geq} (R + 2r)(4R^2 + 4Rr + 3r^2) = 4R^3 + 12R^2r + 11Rr^2 + 6r^3$   
 $\Rightarrow (2) \Rightarrow (1)$  is true

$$\therefore \frac{1}{16} \left( \frac{a^4}{r_a} + \frac{b^4}{r_b} + \frac{c^4}{r_c} \right) \stackrel{(m)}{\geq} R^3 + r^3$$

$$\text{Again, } \sum \frac{a^4}{r_a} \stackrel{\text{Bergstrom}}{\geq} \frac{4(s^2 - 4Rr - r^2)^2}{4R + r} \stackrel{?}{\geq} 144r^3$$

$$\Leftrightarrow s^4 - s^2(8Rr + 2r^2) + 16R^2r^2 - 136Rr^3 - 35r^4 \stackrel{?}{\geq} 0 \quad (3)$$

Now, LHS of (3)  $\stackrel{\text{Gerretsen}}{\geq} s^2(16Rr - 5r^2) - s^2(8Rr + 2r^2) + 16R^2r^2 - 136Rr^3 - 35r^4$

$$\stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(8Rr - 7r^2) + 16R^2r^2 - 136Rr^3 - 35r^4$$

$$= 144Rr^2(R - 2r) \stackrel{\text{Euler}}{\geq} 0 \Rightarrow (3) \text{ is true } \therefore 9r^3 \stackrel{(n)}{\geq} \frac{1}{16} \left( \frac{a^4}{r_a} + \frac{b^4}{r_b} + \frac{c^4}{r_c} \right)$$

$$(m), (n) \Rightarrow 9r^3 \leq \frac{1}{16} \left( \frac{a^4}{r_a} + \frac{b^4}{r_b} + \frac{c^4}{r_c} \right) \leq R^3 + r^3 \text{ (Proved)}$$

**1664. In any  $\Delta ABC$ ,**

$$\min \left( \left( \sum n_a g_a \right) \left( \sum h_a h_b \right)^{-1}, \left( \sum m_a w_a \right) \left( \sum h_a h_b \right)^{-1} \right) \geq \frac{R}{2r}$$

*Proposed by Bogdan Fuștei – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\begin{aligned} & \text{Stewart's theorem} \Rightarrow b^2(s - c) + c^2(s - b) \\ & = an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ & = ag_a^2 + a(s - b)(s - c) \\ & \therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s - a)^2 \\ & \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s - b)(s - c)\} \\ & \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2 \end{aligned}$$

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Let  $s - a = x, s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$  and  $c = x + y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$\Rightarrow$  (a) is true  $\Rightarrow n_a g_a \geq s(s-a)$  and analogs

$$\Rightarrow \sum n_a g_a \geq s \sum (s-a) = s^2 \Rightarrow \left(\sum n_a g_a\right) \left(\sum h_a h_b\right)^{-1} \geq s^2 \left\{\sum \left(\frac{bc}{2R}\right) \left(\frac{ca}{2R}\right)\right\}^{-1}$$

$$= s^2 \left\{\left(\frac{4Rrs}{4R^2}\right)^{-1}\right\} \left\{\left(\sum a\right)^{-1}\right\} = \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r}$$

$$\Rightarrow \left(\sum n_a g_a\right) \left(\sum h_a h_b\right)^{-1} \stackrel{(m)}{\geq} \frac{R}{2r}$$

Again,  $\sum m_a w_a \stackrel{\text{Ioscu}}{\geq} \sum \left\{\left(\frac{b+c}{2}\right) \cos \frac{A}{2} \left(\frac{2bc}{b+c}\right) \cos \frac{A}{2}\right\} = \sum \left[bc \left\{\frac{s(s-a)}{bc}\right\}\right]$

$$= s \sum (s-a) = s^2$$

$$\Rightarrow \left(\sum m_a w_a\right) \left(\sum h_a h_b\right)^{-1} \geq s^2 \left\{\sum \left(\frac{bc}{2R}\right) \left(\frac{ca}{2R}\right)\right\}^{-1} = s^2 \left\{\left(\frac{4Rrs}{4R^2}\right)^{-1}\right\} \left\{\left(\sum a\right)^{-1}\right\}$$

$$= \frac{Rs^2}{2s \cdot rs} = \frac{R}{2r} \Rightarrow \left(\sum m_a w_a\right) \left(\sum h_a h_b\right)^{-1} \stackrel{(n)}{\geq} \frac{R}{2r}$$

$$(m), (n) \Rightarrow \min \left( \left(\sum n_a g_a\right) \left(\sum h_a h_b\right)^{-1}, \left(\sum m_a w_a\right) \left(\sum h_a h_b\right)^{-1} \right) \geq \frac{R}{2r} \text{ (Proved)}$$

**1665. In  $\triangle ABC$  the following relationship holds:**

$$1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right)$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Bogadan Fuștei-Romania**

We know that:  $\frac{1}{2}Rs^2 \geq m_a m_b m_c$

$$r_a r_b r_c = Ss = s^2 r$$

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$$\sin^2 \frac{A}{2} = \frac{r_a - r}{4R} = \frac{r}{2R} \cdot \frac{r_a}{h_a} \text{ and analogs.}$$

$$abc = 4RS = 4Rrs$$

$$2S = ah_a = bh_b = ch_c$$

$$r_a + r_b + r_c = 4R + r. \text{ We have:}$$

$$\frac{1}{2}RS^2 \cdot \frac{1}{r_ar_br_c} \geq \frac{m_am_bm_c}{r_ar_br_c} \Leftrightarrow \frac{1}{2}RS^2 \cdot \frac{1}{rs^2} \geq \frac{m_am_bm_c}{r_ar_br_c} \Leftrightarrow \frac{R}{2r} \geq \frac{m_am_bm_c}{r_ar_br_c}, \quad (1)$$

$$\sum_{cyc} \sin^2 \frac{A}{2} = \frac{4R + r - 3r}{4R} = \frac{4R - 2r}{4R} = \frac{2R - r}{2R}$$

$$\text{But: } \frac{2R-r}{2R} = \frac{r}{2R} \sum_{cyc} \frac{r_a}{h_a} \Rightarrow \sum_{cyc} \frac{r_a}{h_a} = \frac{2R}{r} - 1 \Rightarrow 1 + \sum_{cyc} \frac{r_a}{h_a} = \frac{2R}{r}, \quad (2)$$

$$4S \cdot 2S^2 = h_a h_b h_c \cdot abc = h_a h_b h_c \cdot 4RS \Rightarrow h_a h_b h_c = \frac{2S^2}{R}$$

$$\text{But: } S = sr \Rightarrow S^2 = s^2 r^2 = Ssr = r r_a r_b r_c$$

$$\text{So, } h_a h_b h_c = \frac{2S^2}{R} = \frac{2r}{R} \cdot r_a r_b r_c \Rightarrow \frac{R}{2r} = \frac{r_a r_b r_c}{h_a h_b h_c}$$

$$\text{But: } w_a \leq \sqrt{s(s-a)} = \sqrt{r_b r_c} \text{ and analogs, then } \frac{R}{2r} \geq \frac{w_a w_b w_c}{h_a h_b h_c}, \quad (3)$$

From (1), (2), (3) we get:

$$1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right)$$

### Solution 2 by Tran Hong-Dong Thap-Vietnam

We known Inequality:  $\frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \leq \frac{R}{r}$  (See [www.cut-the-knot.com/triangle/all\\_inclusiveInequality.shtml](http://www.cut-the-knot.com/triangle/all_inclusiveInequality.shtml))

$$\text{So, RHS} = 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right) \stackrel{(*)}{\leq} \frac{2R}{r}$$

$$\begin{aligned} \text{LHS} &= 1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = 1 + \frac{S}{\frac{2S}{a}} + \frac{S}{\frac{2S}{b}} + \frac{S}{\frac{2S}{c}} = \\ &= 1 + \frac{1}{2} \left( \frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right) \\ &= 1 + \frac{1}{2} \cdot \frac{a(s-b)(s-c) + b(s-c)(s-a) + c(s-a)(s-b)}{(s-a)(s-b)(s-c)} \end{aligned}$$

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$$\begin{aligned}
 &= 1 + \frac{1}{2} \cdot \frac{(a+b+c)s^2 - 2(ab+bc+ca)s + 3abc}{sr^2} \\
 &= 1 + \frac{1}{2} \cdot \frac{2s \cdot s^2 - 2(s^2 + 4Rr + r^2)s + 12Rrs}{sr^2} \\
 &= 1 + \frac{1}{2} \cdot \frac{2s(s^2 - s^2 - 4Rr - r^2 + 6Rr)}{sr^2} \\
 &= 1 + \frac{12Rr - r^2}{r} = 1 + \frac{2R}{r} - 1 = \frac{2R}{r} \stackrel{\text{by(*)}}{\geq} RHS.
 \end{aligned}$$

1666. In  $\triangle ABC$  the following relationship holds:

$$\frac{r_a r_b r_c}{w_a w_b w_c} + \frac{2\mu r}{R} \geq 1 + \mu, \quad \mu \leq \frac{1}{8}$$

*Proposed by Marin Chirciu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\begin{aligned}
 w_a &= \frac{2\sqrt{bc}}{b+c} \cdot \sqrt{r_b r_c}, w_b = \frac{2\sqrt{ac}}{a+c} \cdot \sqrt{r_a r_c}, w_c = \frac{2\sqrt{ab}}{a+b} \cdot \sqrt{r_a r_b} \Rightarrow \\
 w_a w_b w_c &= \frac{8abc}{(a+b)(b+c)(c+a)} \cdot r_a r_b r_c \Rightarrow \\
 \frac{r_a r_b r_c}{w_a w_b w_c} &= \frac{(a+b)(b+c)(c+a)}{8abc} = \frac{2s(s^2 + 4Rr + r^2) - 4Rrs}{8 \cdot 4Rrs} \\
 &= \frac{2s(s^2 + 4Rr + r^2 - 2Rrs)}{8 \cdot 4Rrs} = \frac{s^2 + 2Rr + r^2}{16Rr}
 \end{aligned}$$

So,

$$\frac{r_a r_b r_c}{w_a w_b w_c} + \frac{2\mu r}{R} \geq 1 + \mu \Leftrightarrow \frac{s^2 + 2Rr + r^2}{16Rr} + \frac{2\mu r}{R} \stackrel{(*)}{\geq} 1 + \mu$$

$$\text{But: } s^2 \geq 16Rr - 5r^2 \text{ (Gerretsen)} \Rightarrow s^2 + 2Rr + r^2 \geq 18Rr - 4r^2 \Rightarrow$$

$$\frac{s^2 + 2Rr + r^2}{16Rr} + \frac{2\mu r}{R} \geq \frac{9}{8} - \frac{r}{4R} + \frac{2\mu r}{R} \stackrel{(**)}{\geq} 1 + \mu$$

$$(**) \Leftrightarrow \frac{9}{8} + \frac{(1-8\mu)r}{4R} \geq 1 + \mu \Leftrightarrow \frac{1-8\mu}{8} \geq \frac{(1-8\mu)r}{4R} \text{ (true)}$$

$$\text{Because: } \mu \leq \frac{1}{8} \Rightarrow 1 - 8\mu \geq 0, R \geq 2r \Rightarrow \frac{r}{4R} \leq \frac{1}{8} \text{ Proved.}$$

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1667. In  $\triangle ABC$  the following relationship holds:

$$\frac{s^2 - r_a r_b}{s^2 + r_a r_b} + \frac{s^2 - r_b r_c}{s^2 + r_b r_c} + \frac{s^2 - r_c r_a}{s^2 + r_c r_a} \leq 3 \left(1 - \frac{r}{R}\right)$$

Proposed by Marin Chirciu-Romania

**Solution 1 by George Florin Şerban-Romania**

$$\begin{aligned} \sum_{cyc} \frac{s^2 - r_a r_b}{s^2 + r_a r_b} &= \sum_{cyc} \frac{s^2 - \frac{S^2}{(s-b)(s-c)}}{s^2 + \frac{S^2}{(s-b)(s-c)}} = \sum_{cyc} \frac{s^2 - s(s-a)}{s^2 + s(s-a)} = \\ &= \sum_{cyc} \frac{s^2 - s^2 + as}{s(2s-a)} = \sum_{cyc} \frac{as}{s(2s-a)} = \sum_{cyc} \frac{a}{b+c} = \\ &= \frac{2s^2 - 2r^2 - 2Rr}{s^2 + r^2 + 2Rr} \leq 3 \left(1 - \frac{r}{R}\right), R \geq 2r > r \Rightarrow \end{aligned}$$

$$\begin{aligned} 2Rs^2 - 2Rr^2 - 2R^2r &\leq (3R - 3r)s^2 + 3Rr^2 - 3r^3 + 6R^2r - 6Rr^2 \\ (3r - R)s^2 &\leq -Rr^2 - 3r^3 + 8R^2r \end{aligned}$$

If  $3r - R < 0 \Rightarrow (2r - R)s^2 < 0$  we show that  $-Rr^2 - 3r^3 + 8R^2r > 0$

$$8R^2 - Rr - 3r^2 > 0 \xrightarrow{t = \frac{R}{r} \geq 2} 8t^2 - t - 3 > 0$$

$$8t^2 - t - 3 = t(8t - 1) - 3 \geq 2(8 \cdot 2 - 1) - 3 = 27 > 0. \text{ True.}$$

$$\text{If } 3r - R > 0 \Rightarrow 3r > R \Rightarrow t = \frac{R}{r} < 3 \Rightarrow t \in [2, 3)$$

$$(3r - R)s^2 \stackrel{(G)}{\leq} (3r - R)(4R^2 + 4Rr + 3r^2) \leq -Rr^2 - 3r^3 + 8R^2r$$

$$(3 - t)(4t^2 + 4t + 3) \leq -t - 3 + 8t^2$$

$$2t^3 - 5t - 6 \geq 0 \Leftrightarrow (t - 2)(2t^2 + 4t + 3) \geq 0 \text{ true from}$$

$$t \geq 2 \text{ and } 2t^2 + 4t + 3 > 0, \forall t \geq 2.$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\frac{s^2 - r_a r_b}{s^2 + r_a r_b} = \frac{s^2 - \frac{S^2}{(s-a)(s-b)}}{s^2 + \frac{S^2}{(s-a)(s-b)}} = \frac{s^2 - s(s-c)}{s^2 + s(s-c)} = \frac{c}{2s-c} = \frac{c}{a+b}$$

Similarly:

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$$\frac{s^2 - r_b r_c}{s^2 + r_b r_c} = \frac{b}{a+c}; \frac{s^2 - r_c r_a}{s^2 + r_c r_a} = \frac{c}{a+b}$$

$$1 - \frac{r}{R} = 1 - \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc}$$

We need to prove that:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \leq 3 \left[ 1 - \frac{(a+b-c)(b+c-a)(c+a-b)}{2abc} \right]$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \leq \frac{3}{2} \left[ \frac{abc - (a+b-c)(b+c-a)(c+a-b)}{abc} \right]$$

$$\frac{1}{2} \sum_{cyc} \left[ \frac{(b-c)^2}{(a+b)(a+c)} \right] \leq \frac{3}{2} \sum_{cyc} \left[ \frac{b+c-a}{2abc} \cdot (b-c)^2 \right]$$

$$\sum_{cyc} \left[ \left( \frac{3(b+c-a)}{2abc} - \frac{1}{(a+b)(b+c)} \right) (b-c)^2 \right] \geq 0$$

$$S_a = \frac{3(b+c-a)}{2abc} - \frac{1}{(a+b)(b+c)} = \frac{3(b+c-a)(a+b)(b+c) - 2abc}{2abc}$$

We need to prove:  $S_a > 0 \Leftrightarrow 3(b+c-a)(a+b)(a+c) - 2abc \stackrel{(*)}{>} 0$

Let:  $a = y+z, b = z+x, c = x+y, (x, y, z > 0)$

Suppose:  $x = \max\{x, y, z\} \Rightarrow x \geq y; x \geq z$

$$(*) \Leftrightarrow 6x(x+y+2z)(x+z+2y) - 2(x+y)(y+z)(z+x) > 0$$

$$2[3x^3 + 8x^2(y+z) + 5x(y^2+z^2) + 13xyz - yz(y+z)] > 0$$

$$2[3x^3 + 8x^2(y+z) + 5x(y+z)^2 + 3xyz - yz(y+z)] > 0$$

$$2[3x^3 + (8x^2 + 5(xy+xz) - yz) + 3xyz] > 0$$

Which is clearly true because:

$$x^2 + 5(xy+xz) - yz \geq 8yz + 5(zx+yz) - yz = 17yz > 0 \Rightarrow (*) \text{ is true. Proved.}$$

**1668. In  $\triangle ABC$  the following relationship holds:**

$$\frac{r_a - h_a}{r_a + h_a} + \frac{r_b - h_b}{r_b + h_b} + \frac{r_c - h_c}{r_c + h_c} + \frac{2r}{R} \leq 1$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

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**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$\frac{r_a - h_a}{r_a + h_a} + \frac{r_b - h_b}{r_b + h_b} + \frac{r_c - h_c}{r_c + h_c} = \sum_{cyc} \frac{\frac{r_a}{h_a} - 1}{\frac{r_a}{h_a} + 1} = \sum_{cyc} \frac{\frac{a}{2(s-a)} - 1}{\frac{a}{2(s-a)} + 1} = \sum_{cyc} \frac{\frac{a}{b+c-a} - 1}{\frac{a}{b+c-a} + 1}$$

$$\frac{2r}{R} = \frac{(a+b-c)(b+c-a)(a+c-b)}{abc}$$

$$\text{Let: } x = \frac{b+c-a}{a}; y = \frac{a+c-b}{b}; z = \frac{a+b-c}{c}; 0 < xyz \leq 1 \Rightarrow$$

$$\begin{aligned} \frac{1}{x+2} + \frac{1}{y+2} + \frac{1}{z+2} &= \frac{1}{\frac{b+c-a}{a}+2} + \frac{1}{\frac{a+c-b}{b}+2} + \frac{1}{\frac{a+b-c}{c}+2} = \\ &= \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{a+b+c} = 1 \Leftrightarrow xyz + xy + yz + zx = 4 \end{aligned}$$

$$\text{More, } (xy + yz + zx)^2 \geq 3xyz(x + y + z)$$

$$x + y + z \leq \frac{(xy + yz + zx)^2}{3xyz} = \frac{(4 - xyz)^2}{3xyz}$$

Inequality becomes as:

$$\frac{1-x}{1+x} + \frac{1-y}{1+y} + \frac{1-z}{1+z} + 3xyz \leq 1$$

$$(xyz)^2 + xyz(xy + yz + zx + x + y + z - 3) - 2(xy + yz + zx) + 2 \leq 0$$

$$(xyz)^2 + xyz(4 - xyz + x + y + z - 3) - 2(4 - xyz) + 2 \leq 0$$

$$(xyz)^2 + xyz(1 - xyz + x + y + z) + 2xyz - 6 \leq 0$$

$$xyz(x + y + z) + 3xyz - 6 \leq 0$$

Let:  $t = xyz$  ( $0 < t \leq 1$ ). We need to prove:

$$t \cdot \frac{(4-t)^2}{3t} + 3t - 6 \leq 0 \Leftrightarrow \frac{1}{3}(4-t)^2 + 3t - 6 \leq 0 \Leftrightarrow$$

$$(4-t)^2 + 9t - 18 \leq 0 \Leftrightarrow t^2 + t - 2 \leq 0 \Leftrightarrow (t-1)(t+2) \leq 0 \text{ true,}$$

because  $0 < t \leq 1 \rightarrow t-1 \leq 0; t+2 \geq 2 > 0$ . Proved.

**Solution 2 by Avishek Mitra-West Bengal-India**

$$\Omega = \sum_{cyc} \frac{r_a - h_a}{r_a + h_a} = \sum_{cyc} \frac{\frac{S}{s-a} - \frac{2S}{a}}{\frac{S}{s-a} + \frac{2S}{a}} = \sum_{cyc} \frac{3a - 2a}{a + 2s - 2a} = \sum_{cyc} \frac{2a - (b+c)}{b+c} =$$

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$$= \sum_{cyc} \frac{a}{b+c} - 3$$

$$\sum_{cyc} \frac{a}{b+c} = \sum_{cyc} \frac{a+b+c}{b+c} - 3 = 2s \left( \sum_{cyc} \frac{1}{b+c} \right) - 3 = 2s \cdot \frac{\sum(b+c)(c+a)}{\prod(a+b)}$$

$$= 2s \cdot \frac{\sum(ab+bc+ca+c^2)}{\sum ab(a+b) + 2abc} - 3 = 2s \cdot \frac{\sum c^2 + 3\sum ab}{abc + \sum ab(2s-c)} - 3 =$$

$$= 2s \cdot \frac{2(s^2 - 4Rr - r^2) + 3(s^2 + r^2 + 4Rr)}{2s(\sum ab) - abc} - 3 =$$

$$= 2s \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 4Rr) - 4Rrs} - 3 = 2s \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 2Rr)} - 3 =$$

$$= \frac{5s^2 + r^2 + 4Rr}{s^2 + r^2 + 2Rr} - 3 = \frac{2s^2 - 2Rr - 2r^2}{s^2 + r^2 + 2Rr}$$

$$\Omega = \frac{4s^2 - 4Rr - 4r^2}{s^2 + r^2 + 2Rr} - 3 = \frac{s^2 - 10Rr - 7r^2}{s^2 + r^2 + 2Rr}$$

$$\text{Need to show, } \Omega \leq 1 - \frac{2r}{R} = \frac{R-2r}{R}$$

$$\frac{s^2 - 10Rr - 7r^2}{s^2 + r^2 + 2Rr} \leq \frac{R-2r}{R} \Leftrightarrow$$

$$R(s^2 - 10Rr - 7r^2) \leq (R-2r)(s^2 + r^2 + 2Rr) \Leftrightarrow$$

$2s^2 \leq 2R^2 + 2Rr - r^2$  but from Gerretsen  $s^2 \leq 4R^2 + 4Rr + 3r^2$  need to show:

$$4R^2 + 4Rr + 3r^2 \leq 2R^2 + 2Rr - r^2 \Leftrightarrow 2R^2 - 2Rr - 4r^2 \geq 0 \Leftrightarrow$$

$$2(R-2r)(R+r) \geq 0 \text{ true from } R \geq 2r \text{ (Euler). Proved.}$$

**1669. In right  $\triangle ABC$  the following relationship holds:**

$$\frac{m_a^2}{s_a^2} + \frac{m_b^2}{s_b^2} + \frac{m_c^2}{s_c^2} \geq \frac{13}{4}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by George Florin Şerban-Romania**

$$\sum_{cyc} \frac{m_a^2}{s_a^2} = \sum_{cyc} \left( \frac{b^2 + c^2}{2bc} \right)^2 \geq \frac{13}{4}$$

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$$\sum_{cyc} \left( \frac{b^2 + c^2}{bc} \right)^2 \geq 13$$

$$\begin{aligned} \sum_{cyc} \left( \frac{b}{c} + \frac{c}{b} \right)^2 &= (\tan B + \cot B)^2 + \left( \cos B + \frac{1}{\cos B} \right)^2 + \left( \sin B + \frac{1}{\sin B} \right)^2 = \\ &= \left( \frac{\sin B}{\cos B} + \frac{\cos B}{\sin B} \right)^2 + \left( \sin B + \frac{1}{\sin B} \right)^2 + \left( \cos B + \frac{1}{\cos B} \right)^2 = \\ &= \left( \frac{\sin^2 B + \cos^2 B}{\sin B \cos B} \right)^2 + \left( \sin B + \frac{1}{\sin B} \right)^2 + \left( \cos B + \frac{1}{\cos B} \right)^2 = \\ &= \left( \frac{1}{\sin B \cos B} \right)^2 + \left( \sin B + \frac{1}{\sin B} \right)^2 + \left( \cos B + \frac{1}{\cos B} \right)^2 \end{aligned}$$

Let:  $x = \sin B, y = \cos B, x^2 + y^2 = 1, x, y > 0$

$$\begin{aligned} \sum_{cyc} \left( \frac{b}{c} + \frac{c}{b} \right)^2 &= \frac{1}{x^2 y^2} + \left( x + \frac{1}{x} \right)^2 + \left( y + \frac{1}{y} \right)^2 = \\ &= \frac{x^2 + y^2}{x^2 y^2} + \left( x + \frac{1}{x} \right)^2 + \left( y + \frac{1}{y} \right)^2 = \frac{x^2 + y^2}{x^2 y^2} + x^2 + 2 + \frac{1}{x^2} + y^2 + 2 + \frac{1}{y^2} \geq 13 \rightarrow 2 \left( \frac{1}{x^2} + \frac{1}{y^2} \right) \geq \end{aligned}$$

$$8 \rightarrow \frac{1}{x^2} + \frac{1}{y^2} \geq 4 \text{ true from}$$

$$\frac{1}{x^2} + \frac{1}{y^2} \stackrel{\text{Bergstrom}}{\geq} \frac{(1+1)^2}{x^2 + y^2} = 4.$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

WLOG, we may assume  $A = 90^\circ \therefore a^2 = b^2 + c^2$

$$\therefore s_a = \frac{2bcm_a}{b^2 + c^2} \text{ and analogs}$$

$$\begin{aligned} \therefore \frac{m_a^2}{s_a^2} + \frac{m_b^2}{s_b^2} + \frac{m_c^2}{s_c^2} &= \frac{1}{4} \left\{ \left( \frac{b^2 + c^2}{bc} \right)^2 + \left( \frac{c^2 + a^2}{ca} \right)^2 + \left( \frac{a^2 + b^2}{ab} \right)^2 \right\} \\ &= \frac{1}{4} \left\{ \left( \frac{b^2 + c^2}{bc} \right)^2 + \frac{(c^2 + b^2 + c^2)^2}{c^2(b^2 + c^2)} + \frac{(b^2 + c^2 + b^2)^2}{b^2(b^2 + c^2)} \right\} \\ &= \frac{1}{4} \left[ \left( \frac{b^2 + c^2}{bc} \right)^2 + \frac{1}{b^2 + c^2} \left\{ \frac{(b^2 + 2c^2)^2}{c^2} + \frac{(c^2 + 2b^2)^2}{b^2} \right\} \right] \end{aligned}$$

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$$\stackrel{\text{Bergstrom}}{\geq} \frac{1}{4} \left\{ \left( \frac{b^2 + c^2}{bc} \right)^2 + \frac{(b^2 + 2c^2 + c^2 + 2b^2)^2}{(b^2 + c^2)^2} \right\} = \frac{1}{4} \left\{ \left( \frac{b^2 + c^2}{bc} \right)^2 + 9 \right\} \stackrel{\text{A-G}}{\geq} \frac{1}{4} (4 + 9) = \frac{13}{4}$$

1670. In  $\triangle ABC$  the following relationship holds:

$$\mu + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq \frac{(\mu + 3)r_a r_b r_c}{w_a w_b w_c}, \mu \leq 5$$

Proposed by Marin Chirciu-Romania

Solution by Rahim Shahbazov-Baku-Azerbaijan

$$5 + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq \frac{8r_a r_b r_c}{w_a w_b w_c}; \quad (1)$$

$$(1) \rightarrow 5 + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq \frac{(a+b)(b+c)(c+a)}{abc} \rightarrow$$

$$6 + \frac{r_a^2}{r_b^2} + \frac{r_b^2}{r_c^2} + \frac{r_c^2}{r_a^2} \geq (a+b+c) \left( \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right); \quad (2)$$

Let:  $a = x + y, b = y + z, c = z + x \rightarrow$

$$6 + \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq 2(x+y+z) \left( \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \right) \rightarrow$$

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{2}{x+y} + \frac{2}{y+z} + \frac{2}{z+x}; \quad (3) \xrightarrow{a^2 \geq 2a-1}$$

$$\frac{x^2}{y^2} \geq 2 \frac{x}{y} - 1 \xrightarrow{(3)} \frac{x}{y} + \frac{y}{z} + \frac{z}{x} \geq \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{3}{2} \rightarrow$$

$$\frac{xz}{y(y+z)} + \frac{xy}{z(x+z)} + \frac{yz}{x(x+y)} \geq \frac{3}{2}$$

$$\text{Lhs} = \sum_{\text{cyc}} \frac{(xz)^2}{xyz(y+z)} \geq \frac{(xy + yz + zx)^2}{3xyz(x+y+z)} \geq \frac{3}{2}$$

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1671. In  $\triangle ABC$  the following relationship holds:

$$\frac{3(a^2 + b^2 + c^2)}{(a + b + c)^2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \leq \frac{R}{r}$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

**Solution 1 by Bogdan Fuștei-Romania**

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc}; \quad (1)$$

$$x, y, z > 0: \frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \left( \frac{x^2 + y^2 + z^2}{xy + yz + zx} \right)^2; \quad (2)$$

$$\frac{R}{2r} \stackrel{(1)}{\geq} \frac{a^3 + b^3 + c^3}{4abc} + \frac{1}{4} \stackrel{(2)}{\geq} \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2 \rightarrow$$

$$\sqrt{\frac{R}{2r}} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \stackrel{R \geq 2r}{\implies} \sqrt{\frac{R}{2r}} \geq 1 \rightarrow \frac{R}{2r} \geq \sqrt{\frac{R}{2r}} \rightarrow \frac{R}{2r} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca}$$

$$\text{We show that: } \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{3(a^2 + b^2 + c^2)}{(a + b + c)^2} \rightarrow$$

$$\frac{1}{ab + bc + ca} \geq \frac{3}{(a + b + c)^2} \rightarrow (a + b + c)^2 \geq 3(ab + bc + ca)$$

$$a^2 + b^2 + c^2 + 2(ab + bc + ca) \geq 3(ab + bc + ca) \rightarrow$$

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$$

$$\text{So, } \frac{R}{2r} \geq 2 \cdot \frac{a^2 + b^2 + c^2}{ab + bc + ca} \geq \frac{a^2 + b^2 + c^2}{ab + bc + ca} + \frac{3(a^2 + b^2 + c^2)}{(a + b + c)^2}$$

Equality holds if  $a = b = c$ .

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\text{LHS} = 2(s^2 - 4Rr - r^2) \left( \frac{3}{4s^2} + \frac{1}{s^2 + 4Rr + r^2} \right) = \frac{(s^2 - 4Rr - r^2)(7s^2 + 12Rr + 3r^2)}{2s^2(s^2 + 4Rr + r^2)}$$

$$\leq \frac{R}{r}$$

$$\Leftrightarrow s^4(2R - 7r) + s^2(8R^2r + 18Rr^2 + 4r^3) + 3r^3(4R + r)^2 \geq 0$$

$$\Leftrightarrow s^4(2R - 4r) + s^2(8R^2r + 18Rr^2 + 4r^3) + 3r^3(4R + r)^2 \stackrel{(1)}{\geq} 3rs^4$$

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Now, LHS of (1)  $\stackrel{\text{Gerretsen}}{\underset{(a)}{\geq}} s^2(16Rr - 5r^2)(2R - 4r) + s^2(8R^2r + 18Rr^2 + 4r^3)$

$$+ 3r^3(4R + r)^2$$

and RHS of (1)  $\stackrel{\text{Gerretsen}}{\underset{(b)}{\geq}} 3rs^2(4R^2 + 4Rr + 3r^2) \therefore (a), (b)$

$\Rightarrow$  in order to prove (1), it suffices to prove :

$$s^2(16Rr - 5r^2)(2R - 4r) + s^2(8R^2r + 18Rr^2 + 4r^3) + 3r^3(4R + r)^2 \geq 3rs^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(28R^2 - 68Rr + 15r^2) + 3r^2(4R + r)^2 \geq 0$$

$$\Leftrightarrow s^2(28R - 12r)(R - 2r) + 3r^2(4R + r)^2 \stackrel{(2)}{\underset{(c)}{\geq}} 9r^2s^2$$

Now, LHS of (2)  $\stackrel{\text{Gerretsen}}{\underset{(c)}{\geq}} (16Rr - 5r^2)(28R - 12r)(R - 2r)$

+  $3r^2(4R + r)^2$  and RHS of (2)  $\stackrel{\text{Gerretsen}}{\underset{(d)}{\leq}} 9r^2(4R^2 + 4Rr + 3r^2)$

$\therefore (c), (d) \Rightarrow$  in order to prove (2), it suffices to prove

$$: (16Rr - 5r^2)(28R - 12r)(R - 2r) + 3r^2(4R + r)^2$$

$$\geq 9r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 56t^3 - 152t^2 + 89t - 18 \geq 0 \left( \text{where } t = \frac{R}{r} \right) \Leftrightarrow (t - 2)\{(t - 2)(56t + 72) + 153\}$$

$$\geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\underset{(e)}{\geq}} 2 \Rightarrow (2) \Rightarrow (1) \text{ is true}$$

$$\therefore \frac{3(a^2 + b^2 + c^2)}{(a + b + c)^2} + \frac{a^2 + b^2 + c^2}{ab + bc + ca} \leq \frac{R}{r} \text{ (Proved)}$$

**1672. In acute  $\triangle ABC$  the following relationship holds:**

$$2 \left( \frac{h_a^2}{w_a^2} + \frac{h_b^2}{w_b^2} + \frac{h_c^2}{w_c^2} \right) \leq \frac{h_a}{m_a} + \frac{h_b}{m_b} + \frac{h_c}{m_c} + 3$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

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### Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$\text{Lemma 1: } \cos\left(\frac{B-C}{2}\right) = \frac{h_a}{m_a}$$

$$\text{Lemma 2: } \cos(B-C) \leq \frac{h_a}{m_a} \text{ (for acute triangle)}$$

$$(1) \rightarrow \left(\cos^2\frac{B-C}{2} + \cos^2\frac{C-A}{2} + \cos^2\frac{A-B}{2}\right) \leq \sum_{cyc} \frac{h_a}{m_a} + 3$$

$$\cos(A-B) + \cos(B-C) + \cos(C-A) \leq \frac{h_a}{m_a} + \frac{h_b}{m_b} + \frac{h_c}{m_c}$$

$$AA_1 = \frac{h_a}{\cos(B-C)}, AA_1 \geq m_a \rightarrow \cos(B-C) \leq \frac{h_a}{m_a}$$

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$h_a m_a \leq s(s-a) \Leftrightarrow h_a^2 m_a^2 \leq s^2(s-a)^2 \Leftrightarrow \left(\frac{4r^2 s^2}{a^2}\right) \frac{(2b^2 + 2c^2 - a^2)}{4} \leq s^2(s-a)^2$$

$$\Leftrightarrow \frac{16r^2 s^2}{a^2} (2b^2 + 2c^2 - a^2) \leq (a+b+c)^2 (b+c-a)^2$$

$$\Leftrightarrow \left(2 \sum a^2 b^2 - \sum a^4\right) (2b^2 + 2c^2 - a^2) \leq a^2 (a+b+c)^2 (b+c-a)^2$$

$$\Leftrightarrow b^6 + c^6 - b^4 c^2 - b^2 c^4 + a^4 b^2 + a^4 c^2 - 2a^4 bc - 2a^2 b^4 - 2a^2 c^4 + 2a^2 b^3 c + 2a^2 bc^3 \geq 0$$

$$\Leftrightarrow (b^2 + c^2)(b^2 - c^2)^2 + a^4(b-c)^2 - 2a^2(b^4 + c^4 - 2b^2 c^2 + 2b^2 c^2) + 2a^2 bc(b^2 + c^2) \geq 0$$

$$\Leftrightarrow (b^2 + c^2)(b^2 - c^2)^2 + a^4(b-c)^2 - 2a^2(b^2 - c^2)^2 + 2a^2 bc(b-c)^2 \geq 0$$

$$\Leftrightarrow (b^2 + c^2)(b^2 - c^2)^2 + a^4(b-c)^2 - 2a^2(b^2 - c^2)^2$$

$$+ a^2(b-c)^2\{(b+c)^2 - (b^2 + c^2)\} \geq 0$$

$$\Leftrightarrow (b^2 + c^2)(b^2 - c^2)^2 + a^4(b-c)^2 - a^2(b^2 - c^2)^2 - a^2(b^2 + c^2)(b-c)^2 \geq 0$$

$$\Leftrightarrow (b^2 - c^2)^2(b^2 + c^2 - a^2) - a^2(b-c)^2(b^2 + c^2 - a^2) \geq 0$$

$$\Leftrightarrow (b^2 + c^2 - a^2)(b-c)^2(a+b+c)(b+c-a) \geq 0 \rightarrow \text{true}$$

$\therefore b^2 + c^2 - a^2 > 0$  as  $\Delta ABC$  is acute-angled  $\therefore h_a m_a \stackrel{(1)}{\leq} s(s-a)$  and analogs

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$$\begin{aligned} \text{Now, } \sum \frac{h_a}{m_a} &= \sum \frac{h_a^2}{m_a h_a} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum h_a)^2}{\sum m_a h_a} \stackrel{\text{by (1) and analogs}}{\geq} \frac{(s^2 + 4Rr + r^2)^2}{4R^2 \sum s(s-a)} \\ &= \frac{(s^2 + 4Rr + r^2)^2}{4R^2 s^2} \Rightarrow \text{RHS} \stackrel{(i)}{\geq} \frac{12R^2 s^2 + (s^2 + 4Rr + r^2)^2}{4R^2 s^2} \end{aligned}$$

$$\begin{aligned} \text{Again, } 2 \sum \frac{h_a^2}{w_a^2} &= \sum \frac{8r^2 s^2 (b+c)^2 bc}{a^2 \cdot 4b^2 c^2 s(s-a)} = \frac{1}{8R^2} \sum \frac{bc(s+s-a)^2}{s(s-a)} \\ &= \frac{1}{8R^2} \sum \frac{bc\{s^2 + (s-a)^2 + 2s(s-a)\}}{s(s-a)} \\ &= \frac{1}{8R^2} \left\{ s^2 \sum \sec^2 \frac{A}{2} + \sum \frac{bc(s-a)}{s} + 2 \sum ab \right\} \\ &= \frac{1}{8R^2} \{s^2 + (4R+r)^2 + 3(s^2 + 4Rr + r^2) - 12Rr\} \\ &\Rightarrow \text{LHS} \stackrel{(ii)}{\geq} \frac{4s^2 + (4R+r)^2 + 3r^2}{8R^2} \end{aligned}$$

$$(i), (ii) \Rightarrow \text{it suffices to prove : } \frac{12R^2 s^2 + (s^2 + 4Rr + r^2)^2}{4R^2 s^2} \geq \frac{4s^2 + (4R+r)^2 + 3r^2}{8R^2}$$

$$\Leftrightarrow s^4 - s^2(4R^2 + 4Rr) - r^2(4R+r)^2 \stackrel{(a)}{\geq} 0$$

$$\begin{aligned} \text{Now, LHS of (a)} &\stackrel{\text{Gerretsen}}{\geq} s^2(4R^2 + 4Rr + 3r^2) - s^2(4R^2 + 4Rr) - r^2(4R+r)^2 \\ &= r^2\{3s^2 - (4R+r)^2\} \stackrel{\text{Trucht}}{\geq} 0 \end{aligned}$$

**1673. In  $\triangle ABC$  the following relationship holds:**

$$\frac{w_a}{\sqrt{s(s-a)}} + \frac{w_b}{\sqrt{s(s-b)}} + \frac{w_c}{\sqrt{s(s-c)}} \geq \frac{24abc}{(a+b)(b+c)(c+a)}$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution by Avishek Mitra-West Bengal-India*

$$\sum_{cyc} \frac{w_a}{\sqrt{s(s-a)}} = \sum_{cyc} \frac{2\sqrt{bc} \cdot s(s-a)}{b+c} = \sum_{cyc} \frac{2\sqrt{bc}}{b+c} \stackrel{\text{Am-Gm}}{\geq}$$

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$$\geq 3 \sqrt[3]{\frac{8abc}{(a+b)(b+c)(c+a)}}$$

Need to show:

$$3 \cdot \sqrt[3]{\frac{8abc}{(a+b)(b+c)(c+a)}} \geq \frac{24abc}{(a+b)(b+c)(c+a)}$$

$$27 \cdot \frac{8abc}{(a+b)(b+c)(c+a)} \geq \left( \frac{24abc}{(a+b)(b+c)(c+a)} \right)^3$$

$$(a+b)^2(b+c)^2(c+a)^2 \geq \frac{24^3}{27 \cdot 8} \cdot (abc)^2 \text{ which is true from}$$

$$a+b \geq 2\sqrt{ab}, b+c \geq 2\sqrt{bc}, c+a \geq 2\sqrt{ca} \rightarrow$$

$$(a+b)(b+c)(c+a) \geq 8abc \rightarrow$$

$$(a+b)^2(b+c)^2(c+a)^2 \geq 64(abc)^2$$

**1674. In any  $\triangle ABC$ ,  $n_a$  – Nagel’s cevian, the following relationship holds:**

$$\frac{9}{4} \leq \frac{r}{2R} + \frac{R}{r} \leq \frac{n_a^2}{bc} + \frac{n_b^2}{ca} + \frac{n_c^2}{ab} < 2 \left( \frac{m_a^2 + w_a^2}{bc} + \frac{m_b^2 + w_b^2}{ca} + \frac{m_c^2 + w_c^2}{ab} \right)$$

*Proposed by Nguyen Van Canh-Ben Tre-Vietnam*

*Solution by Bogdan Fuștei-Romania*

$$s^2 = n_a^2 + 2r_a h_a \text{ and analogs}$$

“About Nagel and Gergonne cevians-R.M.M.-article”

$$\frac{s^2}{bc} = \frac{n_a^2}{bc} + \frac{2r_a h_a}{bc}, bc = 2Rh_a \text{ and analogs.}$$

$$\frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r} \rightarrow \frac{s^2}{bc} = \frac{n_a^2}{bc} + \frac{2r_a h_a}{2Rh_a} = \frac{n_a^2}{bc} + \frac{r_a}{R}$$

So, we have:

$$\frac{s^2}{bc} + \frac{s^2}{ca} + \frac{s^2}{ab} = \sum \frac{n_a^2}{bc} + \frac{r_a + r_b + r_c}{R} \stackrel{r_a+r_b+r_c=4R+r}{=} \sum \frac{n_a^2}{bc} + \frac{4R+r}{R}$$

$$\frac{s^2}{2R} \left( \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} \right) = \sum \frac{n_a^2}{bc} + \frac{4R+r}{R} \rightarrow \sum \frac{n_a^2}{bc} = \frac{s^2 - 2r(4R+r)}{2Rr}$$

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$$\frac{9}{4} \leq \frac{r}{2R} + \frac{R}{r} = \frac{2R^2 + r^2}{2Rr} \rightarrow \frac{9}{2} \leq \frac{2R^2 + r^2}{Rr} \rightarrow 9Rr \leq 4R^2 + 2r^2$$

$$4R^2 + 2r^2 - 9Rr \geq 0 \xrightarrow[t = \frac{R}{r} \geq 2]{\text{Euler}} 4t^2 - 9t + 2 \geq 0 \leftrightarrow (t-2)(4t-1) \geq 0 \text{ true for } t = \frac{R}{r} \geq 2, 4t-1 > 0$$

$$\frac{9}{4} \leq \frac{r}{2R} + \frac{R}{r}; \quad (1)$$

$$\frac{r}{2R} + \frac{R}{r} = \frac{2R^2 + r^2}{2Rr} \leq \frac{s^2 - 2r(4R+r)}{2Rr} \rightarrow 2R^2 + r^2 \leq s^2 - 2r(4R+r)$$

$$2R^2 + r^2 + 8Rr + 2r^2 \leq s^2 \leftrightarrow 2R^2 + 8Rr + 3r^2 \leq s^2 \text{ (Walker Ineq.)}$$

$$\frac{r}{2R} + \frac{R}{r} \leq \sum \frac{n_a^2}{bc}; \quad (2)$$

$$n_a < m_a + w_a \rightarrow n_a^2 < (m_a + w_a)^2 \text{ and analogs.}$$

$$\sqrt{\frac{x^2 + y^2}{2}} \geq \frac{x+y}{2} \text{ (Qm - Am)} \rightarrow \frac{m_a^2 + w_a^2}{2} \geq \frac{(m_a + w_a)^2}{4} \rightarrow$$

$$2(m_a^2 + w_a^2) \geq (m_a + w_a)^2 \rightarrow n_a^2 \leq 2(m_a^2 + w_a^2) \text{ and analogs.}$$

$$\frac{n_a^2}{bc} < \frac{2(m_a^2 + w_a^2)}{bc}$$

$$\sum \frac{n_a^2}{bc} < 2 \sum \frac{m_a^2 + w_a^2}{bc}; \quad (3)$$

From (1), (2), (3) the inequality is proved.

**1675. In  $\triangle ABC$  the following relationship holds:**

$$7s \sum_{cyc} s_a^3 > (2\sqrt{2} + 1) \left( \sum_{cyc} s_a^2 \right) \left( \sum_{cyc} h_a^2 \right)$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\sum_{cyc} s_a^3 = \sum_{cyc} \frac{(s_a^2)^2}{s_a} \stackrel{\text{Bergstrom}}{\geq} \frac{(\sum s_a^2)^2}{\sum s_a} = \frac{(\sum s_a^2)(\sum s_a^2)}{\sum s_a} \stackrel{s_a \geq h_a}{\geq}$$

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$$\begin{aligned} & \text{www.ssmrmh.ro} \\ & \geq \frac{(\sum s_a^2)(\sum h_a^2)}{\sum s_a} \rightarrow \end{aligned}$$

$$7s \sum_{cyc} s_a^3 \geq \frac{7s}{\sum s_a} \cdot (\sum s_a^2) (\sum h_a^2) \stackrel{(*)}{\geq} (2\sqrt{2} + 1) \left( \sum_{cyc} s_a^2 \right) \left( \sum_{cyc} h_a^2 \right)$$

$$(*) \Leftrightarrow \frac{7s}{\sum s_a} > 2\sqrt{2} + 1 \Leftrightarrow 7s > (2\sqrt{2} + 1) \sum_{cyc} s_a$$

$$\sum_{cyc} s_a \leq \sum_{cyc} \sqrt{s(s-a)} \stackrel{BCS}{\leq} \sqrt{3[s(s-a) + s(s-b) + s(s-c)]} = s\sqrt{3}$$

So, we need to prove:

$$\begin{aligned} 7s > (2\sqrt{2} + 1) \cdot s\sqrt{3} & \Leftrightarrow 7 > (2\sqrt{2} + 1)\sqrt{3} \Leftrightarrow (7 - \sqrt{3})^2 > (2\sqrt{6})^2 \Leftrightarrow \\ & 2 > \sqrt{3} \Leftrightarrow 4 > 3. \text{ True.} \rightarrow (*) \text{ is true. Proved.} \end{aligned}$$

**1676. In acute  $\triangle ABC$  the following relationship holds:**

$$\sum_{cyc} (\sin A)^{2\tan^2 A} + \sum_{cyc} (\sin A)^{2\sec^2 A} < 3$$

*Proposed by Daniel Sitaru-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$\sec^2 A = \frac{1}{\cos^2 A} = 1 + \tan^2 A$$

$$\sec^2 B = \frac{1}{\cos^2 B} = 1 + \tan^2 B$$

$$\sec^2 C = \frac{1}{\cos^2 C} = 1 + \tan^2 C$$

$$\begin{aligned} & \sum_{cyc} (\sin A)^{2\tan^2 A} + \sum_{cyc} (\sin A)^{2\sec^2 A} = \\ & = \sum_{cyc} (\sin A)^{2\tan^2 A} + \sum_{cyc} (\sin A)^{2(1+\tan^2 A)} = \sum_{cyc} (1 + \sin^2 A)(\sin A)^{2\tan^2 A} \\ & = \sum_{cyc} (1 + \sin^2 A)(\sin^2 A)^{\tan^2 A} = \Omega \end{aligned}$$

$$\text{For } 0 < x < \frac{\pi}{2}, \text{ let: } f(x) = (1 + \sin^2 x)(\sin^2 x)^{\tan^2 x}$$

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$$\begin{aligned} f'(x) &= 2\tan x \cdot \sec^2 x \cdot (\sin^2 x)^{\tan^2 x} [2\cos^2 x + (1 + \sin^2 x)\log(\sin^2 x)] = \\ &= 2\tan x \cdot \sec^2 x \cdot (\sin^2 x)^{\tan^2 x} [2(1 - \sin^2 x) + (1 + \sin^2 x)\log(\sin^2 x)] = \\ &= 2\tan x \cdot \sec^2 x \cdot (\sin^2 x)^{\tan^2 x} \cdot \varphi(\sin^2 x) \end{aligned}$$

$$0 < x < \frac{\pi}{2} \rightarrow \tan x > 0, \sec^2 x > 0, \sin^2 x > 0, \tan^2 x > 0 \rightarrow$$

$$2\tan x \cdot \sec^2 x \cdot (\sin^2 x)^{\tan^2 x} > 0$$

$$\varphi(t) = 2(1 - t) + (1 + t)\log t; \left( \because t = \sin^2 x, t \in (0, 1) \right)$$

$$\varphi'(t) = -2 + \log t + \frac{t+1}{t}$$

$$\varphi''(t) = \frac{1}{t} - \frac{1}{t^2} = \frac{t-1}{t^2} < 0, t \in (0, 1)$$

$$\varphi'(t) \downarrow (0, 1) \Rightarrow \varphi'(t) > \varphi'(1) = -2 + 0 + 2 = 0 \Rightarrow \varphi(t) \uparrow (0, 1)$$

$$\Rightarrow \varphi(t) < \varphi(1) = 2(1 - 1) + (1 + 1) \cdot 0 = 0$$

$$\text{Hence, } f'(x) < 0 \left( 0 < x < \frac{\pi}{2} \right) \Rightarrow f(x) \downarrow (0, 1)$$

$$f(x) < f(0^+) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1 + \sin^2 x) \left[ (\sin^2 x)^{\sin^2 x} \right]^{\frac{1}{\cos^2 x}} = 1$$

$$\rightarrow f(x) < 1, \left( 0 < x < \frac{\pi}{2} \right)$$

$$\text{So, } \Omega < 1 + 1 + 1 = 3$$

**1677. In acute  $\triangle ABC$  the following relationship holds:**

$$m_a s_a + m_b s_b + m_c s_c \leq s^2$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution by Tran Hong-Dong Thap-Vietnam*

In acute  $\triangle ABC$

$$m_a s_a \leq s(s - a)$$

$$m_a \cdot \frac{2bc}{b^2 + c^2} \cdot m_a \leq \frac{a + b + c}{2} \cdot \frac{b + c - a}{2} \leftrightarrow$$

$$m_a^2 \cdot \frac{2bc}{b^2 + c^2} \leq \frac{(a + b + c)(b + c - a)}{4} \leftrightarrow$$

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$$\frac{2(b^2 + c^2) - a^2}{4} \cdot \frac{2bc}{b^2 + c^2} \leq \frac{(a + b + c)(b + c - a)}{4} \Leftrightarrow$$

$$2bc[2(b^2 + c^2) - a^2] \leq [(b + c)^2 - a^2](b^2 + c^2) \Leftrightarrow$$

$$[4bc - (b + c)^2](b^2 + c^2) + (b^2 + c^2 - 2bc)a^2 \leq 0$$

$$-(b^2 - 2bc + c^2)(b^2 + c^2) + (b^2 - 2bc + c^2)a^2 \leq 0$$

$$(b^2 - 2bc + c^2)[a^2 - (b^2 + c^2)] \leq 0$$

$$-(b - c)^2(b^2 + c^2 - a^2) \leq 0$$

Which is true, because:  $\triangle ABC$  acute  $\rightarrow b^2 + c^2 > a^2$ ;  $-(b - c)^2 \leq 0$

Similary:

$$m_b s_b \leq s(s - b); m_c s_c \leq s(s - c)$$

$$m_a s_a + m_b s_b + m_c s_c \leq s(s - a) + s(s - b) + s(s - c) = s^2$$

**1678. In  $\triangle ABC$  the following relationship holds:**

$$ab + bc + ca \geq 4\sqrt{3}S \cdot \sqrt{\frac{R}{2r}} \geq 4\sqrt{3}S$$

*Proposed by Rahim Shahbazov-Baku-Azerbaijan*

*Solution by Tran Hong-Dong Thap-Vietnam*

$$ab + bc + ca \geq 4\sqrt{3}S \cdot \sqrt{\frac{R}{2r}} \Leftrightarrow (ab + bc + ca)^2 \geq 48 \cdot S^2 \cdot \frac{R}{2r} \Leftrightarrow$$

$$(ab + bc + ca)^2 \geq 48s(s - a)(s - b)(s - c) \cdot \frac{abc}{(a+b-c)(b+c-a)(c+a-b)}$$

$$(ab + bc + ca)^2 \geq$$

$$\geq 48 \cdot \frac{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}{16} \cdot \frac{abc}{(a+b-c)(b+c-a)(c+a-b)}$$

$$(ab + bc + ca)^2 \geq 3abc(a + b + c)$$

Which is true because:

$$(X + Y + Z)^2 \geq 3(XY + YZ + ZX), X = ab, Y = bc, Z = ca \Rightarrow$$

$$(ab + bc + ca)^2 \geq 3abc(a + b + c)$$

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$$R \geq 2r \rightarrow \frac{R}{2r} \geq 1 \Rightarrow 4\sqrt{3}S \cdot \sqrt{\frac{R}{2r}} \geq 4\sqrt{3}S$$

**1679. In  $\triangle ABC$ ,  $g_a$  –Gergonne’s cevian, the following relationship holds:**

$$a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot \max\left(\frac{g_a}{h_a}, \frac{g_b}{h_b}, \frac{g_c}{h_c}\right) \geq 4S\sqrt{3}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Bogdan Fuștei-Romania*

$$h_a \leq g_a \Rightarrow \frac{g_a}{h_a} \geq 1 \text{ and analogs.}$$

$$g_a \leq AI + r \text{ (triangle inequality)} \leq w_a \leq \sqrt{s(s-a)} \leq m_a \Rightarrow$$

$$\frac{g_a}{h_a} \leq \frac{m_a}{h_a} \text{ and analogs.}$$

$$\text{Let be } \max\left(\frac{g_a}{h_a}, \frac{g_b}{h_b}, \frac{g_c}{h_c}\right) = \frac{m_a}{h_a}$$

$$\text{We show that: } a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot \frac{m_a}{h_a}$$

$$2S = ah_a \Rightarrow a^2 + b^2 + c^2 \geq 2S \cdot 2\sqrt{3} \cdot \frac{m_a}{h_a} = ah_a \cdot 2\sqrt{3} \cdot \frac{m_a}{h_a} = 2\sqrt{3} \cdot am_a$$

$$4m_a^2 + 3a^2 = 2(b^2 + c^2) - a^2 + 3a^2 = 2(a^2 + b^2 + c^2)$$

$$4m_a^2 = (2m_a)^2, 3a^2 = (a\sqrt{3})^2 \text{ then } 4m_a^2 + 3a^2 \stackrel{Am-Gm}{\geq} 4\sqrt{3}am_a$$

$$2(a^2 + b^2 + c^2) \geq 4\sqrt{3}am_a \Rightarrow a^2 + b^2 + c^2 \geq 2\sqrt{3}am_a$$

Simillary:

$$a^2 + b^2 + c^2 \geq 2\sqrt{3}bm_b$$

$$a^2 + b^2 + c^2 \geq 2\sqrt{3}cm_c$$

$$\text{So, } a^2 + b^2 + c^2 \geq 4\sqrt{3} \cdot \max\left(\frac{g_a}{h_a}, \frac{g_b}{h_b}, \frac{g_c}{h_c}\right) \geq 4S\sqrt{3}$$

**1680. In acute  $\triangle ABC$  the following relationship holds:**

$$\cos(A-B)\cos(B-C)\cos(C-A) \leq \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

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### Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$\text{Lemma 1. } \frac{R}{2r} \geq \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a}$$

$$\text{Lemma 2. } \cos(A - B) \leq \frac{h_a}{r_a} \quad (\text{for acute } \triangle ABC)$$

$$\text{Lemma 3. } \frac{h_a h_b h_c}{r_a r_b r_c} = \frac{2r}{R}$$

We have

$$\begin{aligned} \cos(A - B)\cos(B - C)\cos(C - A) &\leq \frac{h_a h_b h_c}{m_a m_b m_c} \leq \frac{h_a h_b h_c}{r_a r_b r_c} = \frac{2r}{R} \leq \frac{\sum m_a^2}{\sum m_a m_b} \\ &\Rightarrow \frac{R}{2r} \geq \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} \end{aligned}$$

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} &\cos(A - B)\cos(B - C)\cos(C - A) \\ &= \left(2\cos^2 \frac{A - B}{2} - 1\right) \left(2\cos^2 \frac{B - C}{2} - 1\right) \left(2\cos^2 \frac{C - A}{2} - 1\right) \\ &\stackrel{(a)}{=} 8 \prod \cos^2 \frac{B - C}{2} - 4 \left(\prod \cos^2 \frac{B - C}{2}\right) \sum \sec^2 \frac{B - C}{2} + 2 \sum \cos^2 \frac{B - C}{2} - 1 \\ &\text{Now, } \sum \cos^2 \frac{B - C}{2} = \sum \frac{(b + c)^2 \sin^2 \frac{A}{2}}{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{16R^2 s} \sum \frac{bc(b + c)^2}{s - a} \\ &= \frac{1}{16R^2 s} \sum \frac{bc(s + s - a)^2}{s - a} \\ &= \frac{1}{16R^2 s} \sum \left\{ \frac{bcs^2}{s - a} + 2sbc + bc(s - a) \right\} = \frac{1}{16R^2 s} \left\{ s^3 \sum \sec^2 \frac{A}{2} + 3s \sum ab - 3abc \right\} \\ &= \frac{1}{16R^2 s} \left[ s^3 \left\{ \frac{s^2 + (4R + r)^2}{s^2} \right\} + 3s(s^2 + 4Rr + r^2) - 12Rrs \right] = \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2} \\ &\Rightarrow \sum \cos^2 \frac{B - C}{2} \stackrel{(1)}{=} \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2} \end{aligned}$$

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$$\text{Again, } \sum \sec^2 \frac{B-C}{2} = \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{(b+c)^2 \sin^2 \frac{A}{2}} = \sum \frac{16R^2 s(s-a)a}{4Rrs(b+c)^2}$$

$$= \frac{2R}{r} \sum \frac{a(b+c-a)}{(b+c)^2} \stackrel{(2)}{\cong} \frac{2R}{r} \left\{ \sum \frac{a}{b+c} - \sum \frac{a^2}{(b+c)^2} \right\}$$

$$\text{Now, } \sum \frac{a}{b+c} = \frac{\sum a(c+a)(a+b)}{\prod(b+c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2 + 2Rr + r^2)}$$

$$= \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \stackrel{(3)}{\cong} \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2}$$

$$\text{and, } \sum \frac{a^2}{(b+c)^2} = \sum \frac{(2s - (b+c))^2}{(b+c)^2}$$

$$= \sum \frac{4s^2 - 4s(b+c) + (b+c)^2}{(b+c)^2} \stackrel{(1)}{\cong} 4s^2 \left[ \frac{\sum \{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2} \right]$$

$$- 4s \left[ \frac{\sum (c+a)(a+b)}{\prod(b+c)} \right] + 3$$

$$\sum \{(c+a)^2(a+b)^2\} = \sum (\sum ab + a^2)^2 = \sum \left\{ (\sum ab)^2 + 2a^2 \sum ab + a^4 \right\}$$

$$= 3(\sum ab)^2 + 2(\sum ab)(\sum a^2) + (\sum a^2)^2 - 2\sum a^2 b^2$$

$$= (\sum ab)^2 + 2(\sum ab)(\sum a^2) + (\sum a^2)^2 + 2\sum a^2 b^2 + 4abc(2s) - 2\sum a^2 b^2$$

$$= (\sum ab + \sum a^2)^2 + 32Rrs^2$$

$$= (3s^2 - 4Rr - r^2)^2 + 32Rrs^2$$

$$\therefore \sum \{(c+a)^2(a+b)^2\} \stackrel{(ii)}{\cong} (3s^2 - 4Rr - r^2)^2 + 32Rrs^2$$

$$\text{Again, } \sum (c+a)(a+b) = \sum (\sum ab + a^2) = 3 \sum ab + \sum a^2$$

$$= \sum a^2 + 2 \sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2$$

$$\therefore \sum (c+a)(a+b) \stackrel{(iii)}{\cong} 5s^2 + 4Rr + r^2$$

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$$\begin{aligned}
 & \because \prod (b+c) = s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum \frac{a^2}{(b+c)^2} \\
 & = \frac{4s^2\{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2\}}{4s^2(s^2 + 2Rr + r^2)^2} - \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} + 3 \\
 & = \frac{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2 - 2(5s^2 + 4Rr + r^2)(s^2 + 2Rr + r^2) + 3(s^2 + 2Rr + r^2)^2}{(s^2 + 2Rr + r^2)^2} \\
 & = \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \\
 & \Rightarrow \sum \frac{a^2}{(b+c)^2} \stackrel{(4)}{=} \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \\
 & \quad (2), (3), (4) \Rightarrow \sum \sec^2 \frac{B-C}{2} \\
 & = \frac{2R}{r} \left\{ \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} - \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \right\} \\
 & \stackrel{(5)}{=} \frac{2R}{r} \left[ \frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right] \\
 & \quad \text{Also, } 8 \prod \cos^2 \frac{B-C}{2} = 8 \prod \frac{(b+c)^2 \sin^2 \frac{A}{2}}{a^2} \\
 & = 8 \left\{ \frac{4s^2(s^2 + 2Rr + r^2)^2}{16R^2r^2s^2} \right\} \left( \frac{r^2}{16R^2} \right) \stackrel{(6)}{=} \frac{(s^2 + 2Rr + r^2)^2}{8R^4} \\
 & (a), (1), (5), (8) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) = \frac{(s^2 + 2Rr + r^2)^2}{8R^4} \\
 & - \left\{ \frac{(s^2 + 2Rr + r^2)^2}{16R^4} \right\} \frac{2R}{r} \left[ \frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right] \\
 & \quad + \frac{4s^2 + (4R+r)^2 + 3r^2}{8R^2} - 1 \\
 & \Rightarrow \cos(A-B)\cos(B-C)\cos(C) \\
 & - A) \stackrel{(m)}{=} \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R+r)^2 + 3r^2\} - 8R^4r}{8R^4r}
 \end{aligned}$$

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(where  $\sigma = (2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2)$ )

$- \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}$

$$\begin{aligned} \text{Now, } \sum m_a m_b &\stackrel{\text{Tereshin}}{\geq} \sum \frac{(b^2 + c^2)(c^2 + a^2)}{16R^2} = \frac{3 \sum a^2 b^2 + \sum a^4}{16R^2} \geq \frac{3 \sum a^2 b^2 + \sum a^2 b^2}{16R^2} \\ &= \sum \frac{b^2 c^2}{4R^2} = \sum h_a^2 \\ \Rightarrow \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} &\leq \frac{\sum m_a^2}{\sum h_a^2} = \frac{\frac{3 \sum a^2}{4}}{\sum \left( \frac{b^2 c^2}{4R^2} \right)} = \frac{3R^2 \sum a^2}{\sum a^2 b^2} \stackrel{?}{\leq} \frac{R}{2r} \end{aligned}$$

$$\Leftrightarrow \sum a^2 b^2 \stackrel{?}{\underset{(p)}{\geq}} 6Rr \sum a^2$$

Let  $s - a = x, s - b = y$  and  $s - c = z \therefore 3s - 2s = \sum x \Rightarrow a = y + z, b = z + x$  and  $c = x + y$

$$\begin{aligned} \therefore (p) &\Leftrightarrow \sum (y + z)^2 (z + x)^2 \geq 6 \frac{\prod (y + z)}{4F} \cdot \frac{F}{\sum x} \cdot \sum (y + z)^2 \\ &\Leftrightarrow 2(\sum x) \left\{ \sum (y + z)^2 (z + x)^2 \right\} \geq 3 \left\{ \prod (y + z) \right\} \left\{ \sum (y + z)^2 \right\} \\ &\Leftrightarrow \sum x^5 + xyz \sum xy - \sum x^3 y^2 - \sum x^2 y^3 \stackrel{(q)}{\geq} 0 \end{aligned}$$

Now,  $F(x, y, z)$

$$\begin{aligned} &= \sum x^5 + xyz \sum xy - \sum x^3 y^2 \\ &- \sum x^2 y^3 \text{ is a homogeneous and symmetric polynomial} \end{aligned}$$

$$F(x, y, 0) = x^5 + y^5 - x^3 y^2 - x^2 y^3 \stackrel{\text{Chebyshev}}{\geq} \frac{1}{2} (x^3 + y^3)(x^2 + y^2) - x^3 y^2$$

$$- x^2 y^3 \stackrel{A-G}{\geq} xy(x^3 + y^3) - x^3 y^2 - x^2 y^3$$

$$\geq x^2 y^2 (x + y) - x^3 y^2 - x^2 y^3 = 0 \Rightarrow \boxed{F(x, y, 0) \geq 0}$$

$$F(x, 1, 1) = x^5 + 1 + 1 + x(x + 1 + x) - x^3 - 1 - x^2 - x^2 - 1 - x^3 = x^5 - 2x^3 + x$$

$$= x(x^2 - 1)^2 \geq 0 \Rightarrow \boxed{F(x, 1, 1) \geq 0}$$

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∴ by SD5 theorem,  $F(x, y, z) \geq 0 \Rightarrow (q) \Rightarrow (p)$  is true  $\Rightarrow \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \leq \frac{R}{2r}$

$$\Rightarrow \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} \stackrel{(n)}{\geq} \frac{2r}{R}$$

∴ (m), (n)  $\Rightarrow$  it suffices to prove

$$: \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2 r \{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4 r}{8R^4 r} - \frac{2r}{R}$$

$\leq 0$

$$\Leftrightarrow \frac{r(s^2 + 2Rr + r^2)^2 - Rr + R^2 r \{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4 r - 16R^3 r^2}{8R^4 r} \leq 0$$

$$\Leftrightarrow s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3 r + 22R^2 r^2 + 8Rr^3 + r^4 \stackrel{(u)}{\geq} 0$$

∴  $\Delta ABC$  is acute – angled, Walker and Gerretsen

$$\Rightarrow (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2) \leq 0$$

$\Rightarrow$  in order to prove (u),

it suffices to prove :  $s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3 r + 22R^2 r^2 + 8Rr^3 + r^4$

$$\leq (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2)$$

$$\stackrel{(v)}{\Leftrightarrow} (R + 2r)s^2 \stackrel{?}{\geq} 8R^3 + 7R^2 r + 7Rr^2 + 2r^3$$

$$\text{Now, } (R + 2r)s^2 \stackrel{\text{Gerretsen}}{\geq} (R + 2r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 8R^3 + 7R^2 r + 7Rr^2 + 2r^3$$

$$\Leftrightarrow 4t^3 - 5t^2 - 4t - 4 \stackrel{?}{\geq} 0 \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(4t^2 + 3t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (v) \Rightarrow (u) \text{ is true}$$

$$\therefore \cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2}$$

**1681. In acute  $\Delta ABC$  the following relationship holds:**

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \left( \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

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### Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

Lemma 1.  $\cos(A - B) \leq \frac{h_a}{m_a}$  (for acute)

Lemma 2.  $m_a m_b m_c \geq r_a r_b r_c$

Lemma 3.  $\frac{h_a h_b h_c}{r_a r_b r_c} = \frac{2r}{R}$

Lemma 4.  $\frac{R}{2r} \geq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^3$

If use lemma's

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{h_a h_b h_c}{m_a m_b m_c} \leq \frac{h_a h_b h_c}{r_a r_b r_c} = \frac{2r}{R} \Rightarrow$$

$$\frac{2r}{R} \leq \left( \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2 \Rightarrow \frac{R}{2r} \geq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2$$

### Solution 2 by Bogdan Fuștei-Romania

We have:

$$1) \frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc}$$

$$2) \forall x, y, z > 0; \frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \left( \frac{x^2 + y^2 + z^2}{xy + yz + zx} \right)^2$$

$$\Rightarrow \frac{R}{2r} \stackrel{(1)}{\geq} \frac{1}{4} + \frac{a^3 + b^3 + c^3}{4abc} \stackrel{(2)}{\geq} \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2$$

How the triangle  $ABC$  is acute, we have:  $\cos(B - C) \leq \frac{h_a}{m_a}$  and analogs.

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{h_a h_b h_c}{m_a m_b m_c}; (i)$$

$$m_a \geq \sqrt{s(s - a)} = \sqrt{r_a r_b} \text{ (and analogs)} \Rightarrow m_a m_b m_c \geq r_a r_b r_c$$

$$\frac{m_a m_b m_c}{h_a h_b h_c} \geq \frac{r_a r_b r_c}{h_a h_b h_c} \Rightarrow \frac{h_a h_b h_c}{r_a r_b r_c} \geq \frac{h_a h_b h_c}{m_a m_b m_c}; (ii)$$

From (i), (ii) we get:

$$\cos(A - B)\cos(B - C)\cos(C - A) \leq \frac{h_a h_b h_c}{r_a r_b r_c}; (iii)$$

From (3)  $r_a r_b r_c = Ss$ ; (4)  $abc = 4RS$ ; (5)  $S^2 = r_a r_b r_c r$  we get:

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$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} = \frac{4S \cdot 2S^2}{abc} = \frac{4S \cdot 2S^2}{4RS} = \frac{2S^2}{R} = \frac{2r}{R} \cdot r_a r_b r_c \Rightarrow$$

$$\frac{h_a h_b h_c}{r_a r_b r_c} = \frac{2r}{R} \leq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2; \quad (iv)$$

From (iii), (iv) the inequality is proved.

### Solution 3 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} & \cos(A - B)\cos(B - C)\cos(C - A) \\ &= \left(2\cos^2 \frac{A - B}{2} - 1\right) \left(2\cos^2 \frac{B - C}{2} - 1\right) \left(2\cos^2 \frac{C - A}{2} - 1\right) \\ &\stackrel{(a)}{=} 8 \prod \cos^2 \frac{B - C}{2} - 4 \left( \prod \cos^2 \frac{B - C}{2} \right) \sum \sec^2 \frac{B - C}{2} + 2 \sum \cos^2 \frac{B - C}{2} - 1 \\ \text{Now, } \sum \cos^2 \frac{B - C}{2} &= \sum \frac{(b + c)^2 \sin^2 \frac{A}{2}}{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}} = \frac{1}{16R^2 s} \sum \frac{bc(b + c)^2}{s - a} \\ &= \frac{1}{16R^2 s} \sum \frac{bc(s + s - a)^2}{s - a} \\ &= \frac{1}{16R^2 s} \sum \left\{ \frac{bcs^2}{s - a} + 2sbc + bc(s - a) \right\} = \frac{1}{16R^2 s} \left\{ s^3 \sum \sec^2 \frac{A}{2} + 3s \sum ab - 3abc \right\} \\ &= \frac{1}{16R^2 s} \left[ s^3 \left\{ \frac{s^2 + (4R + r)^2}{s^2} \right\} + 3s(s^2 + 4Rr + r^2) - 12Rrs \right] = \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2} \\ &\Rightarrow \sum \cos^2 \frac{B - C}{2} \stackrel{(1)}{=} \frac{4s^2 + (4R + r)^2 + 3r^2}{16R^2} \\ \text{Again, } \sum \sec^2 \frac{B - C}{2} &= \sum \frac{16R^2 \sin^2 \frac{A}{2} \cos^2 \frac{A}{2}}{(b + c)^2 \sin^2 \frac{A}{2}} = \sum \frac{16R^2 s(s - a)a}{4Rrs(b + c)^2} \\ &= \frac{2R}{r} \sum \frac{a(b + c - a)}{(b + c)^2} \stackrel{(2)}{=} \frac{2R}{r} \left\{ \sum \frac{a}{b + c} - \sum \frac{a^2}{(b + c)^2} \right\} \\ \text{Now, } \sum \frac{a}{b + c} &= \frac{\sum a(c + a)(a + b)}{\prod (b + c)} = \frac{\sum a(\sum ab + a^2)}{2s(s^2 + 2Rr + r^2)} \\ &= \frac{2s(s^2 + 4Rr + r^2) + 2s(s^2 - 6Rr - 3r^2)}{2s(s^2 + 2Rr + r^2)} \stackrel{(3)}{=} \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} \end{aligned}$$

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$$\begin{aligned}
 & \text{and, } \sum \frac{a^2}{(b+c)^2} = \sum \frac{(2s - (b+c))^2}{(b+c)^2} \\
 & = \sum \frac{4s^2 - 4s(b+c) + (b+c)^2}{(b+c)^2} \stackrel{(i)}{=} 4s^2 \left[ \frac{\sum \{(c+a)^2(a+b)^2\}}{\{\prod(b+c)\}^2} \right] \\
 & \quad - 4s \left[ \frac{\sum(c+a)(a+b)}{\prod(b+c)} \right] + 3 \\
 & \sum \{(c+a)^2(a+b)^2\} = \sum (\sum ab + a^2)^2 = \sum \left\{ (\sum ab)^2 + 2a^2 \sum ab + a^4 \right\} \\
 & = 3(\sum ab)^2 + 2(\sum ab)(\sum a^2) + (\sum a^2)^2 - 2\sum a^2 b^2 \\
 & = (\sum ab)^2 + 2(\sum ab)(\sum a^2) + (\sum a^2)^2 + 2\sum a^2 b^2 + 4abc(2s) - 2\sum a^2 b^2 \\
 & = (\sum ab + \sum a^2)^2 + 32Rrs^2 \\
 & \quad = (3s^2 - 4Rr - r^2)^2 + 32Rrs^2 \\
 & \therefore \sum \{(c+a)^2(a+b)^2\} \stackrel{(ii)}{=} (3s^2 - 4Rr - r^2)^2 + 32Rrs^2 \\
 & \text{Again, } \sum (c+a)(a+b) = \sum (\sum ab + a^2) = 3 \sum ab + \sum a^2 \\
 & = \sum a^2 + 2 \sum ab + \sum ab = 4s^2 + s^2 + 4Rr + r^2 \\
 & \quad \therefore \sum (c+a)(a+b) \stackrel{(iii)}{=} 5s^2 + 4Rr + r^2 \\
 & \therefore \prod (b+c) = s^2 + 2Rr + r^2 \therefore (i), (ii), (iii) \Rightarrow \sum \frac{a^2}{(b+c)^2} \\
 & = \frac{4s^2 \{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2\}}{4s^2 (s^2 + 2Rr + r^2)^2} - \frac{4s(5s^2 + 4Rr + r^2)}{2s(s^2 + 2Rr + r^2)} + 3 \\
 & = \frac{(3s^2 - 4Rr - r^2)^2 + 32Rrs^2 - 2(5s^2 + 4Rr + r^2)(s^2 + 2Rr + r^2) + 3(s^2 + 2Rr + r^2)^2}{(s^2 + 2Rr + r^2)^2} \\
 & = \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \\
 & \Rightarrow \sum \frac{a^2}{(b+c)^2} \stackrel{(4)}{=} \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2}
 \end{aligned}$$

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$$(2), (3), (4) \Rightarrow \sum \sec^2 \frac{B-C}{2}$$

$$= \frac{2R}{r} \left\{ \frac{2s^2 - 2Rr - 2r^2}{s^2 + 2Rr + r^2} - \frac{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4}{(s^2 + 2Rr + r^2)^2} \right\}$$

$$\stackrel{(5)}{=} \frac{2R}{r} \left[ \frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right]$$

$$\text{Also, } 8 \prod \cos^2 \frac{B-C}{2} = 8 \prod \frac{(b+c)^2 \sin^2 \frac{A}{2}}{a^2}$$

$$= 8 \left\{ \frac{4s^2(s^2 + 2Rr + r^2)^2}{16R^2r^2s^2} \right\} \left( \frac{r^2}{16R^2} \right) \stackrel{(6)}{=} \frac{(s^2 + 2Rr + r^2)^2}{8R^4}$$

$$(a), (1), (5), (8) \Rightarrow \cos(A-B)\cos(B-C)\cos(C-A) = \frac{(s^2 + 2Rr + r^2)^2}{8R^4}$$

$$- \left\{ \frac{(s^2 + 2Rr + r^2)^2}{16R^4} \right\} \frac{2R}{r} \left[ \frac{(2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2) - \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\}}{(s^2 + 2Rr + r^2)^2} \right]$$

$$+ \frac{4s^2 + (4R+r)^2 + 3r^2}{8R^2} - 1$$

$$\Rightarrow \cos(A-B)\cos(B-C)\cos(C)$$

$$- A) \stackrel{(m)}{=} \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R+r)^2 + 3r^2\} - 8R^4r}{8R^4r}$$

$$\text{(where } \sigma = (2s^2 - 2Rr - 2r^2)(s^2 + 2Rr + r^2)$$

$$- \{2s^4 - s^2(8Rr + 12r^2) + 12R^2r^2 + 8Rr^3 + 2r^4\})$$

$$\text{Now, } \left( \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2 \geq \frac{2r}{R} \Leftrightarrow R(s^2 + 4Rr + r^2)^2 \geq 8r(s^2 - 4Rr - r^2)^2$$

$$\Leftrightarrow (R - 2r)s^4 + s^2(8R^2r + 66Rr^2 + 16r^3) + 16R^3r^2 - 120R^2r^3 - 63Rr^4$$

$$\stackrel{(u)}{-} 8r^5 \stackrel{(v)}{\geq} 6rs^4$$

$$\text{Now, LHS of (u) } \stackrel{\text{Gerretsen}}{\stackrel{(b)}{\geq}} (R - 2r)(16Rr - 5r^2)s^2 + s^2(8R^2r + 66Rr^2 + 16r^3)$$

$$+ 16R^3r^2 - 120R^2r^3 - 63Rr^4 - 8r^5$$

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and RHS of (u)  $\stackrel{\text{Gerretsen}}{\underset{(c)}{\geq}} 6rs^2(4R^2 + 4Rr + 3r^2) \therefore (b), (c)$

$\Rightarrow$  in order to prove (u), it suffices to prove :

$$(R - 2r)(16Rr - 5r^2)s^2 + s^2(8R^2r + 66Rr^2 + 16r^3) + 16R^3r^2 - 120R^2r^3 - 63Rr^4 - 8r^5 \geq 6rs^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow s^2(5R + 8r) + 16R^3 - 120R^2r - 63Rr^2 - 8r^3 \stackrel{(v)}{\geq} 0$$

Now, LHS of (v)  $\stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2)(5R + 8r) + 16R^3 - 120R^2r - 63Rr^2 - 8r^3 \stackrel{?}{\geq} 0$

$$\Leftrightarrow 2t^3 - 5t^2 + 5t - 6 \stackrel{?}{\geq} 0 \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(2t + 3) + 9t^2\} \stackrel{?}{\geq} 0 \rightarrow \text{true} \therefore t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (v) \Rightarrow (u) \text{ is true}$$

$$\therefore \left( \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2 \stackrel{(n)}{\geq} \frac{2r}{R}$$

$\therefore (m), (n) \Rightarrow$  it suffices to prove

$$\therefore \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r}{8R^4r} - \frac{2r}{R}$$

$$\leq 0$$

$$\Leftrightarrow \frac{r(s^2 + 2Rr + r^2)^2 - R\sigma + R^2r\{4s^2 + (4R + r)^2 + 3r^2\} - 8R^4r - 16R^3r^2}{8R^4r} \leq 0$$

$$\Leftrightarrow s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4 \stackrel{(x)}{\geq} 0$$

$\therefore \Delta ABC$  is acute - angled, Walker and Gerretsen

$$\Rightarrow (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2) \leq 0$$

$\Rightarrow$  in order to prove (x),

it suffices to prove :  $s^4 + 8R^4 - s^2(6R^2 + 8Rr - 2r^2) + 8R^3r + 22R^2r^2 + 8Rr^3 + r^4$

$$\leq (s^2 - 2R^2 - 8Rr - 3r^2)(s^2 - 4R^2 - 4Rr - 3r^2)$$

$$\Leftrightarrow (R + 2r)s^2 \stackrel{(y)}{\geq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$$

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$$\text{Now, } (R + 2r)s^2 \stackrel{\text{Gerretsen}}{\geq} (R + 2r)(4R^2 + 4Rr + 3r^2) \stackrel{?}{\geq} 8R^3 + 7R^2r + 7Rr^2 + 2r^3$$

$$\Leftrightarrow 4t^3 - 5t^2 - 4t - 4 \stackrel{?}{\geq} 0 \quad \left( \text{where } t = \frac{R}{r} \right)$$

$$\Leftrightarrow (t - 2)(4t^2 + 3t + 2) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow (x) \Rightarrow (y) \text{ is true}$$

$$\therefore \cos(A - B)\cos(B - C)\cos(C - A) \leq \left( \frac{ab + bc + ca}{a^2 + b^2 + c^2} \right)^2$$

**1682. In  $\triangle ABC$ ,  $n_a$  – Nagel's cevian,  $g_a$  – Gergonne's cevian the following relationship holds:**

$$\frac{n_a n_b n_c}{r_a r_b r_c} \geq \frac{w_a w_b w_c}{g_a g_b g_c}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\ \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= an_a^2 + a(as - s^2) \end{aligned}$$

$$\begin{aligned} \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \end{aligned}$$

$$= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left( \frac{2\Delta}{a} \right) \left( \frac{\Delta}{s - a} \right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(1)}{\hat{=}} s^2 - 2h_a r_a$$

$$\begin{aligned} \text{Again, Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) \\ &= an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= ag_a^2 + a(s - b)(s - c) \end{aligned}$$

$$\therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s - a)^2$$

$$\begin{aligned} \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} \{b^2(s - b) + c^2(s - c) - a(s - b)(s - c)\} &\stackrel{(a)}{\geq} a^2 s^2 (s - a)^2 \end{aligned}$$

Let  $s - a = x$ ,  $s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z$ ,  $b = z + x$  and  $c = x + y$

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Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$\Rightarrow$  (a) is true  $\Rightarrow n_a g_a \geq s(s-a)$  and analogs

$$\Rightarrow \frac{n_a n_b n_c g_a g_b g_c}{r_a r_b r_c} \geq \frac{s(s-a) \cdot s(s-b) \cdot s(s-a)}{rs^2} = \frac{r^2 s^4}{rs^2} = rs^2 \geq w_a w_b w_c$$

$$= \frac{16Rr^2s^2}{s^2 + 2Rr + r^2} \Leftrightarrow s^2 + 2Rr + r^2 \geq 16Rr$$

$$\Leftrightarrow s^2 - 16Rr + 5r^2 + 2r(R - 2r) \geq 0 \rightarrow \text{true}$$

$$\because s^2 - 16Rr + 5r^2 \stackrel{\text{Gerretsen}}{\geq} 0 \text{ and } R - 2r \stackrel{\text{Euler}}{\geq} 0 \Rightarrow \frac{n_a n_b n_c}{r_a r_b r_c}$$

$$\geq \frac{w_a w_b w_c}{g_a g_b g_c} \text{ (Proved)}$$

### Solution 2 by Bogdan Fuștei-Romania

$$4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c \text{ and analogs.}$$

$$4m_a^2 = 4r_b r_c + (b-c)^2 \text{ and analogs.}$$

$$r_b r_c = s(s-a) \text{ and analogs.}$$

$$b^2 + c^2 + n_a^2 + g_a^2 + 2r_a r \text{ and analogs.}$$

$$n_a^2 = r_b r_c + \frac{(b-c)^2 s}{a} \text{ and analogs.}$$

$$4r_b r_c + (b-c)^2 = r_b r_c + \frac{(b-c)^2 s}{a} + g_a^2 + 2r_b r_c$$

$$4r_b r_c + (b-c)^2 = 3r_b r_c + \frac{(b-c)^2 s}{a} + g_a^2$$

$$r_b r_c + (b-c)^2 = g_a^2 + \frac{(b-c)^2 s}{a} \Rightarrow g_a^2 = s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$$\text{So, } g_a^2 = (s-a) \left( s - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 = s \left( s - a - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 g_a^2 = s(s-a) \left( s - \frac{(b-c)^2}{a} \right) \left( s - a - \frac{(b-c)^2}{a} \right) \geq s(s-a)s(s-b)$$

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$$s(s-a) - s(s-a) \frac{(s-a)(b-c)^2}{a} - \left[ \frac{(b-c)^2}{a} \right]^2 \geq s(s-a)$$

$$\frac{(b-c)^2}{a} (s-s+a) \geq \left[ \frac{(b-c)^2}{a} \right]^2 \Leftrightarrow \frac{(b-c)^2}{a} \cdot a \geq \left[ \frac{(b-c)^2}{a} \right]^2$$

$$(b-c)^2 \geq \frac{(b-c)^2}{a} \cdot \frac{(b-c)^2}{a}$$

If  $b = c \Rightarrow b - c = 0$  the we have equality.

$$\text{If } b \neq c \Rightarrow 1 > \frac{(b-c)^2}{a^2} \Rightarrow a^2 > (b-c)^2 \Rightarrow a > |b-c|$$

$$\begin{cases} a+b > c \\ a+c > b \end{cases} \Rightarrow \text{true.}$$

So,  $n_a g_a \geq r_b r_c$  and analogs, then

$$n_a n_b n_c g_a g_b g_c \geq (r_a r_b r_c)^2 \Rightarrow \frac{n_a n_b n_c}{r_a r_b r_c} \geq \frac{r_a r_b r_c}{g_a g_b g_c}; (1)$$

$$w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c} \text{ and analogs.}$$

$$\frac{2\sqrt{bc}}{b+c} \leq 1 \Rightarrow 2\sqrt{bc} \leq b+c \Rightarrow (\sqrt{b} - \sqrt{c})^2 \geq 0 \text{ true.}$$

So, we have:  $r_a r_b r_c \geq w_a w_b w_c; (2)$

From (1),(2), we get:  $\frac{n_a n_b n_c}{r_a r_b r_c} \geq \frac{w_a w_b w_c}{g_a g_b g_c}$ . Proved.

1683. If  $a, b, c > 0, \mu \leq \frac{1}{2}$  then prove:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{\mu w_a w_b w_c}{r_a r_b r_c} \geq \mu + \frac{3}{2}$$

Proposed by Marin Chirciu-Romania

Solution 1 by Avishek Mitra-West Bengal-India

$$\begin{aligned} \frac{w_a w_b w_c}{r_a r_b r_c} &= \frac{\prod \frac{2}{b+c} \sqrt{bcs(s-a)}}{s^2 r} = \frac{8abc}{(a+b)(b+c)(c+a)} = \\ &= \frac{32Rrs}{2abc + \sum ab(a+b)} = \frac{32Rrs}{2abc + \sum ab(2s-c)} = \\ &= \frac{32Rrs}{2s(\sum ab) - abc} = \frac{32Rrs}{2s(s^2 + r^2 + 4Rr) - 4Rrs} = \frac{16Rr}{s^2 + r^2 + 2Rr} \end{aligned}$$

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$$\begin{aligned}
 \sum \frac{a}{b+c} &= \sum \frac{a+b+c}{b+c} - 3 = 2s \left( \sum \frac{1}{b+c} \right) - 3 = \\
 &= 2s \cdot \frac{\sum (b+c)(c+a)}{\prod (a+b)} - 3 = 2s \cdot \frac{\sum (ab+bc+ca+c^2)}{\sum ab(a+b) + 2abc} - 3 = \\
 &= 2s \cdot \frac{\sum c^2 + 3 \sum ab}{abc + \sum ab(2s-c)} - 3 = \\
 &= 2s \cdot \frac{2(s^2 - 4Rr - r^2) + 3(s^2 + r^2 + 4Rr)}{2s \sum ab - abc} - 3 = \\
 &= 2s \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 4Rr) - 4Rrs} - 3 = 2s \cdot \frac{5s^2 + r^2 + 4Rr}{2s(s^2 + r^2 + 2Rr)} - 3 = \\
 &= \frac{2s^2 - 2Rr - 2r^2}{s^2 + r^2 + 2Rr}
 \end{aligned}$$

Need to show:

$$\begin{aligned}
 \frac{2s^2 - 2Rr - 2r^2}{s^2 + r^2 + 2Rr} + \mu \cdot \frac{16Rr}{s^2 + r^2 + 2Rr} &\geq \mu + \frac{3}{2} \\
 \frac{2s^2 - Rr(2 - 16\mu) - 2r^2}{s^2 + r^2 + 2Rr} &\geq \mu + \frac{3}{2} \\
 2s^2 - Rr(2 - 16\mu) - 2r^2 &\geq \left(\mu + \frac{3}{2}\right)s^2 + \left(\mu + \frac{3}{2}\right)2Rr + \left(\mu + \frac{3}{2}\right)r^2 \\
 s^2 \left(\frac{1}{2} - \mu\right) &\geq Rr(5 - 14\mu) + r^2 \left(\frac{7}{2} + \mu\right) \\
 s^2 &\geq 16Rr - 5r^2 \text{ (Gerretsen)}
 \end{aligned}$$

Need to show:

$$\begin{aligned}
 \left(\frac{1}{2} - \mu\right) 16Rr - \left(\frac{1}{2} - \mu\right) 5r^2 &\geq Rr(5 - 14\mu) + r^2 \left(\frac{7}{2} + \mu\right) \\
 Rr(8 - 16\mu - 5 + 14\mu) &\geq r^2 \left(\frac{5}{2} - 5\mu + \frac{7}{2} + \mu\right) \\
 R(3 - 2\mu) &\geq r(6 - 4\mu) \Leftrightarrow R \geq 2r \text{ true by Euler. Proved.}
 \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned}
 w_a &= \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c}, w_b = \frac{2\sqrt{ac}}{a+c} \sqrt{r_a r_c}, w_c = \frac{2\sqrt{ab}}{a+b} \sqrt{r_a r_b} \\
 w_a w_b w_c &= \frac{8abc}{(a+b)(b+c)(c+a)} \cdot r_a r_b r_c
 \end{aligned}$$

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$$\frac{w_a w_b w_c}{r_a r_b r_c} = \frac{8abc}{(a+b)(b+c)(c+a)}$$

Inequality becomes as:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \mu \cdot \frac{8abc}{(a+b)(b+c)(c+a)} \geq \mu + \frac{3}{2}, \left( \mu \leq \frac{1}{2} \right)$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \stackrel{(*)}{\geq} \mu + \frac{3}{2} - 8\mu \cdot \frac{abc}{(a+b)(b+c)(c+a)}$$

By Schur's inequality:

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} + \frac{4abc}{(a+b)(b+c)(c+a)} \geq 2 \Leftrightarrow$$

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \stackrel{(**)}{\geq} 2 - \frac{4abc}{(a+b)(b+c)(c+a)}$$

From (\*), (\*\*) we need to prove:

$$2 - \frac{4abc}{(a+b)(b+c)(c+a)} \geq \mu + \frac{3}{2} - \frac{8\mu abc}{(a+b)(b+c)(c+a)} \Leftrightarrow$$

$$\frac{1}{2} - \mu \geq \frac{4abc}{(a+b)(b+c)(c+a)} (1 - 2\mu) \Leftrightarrow$$

$$(1 - 2\mu) \left( 1 - \frac{8abc}{(a+b)(b+c)(c+a)} \right) \geq 0$$

Which is true because:

$$\mu \leq \frac{1}{2} \Rightarrow 1 - 2\mu \geq 0$$

$$(a+b)(b+c)(c+a) \stackrel{AGM}{\geq} 2\sqrt{ab} \cdot 2\sqrt{bc} \cdot 2\sqrt{ca} = 8abc$$

$$\frac{8abc}{(a+b)(b+c)(c+a)} \leq 1 \Leftrightarrow 1 - \frac{8abc}{(a+b)(b+c)(c+a)} \geq 0$$

Proved.

**1684. In  $\triangle ABC$ ,  $n_a$  – Nagel's cevian,  $g_a$  – Gergonne's cevian the following relationship holds:**

$$\frac{R}{r} \cdot \frac{n_a n_b n_c g_a g_b g_c}{h_a^2 h_b^2 h_c^2} \geq \frac{2m_a m_b m_c}{s_a s_b s_c}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

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### Solution 1 by Bogdan Fuștei-Romania

$$4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c \text{ and analogs.}$$

$$4m_a^2 = 4r_b r_c + (b - c)^2 \text{ and analogs.}$$

$$r_b r_c = s(s - a) \text{ and analogs.}$$

$$b^2 + c^2 + n_a^2 + g_a^2 + 2r_a r \text{ and analogs.}$$

$$n_a^2 = r_b r_c + \frac{(b-c)^2 s}{a} \text{ and analogs.}$$

$$4r_b r_c + (b - c)^2 = r_b r_c + \frac{(b-c)^2 s}{a} + g_a^2 + 2r_b r_c$$

$$4r_b r_c + (b - c)^2 = 3r_b r_c + \frac{(b - c)^2 s}{a} + g_a^2$$

$$r_b r_c + (b - c)^2 = g_a^2 + \frac{(b - c)^2 s}{a} \Rightarrow g_a^2 = s(s - a) - \frac{(s - a)(b - c)^2}{a}$$

$$\text{So, } g_a^2 = (s - a) \left( s - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 = s \left( s - a - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 g_a^2 = s(s - a) \left( s - \frac{(b-c)^2}{a} \right) \left( s - a - \frac{(b-c)^2}{a} \right) \geq s(s - a)s(s - b)$$

$$s(s - a) - s(s - a) \frac{(s - a)(b - c)^2}{a} - \left[ \frac{(b - c)^2}{a} \right]^2 \geq s(s - a)$$

$$\frac{(b - c)^2}{a} (s - s + a) \geq \left[ \frac{(b - c)^2}{a} \right]^2 \Leftrightarrow \frac{(b - c)^2}{a} \cdot a \geq \left[ \frac{(b - c)^2}{a} \right]^2$$

$$(b - c)^2 \geq \frac{(b - c)^2}{a} \cdot \frac{(b - c)^2}{a}$$

If  $b = c \Rightarrow b - c = 0$  the we have equality.

$$\text{If } b \neq c \Rightarrow 1 > \frac{(b-c)^2}{a^2} \Rightarrow a^2 > (b - c)^2 \Rightarrow a > |b - c|$$

$$\begin{cases} a + b > c \\ a + c > b \end{cases} \Rightarrow \text{true.}$$

So,  $n_a g_a \geq r_b r_c$  and analogs  $\Rightarrow n_a g_a r_a \geq r_a r_b r_c$

$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} \stackrel{abc=4RS}{=} \frac{2r}{R} \cdot r_a r_b r_c;$$

$$r_a r_b r_c = Ss; S = sr$$

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So, we have:

$$\frac{n_a g_a r_a}{h_a h_b h_c} \geq \frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$$

$$w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c} \text{ and analogs, then}$$

$$w_a w_b w_c = \frac{8abc}{(a+b)(b+c)(c+a)} \cdot r_a r_b r_c \Rightarrow$$

$$\frac{r_a r_b r_c}{w_a w_b w_c} = \frac{(a+b)(b+c)(c+a)}{8abc}$$

$$\text{But } h_a \leq w_a \text{ and analogs} \Rightarrow h_a h_b h_c \leq w_a w_b w_c \Rightarrow \frac{h_a h_b h_c}{r_a r_b r_c} \leq \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow$$

$$\frac{2r}{R} \leq \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow \frac{R}{2r} \geq \frac{r_a r_b r_c}{w_a w_b w_c} = \frac{(a+b)(b+c)(c+a)}{8abc}$$

$$(x-1)^2 \geq 0, \forall x \in \mathbb{R} \Rightarrow x^2 + 1 \geq 2x \Rightarrow x + \frac{1}{x} \geq 2$$

$$x = \frac{r_a r_b r_c}{w_a w_b w_c}; \frac{1}{x} = \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow \frac{r_a r_b r_c}{w_a w_b w_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2 \Leftrightarrow$$

$$\frac{R}{2r} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2 \Leftrightarrow \frac{n_a g_a r_a}{h_a h_b h_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2$$

$$\text{From } n_a g_a \geq r_b r_c \Rightarrow n_a n_b n_c g_a g_b g_c \geq (r_a r_b r_c)^2$$

$$\Rightarrow \frac{n_a n_b n_c}{r_a r_b r_c} \geq \frac{r_a r_b r_c}{g_a g_b g_c}; (1)$$

$$w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c} \text{ and analogs.}$$

$$\frac{2\sqrt{bc}}{b+c} \leq 1 \Leftrightarrow 2\sqrt{bc} \leq b+c \Leftrightarrow (\sqrt{b} - \sqrt{c})^2 \geq 0, \text{true.}$$

$$\text{So, we have: } w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c} \leq 1 \cdot \sqrt{r_b r_c} = \sqrt{r_b r_c}$$

$$\Rightarrow r_a r_b r_c \geq w_a w_b w_c; (2)$$

$$\text{From (1), (2) we have: } \frac{r_a r_b r_c}{g_a g_b g_c} \geq \frac{w_a w_b w_c}{g_a g_b g_c}$$

$$\text{From } n_a g_a \geq r_b r_c \Rightarrow n_a n_b n_c g_a g_b g_c \geq (r_a r_b r_c)^2 \Rightarrow$$

$$\frac{n_a n_b n_c g_a g_b g_c}{h_a^2 h_b^2 h_c^2} \geq \left( \frac{r_a r_b r_c}{h_a h_b h_c} \right)^2$$

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$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} \stackrel{abc=4RS}{=} \frac{2r}{R} \cdot r_a r_b r_c$$

$$\frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r} \Rightarrow \left( \frac{r_a r_b r_c}{h_a h_b h_c} \right)^2 = \frac{R^2}{4r^2}$$

But  $s_a = \frac{2bc}{b^2+c^2} \cdot m_a$  and analogs  $\Rightarrow \frac{b^2+c^2}{2bc} = \frac{m_a}{s_a} \Leftrightarrow \frac{1}{2} \left( \frac{b}{c} + \frac{c}{b} \right) = \frac{m_a}{s_a}$

$h_a \leq s_a$  and analogs  $\Rightarrow \frac{m_a}{s_a} \leq \frac{m_a}{h_a} \Rightarrow \frac{m_a}{h_a} \geq \frac{1}{2} \left( \frac{b}{c} + \frac{c}{b} \right)$  and analogs.

From  $\frac{R}{2r} \geq \frac{m_a}{h_a}$  (Panaitopol Inequality) we have:

$$\frac{R}{r} \geq \frac{b}{c} + \frac{c}{b} \text{ (Bădilă Inequality)}$$

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s-c) + c^2(s-b) = an_a^2 + a(s-b)(s-c) \\ &\Rightarrow s(b^2+c^2) - bc(2s-a) = an_a^2 + a(s^2 - s(2s-a) + bc) \Rightarrow s(b^2+c^2) - 2sbc \\ &= an_a^2 + a(as-s^2) \\ &\Rightarrow s(b^2+c^2 - a^2 - 2bc) = an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s-b)(s-c)(s-a)}{bc(s-a)} \\ &= as^2 - \frac{4\Delta^2}{s-a} = as^2 - 2a \left( \frac{2\Delta}{a} \right) \left( \frac{\Delta}{s-a} \right) = as^2 - 2ah_a r_a \stackrel{(1)}{\therefore} n_a^2 \cong s^2 - 2h_a r_a \end{aligned}$$

$$\begin{aligned} \text{Again, Stewart's theorem} &\Rightarrow b^2(s-c) + c^2(s-b) \\ &= an_a^2 + a(s-b)(s-c) \text{ and } b^2(s-b) + c^2(s-c) \\ &= ag_a^2 + a(s-b)(s-c) \\ \therefore an_a^2 \cdot ag_a^2 &\geq a^2 s^2 (s-a)^2 \\ &\Leftrightarrow \{b^2(s-c) + c^2(s-b) - a(s-b)(s-c)\} \{b^2(s-b) + c^2(s-c) - a(s-b)(s-c)\} \\ &\stackrel{(a)}{\geq} a^2 s^2 (s-a)^2 \end{aligned}$$

Let  $s-a = x, s-b = y$  and  $s-c = z \therefore s = x+y+z \Rightarrow a = y+z, b = z+x$  and  $c = x+y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\} \{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

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$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$\Rightarrow$  (a) is true  $\Rightarrow n_a g_a \geq s(s-a)$  and analogs

$$\begin{aligned} \Rightarrow \frac{R}{r} \cdot \frac{n_a n_b n_c g_a g_b g_c}{h_a^2 h_b^2 h_c^2} &\geq \left(\frac{R}{r}\right) \frac{16R^2 r^2 s^2 \cdot s(s-a) \cdot s(s-b) \cdot s(s-a)}{64r^6 s^6} \\ &= \left(\frac{R}{r}\right) \left(\frac{16R^2 r^2 s^4 \cdot r^2 s^2}{64r^6 s^6}\right) = \frac{R^3}{4r^3} \geq \frac{2m_a m_b m_c}{s_a s_b s_c} \end{aligned}$$

$$\Leftrightarrow \frac{R^3}{4r^3} \geq 2 \prod \left( \frac{m_a}{2bcm_a} \right) \Leftrightarrow \frac{R^3}{r^3} \geq \prod \left( \frac{b}{c} + \frac{c}{b} \right) \rightarrow \text{true} \because \frac{b}{c} + \frac{c}{b} \stackrel{\text{Bandila}}{\geq} \frac{R}{r} \text{ and analogs}$$

$$\therefore \frac{R}{r} \cdot \frac{n_a n_b n_c g_a g_b g_c}{h_a^2 h_b^2 h_c^2} \geq \frac{2m_a m_b m_c}{s_a s_b s_c} \text{ (Proved)}$$

**1685. In  $\triangle ABC$  the following relationship holds:**

$$2 + \left( 1 + \frac{4m_a m_b m_c (m_a + m_b + m_c)}{9F^2} \right)^2 \geq \frac{(m_a + m_b + m_c)^4}{9F^2}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$2 + \left( 1 + \frac{2R}{r} \right)^2 \geq \frac{(a+b+c)^4}{16F^2} \Leftrightarrow 2 + \frac{(2R+r)^2}{r^2} \geq \frac{16s^4}{s^2 r^2} \Leftrightarrow \frac{4R^2 + 4Rr + 3r^2}{r^2} \geq \frac{s^2}{r^2}$$

$\rightarrow$  true (Gerretsen)

$$\therefore 2 + \left( 1 + \frac{2R}{r} \right)^2 \geq \frac{(a+b+c)^4}{16F^2} \Rightarrow 2 + \left( 1 + \frac{abc(a+b+c)}{4F^2} \right)^2$$

$$\geq \frac{(a+b+c)^4}{16F^2} \text{ applying which on a triangle with sides}$$

$$\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3} \text{ whose area of course } = \frac{F}{3}, \text{ we get}$$

$$\therefore 2 + \left( 1 + \frac{\left(\frac{16}{81}\right) m_a m_b m_c (m_a + m_b + m_c)}{4\left(\frac{F^2}{9}\right)} \right)^2 \geq \frac{\left(\frac{16}{81}\right) (m_a + m_b + m_c)^4}{16\left(\frac{F^2}{9}\right)}$$

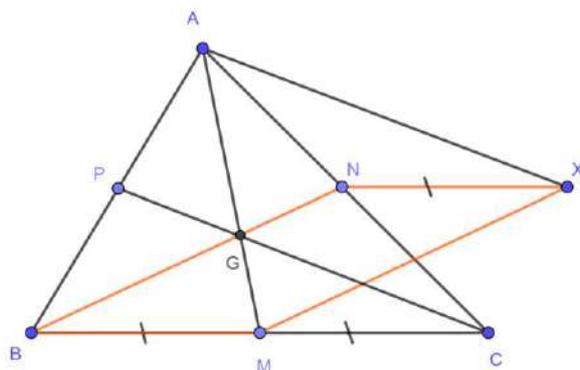
$$\Rightarrow 2 + \left( 1 + \frac{4m_a m_b m_c (m_a + m_b + m_c)}{9F^2} \right)^2 \geq \frac{(m_a + m_b + m_c)^4}{9F^2} \text{ (Proved)}$$

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**Solution 2 by Tran Hong-Dong Thap-Vietnam**



We have:  $MB = MC, PA = PB, BMXN$  –parallelogram, then

$$AM = m_a, BN = MX = m_b, CP = AX = m_c$$

$$\text{Choose: } F \equiv A, D \equiv M, E \equiv X \Rightarrow S_{\Delta DEF} = \frac{3}{4}S_{\Delta ABC}$$

$$R_m = \frac{m_a m_b m_c}{3S_{\Delta ABC}}; r_m = \frac{3S_{\Delta ABC}}{2(m_a + m_b + m_c)} \Rightarrow \frac{R_m}{2r_m} = \frac{1}{9} \cdot \frac{(m_a + m_b + m_c)m_a m_b m_c}{S^2}$$

So, we have inequality:

$$\frac{(m_a + m_b + m_c)^4}{9F^2} \leq 4 \left( \frac{R_m}{r_m} \right)^2 + 4 \cdot \frac{R_m}{r_m} + 3; \quad (1)$$

Proof.  $F = \frac{4}{3}S_{\Delta DEF}$  then (1) becomes as:

$$\frac{(m_a + m_b + m_c)^4}{9 \left( \frac{4}{3}S_{\Delta DEF} \right)^2} \leq 4 \left( \frac{R_m}{r_m} \right)^2 + 4 \cdot \frac{R_m}{r_m} + 3 \Leftrightarrow$$

$$\frac{(m_a + m_b + m_c)^4}{16S_{\Delta DEF}^2} \leq 4 \left( \frac{R_m}{r_m} \right)^2 + 4 \cdot \frac{R_m}{r_m} + 3$$

In any  $\Delta ABC$ , we need to prove:

$$\frac{(a + b + c)^4}{16s^2 r^2} \leq 4 \left( \frac{R}{r} \right)^2 + 4 \cdot \frac{R}{r} + 3 \Leftrightarrow$$

$$\frac{(2s)^4}{16s^2 r^2} \leq 4 \left( \frac{R}{r} \right)^2 + 4 \cdot \frac{R}{r} + 3 \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen)}$$

Now,

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$$2 + \left(1 + \frac{4m_a m_b m_c (m_a + m_b + m_c)}{9F^2}\right)^2 \geq \frac{(m_a + m_b + m_c)^4}{9F^2} \Leftrightarrow$$

$$2 + \left(1 + 4 \cdot \frac{R_m}{2r_m}\right)^2 \geq \frac{(m_a + m_b + m_c)^4}{9F^2}; \quad (2)$$

From (1),(2) we need to prove:

$$2 + (1 + 2t)^2 = 4t^2 + 4t + 3; \left(\because t = \frac{R_m}{2r_m}\right) \Leftrightarrow$$

$$2 + 1 + 4t + 4t^2 = 4t^2 + 4t + 3 \Leftrightarrow 4t^2 + 4t + 3 = 4t^2 + 4t + 3 \Rightarrow (2) \text{ is true.}$$

**1686. In  $\triangle ABC$ ,  $n_a$  – Nagel's cevian,  $g_a$  – Gergonne's cevian the following relationship holds:**

$$\frac{n_a g_a r_a}{h_a h_b h_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \text{Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) = an_a^2 + a(s - b)(s - c) \\ \Rightarrow s(b^2 + c^2) - bc(2s - a) &= an_a^2 + a(s^2 - s(2s - a) + bc) \Rightarrow s(b^2 + c^2) - 2sbc \\ &= an_a^2 + a(as - s^2) \\ \Rightarrow s(b^2 + c^2 - a^2 - 2bc) &= an_a^2 - as^2 \Rightarrow an_a^2 = as^2 + s(2bccosA - 2bc) \\ &= as^2 - 4sbc \sin^2 \frac{A}{2} = as^2 - \frac{4sbc(s - b)(s - c)(s - a)}{bc(s - a)} \\ &= as^2 - \frac{4\Delta^2}{s - a} = as^2 - 2a \left(\frac{2\Delta}{a}\right) \left(\frac{\Delta}{s - a}\right) = as^2 - 2ah_a r_a \therefore n_a^2 \stackrel{(1)}{=} s^2 - 2h_a r_a \end{aligned}$$

$$\begin{aligned} \text{Again, Stewart's theorem} &\Rightarrow b^2(s - c) + c^2(s - b) \\ &= an_a^2 + a(s - b)(s - c) \text{ and } b^2(s - b) + c^2(s - c) \\ &= ag_a^2 + a(s - b)(s - c) \\ &\therefore an_a^2 \cdot ag_a^2 \geq a^2 s^2 (s - a)^2 \\ \Leftrightarrow \{b^2(s - c) + c^2(s - b) - a(s - b)(s - c)\} &\{b^2(s - b) + c^2(s - c) - a(s - b)(s - c)\} \stackrel{(a)}{\geq} a^2 s^2 (s - a)^2 \end{aligned}$$

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Let  $s - a = x, s - b = y$  and  $s - c = z \therefore s = x + y + z \Rightarrow a = y + z, b = z + x$  and  $c = x + y$

Using these substitutions, (a)

$$\Leftrightarrow \{z(z+x)^2 + y(x+y)^2 - yz(y+z)\}\{y(z+x)^2 + z(x+y)^2 - yz(y+z)\} \geq x^2(y+z)^2(x+y+z)^2$$

$$\Leftrightarrow xy^2 + xz^2 + y^3 + z^3 \geq 2xyz + yz(y+z) \Leftrightarrow x(y-z)^2 + (y+z)(y-z)^2 \geq 0 \rightarrow \text{true}$$

$$\Rightarrow \text{(a) is true} \Rightarrow n_a g_a \geq s(s-a)$$

$$\Rightarrow n_a g_a r_a \geq s(s-a) \left( \frac{rs}{s-a} \right) = rs^2 \Rightarrow n_a g_a r_a \geq rs^2 \Rightarrow \frac{n_a g_a r_a}{h_a h_b h_c} \geq \frac{4Rrs \cdot rs^2}{8r^3 s^3} = \frac{R}{2r}$$

$$\Rightarrow \frac{n_a g_a r_a}{h_a h_b h_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq \frac{R}{2r} + \frac{16Rr^2 s^2}{(s^2 + 2Rr + r^2)rs^2}$$

$$= \frac{R}{2r} + \frac{16Rr}{s^2 + 2Rr + r^2} \stackrel{?}{\geq} 2 \Leftrightarrow \frac{R}{2r} \geq \frac{2s^2 - 14Rr + 2r^2}{s^2 + 2Rr + r^2}$$

$$\Leftrightarrow (R-2r)s^2 + R(2Rr+r^2) + 2r(12Rr-2r^2) \stackrel{?}{\geq} 2rs^2 \quad \text{(i)}$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} \underset{\text{(m)}}{(R-2r)(16Rr-5r^2) + R(2Rr+r^2) + 2r(12Rr-2r^2)}$

$- 2r^2)$  and RHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} \underset{\text{(n)}}{2r(4R^2 + 4Rr + 3r^2)}$

(m), (n)  $\Rightarrow$  in order to prove (i), it suffices to prove :

$$(R-2r)(16Rr-5r^2) + R(2Rr+r^2) + 2r(12Rr-2r^2) \geq 2r(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 10r(R-2r) \geq 0 \rightarrow \text{true by Euler} \Rightarrow \text{(i) is true}$$

$$\therefore \frac{n_a g_a r_a}{h_a h_b h_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2 \text{ (Proved)}$$

### Solution 2 by Bogdan Fuștei-Romania

$$4m_a^2 = n_a^2 + g_a^2 + 2r_b r_c \text{ and analogs.}$$

$$4m_a^2 = 4r_b r_c + (b-c)^2 \text{ and analogs.}$$

$$r_b r_c = s(s-a) \text{ and analogs.}$$

$$b^2 + c^2 + n_a^2 + g_a^2 + 2r_a r \text{ and analogs.}$$

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$$n_a^2 = r_b r_c + \frac{(b-c)^2 s}{a} \text{ and analogs.}$$

$$4r_b r_c + (b-c)^2 = r_b r_c + \frac{(b-c)^2 s}{a} + g_a^2 + 2r_b r_c$$

$$4r_b r_c + (b-c)^2 = 3r_b r_c + \frac{(b-c)^2 s}{a} + g_a^2$$

$$r_b r_c + (b-c)^2 = g_a^2 + \frac{(b-c)^2 s}{a} \Rightarrow g_a^2 = s(s-a) - \frac{(s-a)(b-c)^2}{a}$$

$$\text{So, } g_a^2 = (s-a) \left( s - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 = s \left( s - a - \frac{(b-c)^2}{a} \right) \text{ and analogs.}$$

$$n_a^2 g_a^2 = s(s-a) \left( s - \frac{(b-c)^2}{a} \right) \left( s - a - \frac{(b-c)^2}{a} \right) \geq s(s-a)s(s-b)$$

$$s(s-a) - s(s-a) \frac{(s-a)(b-c)^2}{a} - \left[ \frac{(b-c)^2}{a} \right]^2 \geq s(s-a)$$

$$\frac{(b-c)^2}{a} (s-s+a) \geq \left[ \frac{(b-c)^2}{a} \right]^2 \Leftrightarrow \frac{(b-c)^2}{a} \cdot a \geq \left[ \frac{(b-c)^2}{a} \right]^2$$

$$(b-c)^2 \geq \frac{(b-c)^2}{a} \cdot \frac{(b-c)^2}{a}$$

If  $b = c \Rightarrow b - c = 0$  the we have equality.

$$\text{If } b \neq c \Rightarrow 1 > \frac{(b-c)^2}{a^2} \Rightarrow a^2 > (b-c)^2 \Rightarrow a > |b-c|$$

$$\begin{cases} a + b > c \\ a + c > b \end{cases} \Rightarrow \text{true.}$$

So,  $n_a g_a \geq r_b r_c$  and analogs  $\Rightarrow n_a g_a r_a \geq r_a r_b r_c$

$$h_a h_b h_c = \frac{2S}{a} \cdot \frac{2S}{b} \cdot \frac{2S}{c} \stackrel{abc=4RS}{=} \frac{2R}{R} \cdot r_a r_b r_c$$

$$r_a r_b r_c = Ss; S = sr$$

So, we have:

$$\frac{n_a g_a r_a}{h_a h_b h_c} \geq \frac{r_a r_b r_c}{h_a h_b h_c} = \frac{R}{2r}$$

$$w_a = \frac{2\sqrt{bc}}{b+c} \sqrt{r_b r_c} \text{ and analogs, then}$$

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$$w_a w_b w_c = \frac{8abc}{(a+b)(b+c)(c+a)} \cdot r_a r_b r_c \Rightarrow$$

$$\frac{r_a r_b r_c}{w_a w_b w_c} = \frac{(a+b)(b+c)(c+a)}{8abc}$$

But  $h_a \leq w_a$  and analogs  $\Rightarrow h_a h_b h_c \leq w_a w_b w_c \Rightarrow \frac{h_a h_b h_c}{r_a r_b r_c} \leq \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow$

$$\frac{2r}{R} \leq \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow \frac{R}{2r} \geq \frac{r_a r_b r_c}{w_a w_b w_c} = \frac{(a+b)(b+c)(c+a)}{8abc}$$

$$(x-1)^2 \geq 0, \forall x \in \mathbb{R} \Rightarrow x^2 + 1 \geq 2x \Rightarrow x + \frac{1}{x} \geq 2$$

$$x = \frac{r_a r_b r_c}{w_a w_b w_c}; \frac{1}{x} = \frac{w_a w_b w_c}{r_a r_b r_c} \Rightarrow \frac{r_a r_b r_c}{w_a w_b w_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2 \Leftrightarrow$$

$$\frac{R}{2r} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2 \Leftrightarrow \frac{n_a g_a r_a}{h_a h_b h_c} + \frac{w_a w_b w_c}{r_a r_b r_c} \geq 2$$

**1687. In  $\triangle ABC$  the following relationship holds:**

$$1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right)$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by Bogdan Fuștei-Romania**

$$1) \frac{1}{2} R s^2 \geq m_a m_b m_c$$

$$2) r_a r_b r_c = S s = s^2 r$$

$$3) \sin^2 \frac{A}{2} = \frac{r_a - r}{4R} = \frac{R}{2R} \cdot \frac{r_a}{h_a}$$

$$4) abc = 4RS = 4Rrs$$

$$5) 2S = ah_a = bh_b = ch_c$$

$$6) r_a + r_b + r_c = 4R + r$$

$$\text{We have: } \frac{1}{2} R s^2 \cdot \frac{1}{r_a r_b r_c} \geq \frac{m_a m_b m_c}{r_a r_b r_c}$$

$$\frac{1}{2} R s^2 \cdot \frac{1}{s^2 r} \geq \frac{m_a m_b m_c}{r_a r_b r_c} \Rightarrow \frac{r}{2r} \geq \frac{m_a m_b m_c}{r_a r_b r_c}; (1)$$

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$$\sum_{cyc} \sin^2 \frac{A}{2} = \frac{4R + r - 3r}{4R} = \frac{4R - 2r}{4R} = \frac{2R - r}{2R}$$

$$\frac{2R - r}{2R} = \frac{r}{2R} \cdot \sum_{cyc} \frac{r_a}{h_a} \Rightarrow \sum_{cyc} \frac{r_a}{h_a} = \frac{2R}{r} - 1 \Rightarrow 1 + \sum_{cyc} \frac{r_a}{h_a} = \frac{2R}{r}; (2)$$

$$4S \cdot 2S^2 = h_a h_b h_c \cdot abc = h_a h_b h_c \cdot 4RS \Rightarrow h_a h_b h_c = \frac{2S^2}{R}$$

$$\text{But } S = sr \Rightarrow S^2 = s^2 r^2 = Ssr = rr_a r_b r_c \Rightarrow h_a h_b h_c = \frac{2r}{R} \cdot r_a r_b r_c$$

$$\frac{R}{2r} = \frac{r_a r_b r_c}{h_a h_b h_c}$$

$$\text{But } w_a \leq \sqrt{s(s-a)} = \sqrt{r_b r_c} \text{ and analogs, then } w_a w_b w_c \leq r_a r_b r_c$$

$$\frac{R}{2r} \geq \frac{w_a w_b w_c}{h_a h_b h_c}; (3)$$

From (1), (2), (3) we get:

$$1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right)$$

### Solution 2 by Soumava Chakraborty-Kolkata-India

$$m_a^2 m_b^2 m_c^2 = \frac{1}{64} (2b^2 + 2c^2 - 2a^2)(2c^2 + 2a^2 - 2b^2)(2a^2 + 2b^2 - 2c^2) \stackrel{(1)}{=} \frac{1}{64} \{-4\sum a^6 + 6(\sum a^4 b^2 + \sum a^2 b^4) + 3a^2 b^2 c^2\}$$

$$\text{Now, } \sum a^6 = (\sum a^2)^3 - 3(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)$$

$$= (\sum a^2)^3 - 3(2a^2 b^2 c^2 + \sum a^2 b^2 (\sum a^2 - c^2))$$

$$= (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2 \therefore \sum a^6 \stackrel{(2)}{=} (\sum a^2)^3 + 3a^2 b^2 c^2 - 3(\sum a^2 b^2) \sum a^2$$

$$\text{Again, } \sum a^4 b^2 + \sum a^2 b^4 = \sum a^2 b^2 (\sum a^2 - c^2) \stackrel{(3)}{=} (\sum a^2 b^2) \sum a^2 - 3a^2 b^2 c^2$$

$$\therefore (1), (2), (3) \Rightarrow m_a^2 m_b^2 m_c^2$$

$$= \frac{1}{64} \{-4(\sum a^2)^3 - 12a^2 b^2 c^2 + 12(\sum a^2 b^2) \sum a^2 + 6(\sum a^2 b^2) \sum a^2$$

$$- 18a^2 b^2 c^2 + 3a^2 b^2 c^2\}$$

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$$\begin{aligned}
 &= \frac{1}{64} \{-4(\sum a^2)^3 + 18(\sum a^2 b^2) \sum a^2 - 27a^2 b^2 c^2\} \\
 &= \frac{1}{64} \{-4(\sum a^2)^3 + 18((\sum ab)^2 - 2abc(2s))(\sum a^2) - 27a^2 b^2 c^2\} \\
 &= \frac{1}{64} \{-32(s^2 - 4Rr - r^2)^3 + 36(s^2 - 4Rr - r^2)(s^2 + 4Rr + r^2)^2 \\
 &\quad - 576Rrs^2(s^2 - 4Rr - r^2) - 432R^2 r^2 s^2\} \\
 &= \frac{1}{16} \{s^6 - s^4(12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3\} \leq \frac{R^2 s^4}{4}
 \end{aligned}$$

$$\Leftrightarrow s^6 - s^4(4R^2 + 12Rr - 33r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{(i)}{\leq} 0$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} -s^4(8Rr - 36r^2) - s^2(60R^2 r^2 + 120Rr^3 + 33r^4) - r^3(4R + r)^3 \stackrel{?}{\geq} 0$

$$\Leftrightarrow s^4(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3 \stackrel{?}{\stackrel{(ii)}{\geq}} 20rs^4$$

Now, LHS of (ii)  $\stackrel{\text{Gerretsen}}{\stackrel{(a)}{\geq}} s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) + r^2(4R + r)^3$

and RHS of (ii)  $\stackrel{\text{Gerretsen}}{\stackrel{(b)}{\geq}} 20rs^2(4R^2 + 4Rr + 3r^2)$

(a), (b)  $\Rightarrow$  in order to prove (ii), it suffices to prove

$$\begin{aligned}
 &: s^2(16Rr - 5r^2)(8R - 16r) + s^2(60R^2 r + 120Rr^2 + 33r^3) \\
 &+ r^2(4R + r)^3
 \end{aligned}$$

$$\geq 20rs^2(4R^2 + 4Rr + 3r^2) \Leftrightarrow s^2(108R^2 - 256Rr + 53r^2) + r(4R + r)^3 \geq 0$$

$$\Leftrightarrow s^2(108R^2 - 256Rr + 80r^2) + r(4R + r)^3 \stackrel{(iii)}{\geq} 27r^2 s^2$$

Now, LHS of (iii)  $\stackrel{\text{Gerretsen}}{\stackrel{(c)}{\geq}} (108R^2 - 256Rr + 80r^2)(16Rr - 5r^2)$

+  $r(4R + r)^3$  and RHS of (iii)  $\stackrel{\text{Gerretsen}}{\stackrel{(d)}{\geq}} 27r^2(4R^2 + 4Rr + 3r^2)$

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(c), (d)  $\Rightarrow$  in order to prove (iii), it suffices to prove :

$$(108R^2 - 256Rr + 80r^2)(16Rr - 5r^2) + r(4R + r)^3 \geq 27r^2(4R^2 + 4Rr + 3r^2)$$

$$\Leftrightarrow 224t^3 - 587t^2 + 308t - 60 \geq 0 \quad \left(\text{where } t = \frac{R}{r}\right)$$

$$\Leftrightarrow (t - 2)\{(t - 2)(224t + 309) + 648\} \geq 0 \rightarrow \text{true} \because t \stackrel{\text{Euler}}{\geq} 2 \Rightarrow \text{(iii)} \Rightarrow \text{(ii)}$$

$$\Rightarrow \text{(i) is true} \Rightarrow m_a^2 m_b^2 m_c^2 \leq \frac{R^2 s^4}{4} \Rightarrow \frac{m_a m_b m_c}{r_a r_b r_c} \leq \frac{R s^2}{2rs^2}$$

$$\Rightarrow \frac{2m_a m_b m_c}{r_a r_b r_c} \leq \frac{R}{r} \Rightarrow \frac{2m_a m_b m_c}{r_a r_b r_c} + \frac{2w_a w_b w_c}{h_a h_b h_c} \leq \frac{R}{r} + \frac{32Rr^2 s^2 \cdot 4Rrs}{(s^2 + 2Rr + r^2) \cdot 8r^3 s^3}$$

$$= \frac{R}{r} + \frac{16R^2}{s^2 + 2Rr + r^2} \Rightarrow \text{RHS} \stackrel{(4)}{\geq} \frac{R}{r} + \frac{16R^2}{s^2 + 2Rr + r^2}$$

$$\text{Now, } 1 + \sum \frac{r_a}{h_a} = 1 + \sum \frac{4R \sin \frac{A}{2} \cos \frac{A}{2} \tan \frac{A}{2}}{2rs} = 1 + \frac{R}{r} \sum 2 \sin^2 \frac{A}{2}$$

$$= 1 + \frac{R}{r} \sum (1 - \cos A) = 1 + \frac{R}{r} \left( \frac{2R - r}{R} \right) = 1 + \frac{2R - r}{r} \stackrel{(5)}{=} \frac{2R}{r}$$

$$(4), (5) \Rightarrow \text{it suffices to prove : } \frac{2R}{r} \geq \frac{R}{r} + \frac{16R^2}{s^2 + 2Rr + r^2} \Leftrightarrow \frac{1}{r} \geq \frac{16R}{s^2 + 2Rr + r^2}$$

$$\Leftrightarrow s^2 - 14Rr + r^2 \geq 0$$

$$\Leftrightarrow s^2 - 16Rr + 5r^2 + 2r(R - 2r) \geq 0 \rightarrow \text{true}$$

$$\because s^2 - 16Rr + 5r^2 \stackrel{\text{Gerretsen}}{\geq} 0 \text{ and } 2r(R - 2r) \stackrel{\text{Euler}}{\geq} 0$$

$$\therefore 1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} \geq 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right) \text{ (Proved)}$$

### Solution 3 by Tran Hong-Dong Thap-Vietnam

$$\text{We know inequality: } \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \leq \frac{R}{r}$$

So,

$$\text{RHS} = 2 \left( \frac{m_a m_b m_c}{r_a r_b r_c} + \frac{w_a w_b w_c}{h_a h_b h_c} \right) \stackrel{(*)}{\leq} \frac{2R}{r}$$

$$\text{LHS} = 1 + \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = 1 + \frac{S}{s-a} + \frac{S}{s-b} + \frac{S}{s-c} = 1 + \frac{2S}{a} + \frac{2S}{b} + \frac{2S}{c} =$$

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$$\begin{aligned}
 &= 1 + \frac{1}{2} \left( \frac{a}{s-a} + \frac{b}{s-b} + \frac{c}{s-c} \right) = \\
 &= 1 + \frac{1}{2} \cdot \frac{a(s-b)(s-c) + b(s-a)(s-c) + c(s-a)(s-b)}{(s-a)(s-b)(s-c)} = \\
 &= 1 + \frac{1}{2} \cdot \frac{(a+b+c)s^2 - 2(ab+bc+ca)s + 3abc}{sr^2} = \\
 &= 1 + \frac{1}{2} \cdot \frac{2s(s^2 - s^2 - 4Rr - r^2 + 6Rr)}{sr^2} = \\
 &= 1 + \frac{2Rr - r^2}{r} = 1 + \frac{2R}{r} - 1 = \frac{2R}{r} \stackrel{(*)}{\geq} RHS
 \end{aligned}$$

1688. In  $\triangle ABC$  the following relationship holds:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{2(m_a m_b + m_b m_c + m_c m_a)}{m_a^2 + m_b^2 + m_c^2} \geq 5$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

**Solution 1 by Bogdan Fuștei-Romania**

$$\frac{R}{2r} \geq \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \Rightarrow \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} \leq \frac{2r}{R}$$

Well known that:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \geq \frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c}$$

$$\sin^2 \frac{A}{2} = \frac{r}{2R} \cdot \frac{r_a}{h_a} = \frac{r_a - r}{4R} \Rightarrow \frac{r_a}{h_a} = \frac{r_a - r}{2r} \text{ and analogs, then}$$

$$\frac{r_a}{h_a} + \frac{r_b}{h_b} + \frac{r_c}{h_c} = \frac{r_a - r + r_b - r + r_c - r}{2r} = \frac{4R + r - 3r}{2r} = \frac{4R - 2r}{2r} = \frac{2R}{r} - 1$$

$$\text{So, we have: } \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} \geq \frac{2R}{r} - 1; (1)$$

$$\text{From } x + \frac{1}{x} \geq 2, \forall x > 0 \text{ and } x = \frac{R}{2r} \Rightarrow \frac{R}{2r} + \frac{r}{2R} \geq 2 \Rightarrow \frac{2R}{r} + \frac{8r}{R} \geq 8$$

$$\frac{2R}{r} - 1 + \frac{4r}{R} + \frac{4r}{R} \geq 7 \Rightarrow \frac{2R}{r} - 1 + \frac{4r}{R} \geq 7 - \frac{4r}{R}; (2)$$

From (1), (2) we have:

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{4r}{R} \geq 7 - \frac{4r}{R} \text{ and from } \frac{2(m_a m_b + m_b m_c + m_c m_a)}{m_a^2 + m_b^2 + m_c^2} \geq \frac{4r}{R} \text{ we get:}$$

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$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{2(m_a m_b + m_b m_c + m_c m_a)}{m_a^2 + m_b^2 + m_c^2} \geq 7 - \frac{4r}{R}; \quad (3)$$

We must show that:  $7 - \frac{4r}{R} \geq 5 \Leftrightarrow 7 - 5 \geq \frac{4r}{R} \Leftrightarrow 2R \geq 4r \Leftrightarrow R \geq 2r$  true by Euler.

So,

$$\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + \frac{2(m_a m_b + m_b m_c + m_c m_a)}{m_a^2 + m_b^2 + m_c^2} \geq 5$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \sum m_a m_b &\stackrel{\text{Tereshin}}{\geq} \sum \frac{(b^2 + c^2)(c^2 + a^2)}{16R^2} = \frac{3 \sum a^2 b^2 + \sum a^4}{16R^2} \geq \frac{3 \sum a^2 b^2 + \sum a^2 b^2}{16R^2} \\ &= \sum \frac{b^2 c^2}{4R^2} = \sum h_a^2 \\ \Rightarrow \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} &\leq \frac{\sum m_a^2}{\sum h_a^2} = \frac{\frac{3 \sum a^2}{4}}{\sum \left( \frac{b^2 c^2}{4R^2} \right)} = \frac{3R^2 \sum a^2}{\sum a^2 b^2} \stackrel{?}{\geq} \frac{R}{2r} \\ &\Leftrightarrow \sum a^2 b^2 \stackrel{?}{\geq} 6Rr \sum a^2 \end{aligned}$$

Let  $s - a = x, s - b = y$  and  $s - c = z \therefore 3s - 2s = \sum x \Rightarrow a = y + z, b = z + x$  and  $c = x + y$

$$\begin{aligned} \therefore (p) &\Leftrightarrow \sum (y + z)^2 (z + x)^2 \geq 6 \frac{\prod (y + z)}{4F} \cdot \frac{F}{\sum x} \cdot \sum (y + z)^2 \\ &\Leftrightarrow 2(\sum x) \left\{ \sum (y + z)^2 (z + x)^2 \right\} \geq 3 \left\{ \prod (y + z) \right\} \left\{ \sum (y + z)^2 \right\} \\ &\Leftrightarrow \sum x^5 + xyz \sum xy - \sum x^3 y^2 - \sum x^2 y^3 \stackrel{(ii)}{\geq} 0 \end{aligned}$$

Now,  $F(x, y, z)$

$$\begin{aligned} &= \sum x^5 + xyz \sum xy - \sum x^3 y^2 \\ &- \sum x^2 y^3 \text{ is a homogeneous and symmetric polynomial} \end{aligned}$$

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$$F(x, y, 0) = x^5 + y^5 - x^3y^2 - x^2y^3 \stackrel{\text{Chebyshev}}{\geq} \frac{1}{2}(x^3 + y^3)(x^2 + y^2) - x^3y^2$$

$$\begin{aligned} & - x^2y^3 \stackrel{\text{A-G}}{\geq} xy(x^3 + y^3) - x^3y^2 - x^2y^3 \\ & \geq x^2y^2(x + y) - x^3y^2 - x^2y^3 = 0 \Rightarrow \boxed{F(x, y, 0) \geq 0} \end{aligned}$$

$$\begin{aligned} F(x, 1, 1) &= x^5 + 1 + 1 + x(x + 1 + x) - x^3 - 1 - x^2 - x^2 - 1 - x^3 = x^5 - 2x^3 + x \\ &= x(x^2 - 1)^2 \geq 0 \Rightarrow \boxed{F(x, 1, 1) \geq 0} \end{aligned}$$

$$\therefore \text{by SD5 theorem, } F(x, y, z) \geq 0 \Rightarrow \text{(ii)} \Rightarrow \text{(i) is true} \Rightarrow \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \leq \frac{R}{2r}$$

$$\Rightarrow \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} \geq \frac{2r}{R}$$

$$\Rightarrow \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 2 \left( \frac{m_a m_b + m_b m_c + m_c m_a}{m_a^2 + m_b^2 + m_c^2} \right) \geq \frac{y}{x} + \frac{x}{z} + \frac{z}{y} + \left( \frac{4S}{s} \right) \left( \frac{4S}{abc} \right)$$

$$= \frac{y+x}{x} + \frac{x+z}{z} + \frac{z+y}{y} + \frac{16xyz}{(x+y)(y+z)(z+x)} - 3$$

$$\stackrel{\text{A-G}}{\geq} 4 \sqrt[4]{\frac{16xyz(x+y)(y+z)(z+x)}{xyz(x+y)(y+z)(z+x)}} - 3 = 5 \text{ (Proved)}$$

1689. In  $\triangle ABC$  the following relationship holds:

$$3 \cdot \sqrt{\frac{2r}{R}} \leq \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq \left( \frac{R}{r} \right)^2 - 1$$

Proposed by Marin Chirciu-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$\text{Let } \Omega = \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \stackrel{\text{AGM}}{\geq} 3 \sqrt[3]{\frac{h_a h_b h_c}{w_a w_b w_c}} \stackrel{w_a \leq \sqrt{s(s-a)}}{\geq} 3 \sqrt[3]{\frac{h_a h_b h_c}{s \sqrt{s(s-a)(s-b)(s-c)}}} =$$

$$= 3 \sqrt[3]{\frac{2s^2 r^2}{s \cdot sr}} = 3 \sqrt[3]{\frac{2r}{R}} \stackrel{(*)}{\geq} 3 \sqrt{\frac{2r}{R}}$$

$$(*) \Leftrightarrow \left( \frac{2r}{R} \right)^2 \geq \left( \frac{2r}{R} \right)^3 \Leftrightarrow 1 \geq \frac{2r}{R} \Leftrightarrow R \geq 2r \text{ (Euler)}$$

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$$h_a \leq w_a; h_b \leq w_b; h_c \leq w_c \Rightarrow$$

$$\Omega = \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq 1 + 1 + 1 = 3 \stackrel{(**)}{\leq} \left(\frac{R}{r}\right)^2 - 1$$

$$(**) \Leftrightarrow 4 \leq \left(\frac{R}{r}\right)^2 \Leftrightarrow 4r^2 \leq R^2 \Leftrightarrow 2r \leq R \text{ (Euler)}$$

1690. In  $\triangle ABC$  the following relationship holds:

$$3 \cdot \sqrt{\frac{2r}{R}} \leq \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq \left(\frac{R}{r}\right)^2 - 1$$

Proposed by Marin Chirciu-Romania

**Solution 1 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \text{Let } \Omega &= \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \stackrel{AGM}{\geq} 3 \sqrt[3]{\frac{h_a h_b h_c}{w_a w_b w_c}} \stackrel{w_a \leq \sqrt{s(s-a)}}{\geq} 3 \sqrt[3]{\frac{h_a h_b h_c}{s \sqrt{s(s-a)(s-b)(s-c)}}} = \\ &= 3 \sqrt[3]{\frac{2s^2 r^2}{s \cdot sr}} = 3 \sqrt[3]{\frac{2r}{R}} \stackrel{(*)}{\geq} 3 \sqrt{\frac{2r}{R}} \end{aligned}$$

$$(*) \Leftrightarrow \left(\frac{2r}{R}\right)^2 \geq \left(\frac{2r}{R}\right)^3 \Leftrightarrow 1 \geq \frac{2r}{R} \Leftrightarrow R \geq 2r \text{ (Euler).}$$

$$h_a \leq w_a; h_b \leq w_b; h_c \leq w_c \Rightarrow$$

$$\Omega = \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq 1 + 1 + 1 = 3 \stackrel{(**)}{\leq} \left(\frac{R}{r}\right)^2 - 1$$

$$(**) \Leftrightarrow 4 \leq \left(\frac{R}{r}\right)^2 \Leftrightarrow 4r^2 \leq R^2 \Leftrightarrow 2r \leq R \text{ (Euler). Proved.}$$

**Solution 2 by Soumava Chakraborty-Kolkata-India**

$$r_b + r_c = s \left( \frac{\sin \frac{B}{2}}{\cos \frac{B}{2}} + \frac{\sin \frac{C}{2}}{\cos \frac{C}{2}} \right) = \frac{s \sin \left( \frac{B+C}{2} \right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}} = \frac{s \cos^2 \frac{A}{2}}{\left( \frac{s}{4R} \right)} = 4R \cos^2 \frac{A}{2}$$

$$\therefore r_b + r_c \stackrel{(i)}{=} 4R \cos^2 \frac{A}{2}$$

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$$\begin{aligned}
 \text{Now, } (b+c)^2 &\geq 32Rr\cos^2\frac{A}{2} \stackrel{\text{by (i)}}{=} 8r(r_b+r_c) = 8r^2s\left(\frac{1}{s-b} + \frac{1}{s-c}\right) \\
 &= 8(s-a)(s-b)(s-c)\frac{a}{(s-b)(s-c)} = 4a(b+c-a) \\
 \Leftrightarrow (b+c)^2 + 4a^2 - 4a(b+c) &\geq 0 \Leftrightarrow (b+c-2a)^2 \geq 0 \rightarrow \text{true} \therefore b+c \\
 &\geq 4\sqrt{2Rr}\cos\frac{A}{2} \Rightarrow \frac{h_a}{w_a} = \frac{2rs(b+c)}{2bcc\cos\frac{A}{2}\cdot a} \geq \frac{rs\cdot 4\sqrt{2Rr}\cos\frac{A}{2}}{4Rrs\cdot\cos\frac{A}{2}} \\
 &= \sqrt{\frac{2r}{R}} \Rightarrow \frac{h_a}{w_a} \geq \sqrt{\frac{2r}{R}} \text{ and analogs} \Rightarrow \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \geq 3\sqrt{\frac{2r}{R}} \text{ and } \therefore \frac{h_a}{w_a} \\
 &\leq 1 \text{ and analogs} \therefore \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq 3 = 4 - 1 \stackrel{\text{Euler}}{\cong} \left(\frac{R}{r}\right)^2 - 1 \\
 \therefore 3\sqrt{\frac{2r}{R}} &\leq \frac{h_a}{w_a} + \frac{h_b}{w_b} + \frac{h_c}{w_c} \leq \left(\frac{R}{r}\right)^2 - 1 \text{ (Proved)}
 \end{aligned}$$

**1691. In any  $\triangle ABC$  the following relationship holds:**

$$\frac{1}{1 - \sin^2\frac{A}{2}\sin^2\frac{B}{2}} + \frac{1}{1 - \sin^2\frac{B}{2}\sin^2\frac{C}{2}} + \frac{1}{1 - \sin^2\frac{C}{2}\sin^2\frac{A}{2}} \leq \frac{36R^2}{11R^2 + Rr - r^2}$$

*Proposed by Nguyen Van Canh-Ben Tre-Vietnam*

*Solution by Tran Hong-Dong Thap-Vietnam*

**Lemma:** If  $0 < x, y, z < 1$  then:

$$\frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{3}{1 - \frac{xy+yz+zx}{3}}$$

**Proof:** Let  $\varphi(x) = \frac{1}{1-t}$ ; ( $0 < t < 1$ )

$$\varphi'(t) = \frac{1}{(1-t)^2}; (0 < t < 1), \varphi''(t) = -\frac{2}{(1-t)^3} < 0; (0 < t < 1)$$

$$\varphi(xy) + \varphi(yz) + \varphi(zx) \stackrel{\text{Jensen}}{\leq} 3\varphi\left(\frac{xy+yz+zx}{3}\right) \Leftrightarrow$$

$$\frac{1}{1-xy} + \frac{1}{1-yz} + \frac{1}{1-zx} \leq \frac{3}{1 - \frac{xy+yz+zx}{3}}$$

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Now, choose:  $x = \sin^2 \frac{A}{2}$ ;  $y = \sin^2 \frac{B}{2}$ ;  $z = \sin^2 \frac{C}{2} \Rightarrow$

$$\begin{aligned} LHS &= \sum_{cyc} \frac{1}{1 - \sin^2 \frac{A}{2} \sin^2 \frac{B}{2}} \leq \frac{3}{1 - \frac{\sum_{cyc} \sin^2 \frac{A}{2} \sin^2 \frac{B}{2}}{3}} = \\ &= \frac{3}{1 - \frac{\frac{s^2 + r^2 - 8Rr}{16R^2}}{3}} = \frac{3}{1 - \frac{s^2 + r^2 - 8Rr}{48R^2}} = \\ &= \frac{48 \cdot 3 \cdot R^2}{48R^2 + 8Rr - s^2 - r^2} \stackrel{(*)}{\leq} \frac{36R^2}{11R^2 + Rr - r^2} \end{aligned}$$

$$(*) \Leftrightarrow 4(11R^2 + Rr - r^2) \leq 48R^2 + 8Rr - s^2 - r^2 \Leftrightarrow$$

$s^2 \leq 4R^2 + 4Rr + 3r^2$  true by Gerretsen inequality  $\Rightarrow (*)$  is true. Proved.

**1692. In  $\triangle ABC$  the following relationship holds:**

$$\left( \frac{m_a m_b}{m_a + m_b} \right)^2 + \left( \frac{m_b m_c}{m_b + m_c} \right)^2 + \left( \frac{m_c m_a}{m_c + m_a} \right)^2 \geq \frac{27r^2}{4}$$

*Proposed by Marian Ursărescu-Romania*

*Solution by Tran Hong-Dong Thap-Vietnam*

Let:  $x = m_a, y = m_b, z = m_c; x, y, z > 0 \Rightarrow$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \leq \frac{1}{h_a} + \frac{1}{h_b} + \frac{1}{h_c} = \frac{1}{r}$$

$$\text{Now, } \left( \frac{xy}{x+y} \right)^2 + \left( \frac{yz}{y+z} \right)^2 + \left( \frac{zx}{z+x} \right)^2 = \left( \frac{1}{\frac{1}{x} + \frac{1}{y}} \right)^2 + \left( \frac{1}{\frac{1}{y} + \frac{1}{z}} \right)^2 + \left( \frac{1}{\frac{1}{z} + \frac{1}{x}} \right)^2 \stackrel{CBS}{\geq}$$

$$\geq \frac{1}{3} \left( \frac{1}{\frac{1}{x} + \frac{1}{y}} + \frac{1}{\frac{1}{y} + \frac{1}{z}} + \frac{1}{\frac{1}{z} + \frac{1}{x}} \right)^2 \stackrel{\text{Bergstrom}}{\geq} \frac{1}{3} \left( \frac{9}{2 \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)} \right)^2 \geq \frac{27}{r^2} = \frac{27r^2}{4}$$

**1693. In  $\triangle ABC$  the following relationship holds:**

$$\left( \frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^2 + m_b^2 + m_c^2}{4m_a m_b m_c} \geq$$

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$$\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Bogdan Fuștei-Romania*

$$\text{From } \frac{1}{2} R s^2 \geq m_a m_b m_c \text{ we have: } \frac{R}{2r} \geq \frac{m_a m_b m_c}{r_a r_b r_c}$$

$$4R + r \geq m_a + m_b + m_c$$

$$\frac{9}{2} R \geq 4R + r \Leftrightarrow R \geq 2r \text{ (Euler)} \Rightarrow \frac{R}{2r} \geq \frac{m_a + m_b + m_c}{9r}$$

$$\left( \frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9r r_a r_b r_c} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}; \quad (1)$$

We know that:

$$\frac{R}{r} \geq \frac{abc + a^3 + b^3 + c^3}{2abc}$$

$$\frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \left( \frac{x^2 + y^2 + z^2}{xy + yz + zx} \right)^2, \forall x, y, z > 0$$

Let the triangle  $m_a m_b m_c$  with  $R_m, r_m, S_m$

$$S_m = \frac{3}{4} S; \text{ (} S \text{ - area of } \triangle ABC \text{); } abc = 4RS \Rightarrow m_a m_b m_c = 4R_m S_m$$

$$m_a m_b m_c = 4R_m \cdot \frac{3}{4} S \Rightarrow R_m = \frac{m_a m_b m_c}{3S}$$

$$S_m = s_m \cdot r_m; s_m = \frac{m_a + m_b + m_c}{3} \Rightarrow \frac{3}{4} S = \frac{(m_a + m_b + m_c) \cdot r_m}{2} \Rightarrow$$

$$r_m = \frac{3S}{2(m_a + m_b + m_c)} \Rightarrow 2r_m = \frac{3S}{m_a + m_b + m_c}$$

$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c}{3S} \cdot \frac{m_a + m_b + m_c}{3S} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

$$\frac{R}{2r} \geq \frac{1}{4} + \frac{a^3 + b^3 + c^3}{4abc} \geq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2$$

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \geq$$

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$$\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2; (2)$$

From (1),(2) we get:

$$\begin{aligned} \left( \frac{R}{2r} \right)^2 &\geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^2 + m_b^2 + m_c^2}{4m_a m_b m_c} \geq \\ &\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2 \end{aligned}$$

1694. In  $\triangle ABC$  the following relationship holds:

$$\left( \frac{R}{r} \right)^2 \geq 2 \left( \frac{m_a}{m_b} + \frac{m_b}{m_a} \right)$$

Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Marian Ursărescu-Romania

In any  $\triangle ABC$  we have:  $\frac{m_a}{h_a} \leq \frac{R}{2r}$  (Panaitopol inequality)

$$2m_a \leq \frac{R}{r} \cdot h_a \Rightarrow \frac{2m_a}{m_b} \leq \frac{R}{r} \cdot \frac{h_a}{m_b} \leq \frac{R}{r} \cdot \frac{h_a}{h_b} = \frac{R}{r} \cdot \frac{b}{a}; (m_b \geq m_a) \Rightarrow$$

$$\frac{2m_a}{m_b} \leq \frac{R}{r} \cdot \frac{b}{a} \text{ and simillary } \frac{2m_b}{m_a} \leq \frac{R}{r} \cdot \frac{a}{b} \Rightarrow 2 \left( \frac{m_a}{m_b} + \frac{m_b}{m_a} \right) \leq \frac{R}{r} \left( \frac{a}{b} + \frac{b}{a} \right); (1)$$

$$\text{But in any } \triangle ABC \text{ we have: } \frac{a}{b} + \frac{b}{a} \leq \frac{R}{r}; (2)$$

$$\text{From (1),(2) we get: } 2 \left( \frac{m_a}{m_b} + \frac{m_b}{m_a} \right) \leq \left( \frac{R}{r} \right)^2$$

1695. In any  $\triangle ABC, \triangle A'B'C'$  the following relationship holds:

$$\sum_{cyc} (a^2 + a'^2) + 2 \cdot \sqrt{\left( \sum_{cyc} a^2 \right) \left( \sum_{cyc} a'^2 \right)} \geq 36(r + r')^2$$

Proposed by Daniel Sitaru-Romania

Solution 1 by Adrian Popa-Romania

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$$\begin{aligned}
 \left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a'^2\right) &= (a^2 + b^2 + c^2)(a'^2 + b'^2 + c'^2) \stackrel{CBS}{\geq} \\
 &\geq (aa' + bb' + cc')^2 \Rightarrow \\
 \sum_{cyc} (a^2 + a'^2) + 2 \cdot \sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a'^2\right)} &\geq \\
 \geq a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2 + 2aa' + 2bb' + 2cc' &= \\
 = (a + a')^2 + (b + b')^2 + (c + c')^2 &\stackrel{Bergstrom}{\geq} \\
 \geq \frac{(a + b + c + a' + b' + c')^2}{3} = \frac{(2s + 2s')^2}{3} = \frac{4(s + s')^2}{3} &\stackrel{Mitrinovic}{\geq} \\
 \geq \frac{4(3\sqrt{3}r + 3\sqrt{3}r')^2}{3} = \frac{4 \cdot 9 \cdot 3(r + r')^2}{3} &= 36(r + r')^2
 \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

In any  $\triangle ABC, \triangle A'B'C'$ :

$$a^2 + b^2 + c^2 \geq 4\sqrt{3}S = 4\sqrt{3}sr \stackrel{s \geq 3\sqrt{3}r}{\geq} 36r^2$$

$$a'^2 + b'^2 + c'^2 \geq 4\sqrt{3}S' = 4\sqrt{3}s'r' \stackrel{s' \geq 3\sqrt{3}r'}{\geq} 36r'^2$$

$$\begin{aligned}
 \sum_{cyc} (a^2 + a'^2) + 2 \cdot \sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a'^2\right)} &= \sum_{cyc} a^2 + \sum_{cyc} a'^2 + 2 \cdot \sqrt{\left(\sum_{cyc} a^2\right)\left(\sum_{cyc} a'^2\right)} \\
 &\geq 36r^2 + 36r'^2 + 2\sqrt{36r^2 \cdot 36r'^2} = 36(r^2 + r'^2 + 2rr') = 36(r + r')^2
 \end{aligned}$$

**1696. In  $\triangle ABC$  the following relationship holds:**

$$\begin{aligned}
 \left(\frac{R}{2r}\right)^2 &\geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \geq \\
 &\geq \left(\frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a}\right)^2
 \end{aligned}$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

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**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\sum xy \leq \sum x^2 \Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \leq \frac{m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2}{9S^2}$$

$$= \frac{\left(\frac{9}{16}\right) \sum a^2 b^2 \text{ Goldstone } \left(\frac{9}{16}\right) 4R^2 s^2}{9S^2} \stackrel{\geq}{=} \frac{\left(\frac{9}{16}\right) 4R^2 s^2}{9r^2 s^2} = \left(\frac{R}{2r}\right)^2$$

$$\therefore \left(\frac{R}{2r}\right)^2 \stackrel{(1)}{\geq} \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

$$\text{Now, } \frac{R}{2r} \geq \frac{1}{4} + \frac{\sum a^3}{4abc} = \frac{4Rrs + 2s(s^2 - 6Rr - 3r^2)}{16Rrs} = \frac{s^2 - 4Rr - 3r^2}{8Rr}$$

$$\Leftrightarrow s^2 - 4Rr - 3r^2 \leq 4R^2 \Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \rightarrow \text{true}$$

$$\text{(Gerretsen)} \therefore \frac{abc(a+b+c)}{16S^2} \geq$$

$\frac{1}{4} + \frac{\sum a^3}{4abc}$  applying which on a triangle with sides  $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$  whose area of course =  $\frac{S}{3}$

$$\text{we get : } \frac{\frac{8}{27} m_a m_b m_c \left\{ \frac{2}{3} (m_a + m_b + m_c) \right\}}{16 \left( \frac{S^2}{9} \right)} \geq \frac{1}{4} + \frac{\frac{8}{27} (m_a^3 + m_b^3 + m_c^3)}{4 \left( \frac{8}{27} \right) m_a m_b m_c}$$

$$\Rightarrow \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c}$$

$$\therefore \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \stackrel{(2)}{\geq} \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c}$$

$$\text{Again, } \frac{1}{4} + \frac{\sum a^3}{4abc} \geq \left( \frac{\sum a^2}{\sum ab} \right)^2 \Leftrightarrow \frac{1}{4} + \frac{3abc + (\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \left( \frac{\sum a^2}{\sum ab} \right)^2$$

$$\Leftrightarrow \frac{(\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \left( \frac{\sum a^2}{\sum ab} \right)^2 - 1$$

$$\Leftrightarrow \frac{(\sum a)(\sum a^2 - \sum ab)}{4abc} \geq \frac{(\sum a^2 - \sum ab)(\sum a^2 + \sum ab)}{(\sum ab)^2}$$

$$\Leftrightarrow \left( \sum a^2 - \sum ab \right) \left\{ \frac{\sum a}{4abc} - \frac{\sum a^2 + \sum ab}{(\sum ab)^2} \right\} \geq 0 \therefore \text{in order to prove :}$$

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$$\frac{1}{4} + \frac{\sum a^3}{4abc} \geq \left( \frac{\sum a^2}{\sum ab} \right)^2, \text{ it suffices to prove : } \frac{\sum a}{4abc} > \frac{\sum a^2 + \sum ab}{(\sum ab)^2} \Leftrightarrow (s^2 + 4Rr + r^2)^2 > 8Rr(3s^2 - 4Rr - r^2)$$

$$\Leftrightarrow s^4 - s^2(16Rr - 2r^2) + r^2(48R^2 + 16Rr + r^2) \stackrel{(i)}{\geq} 0$$

Now, LHS of (i)  $\stackrel{\text{Gerretsen}}{\geq} s^2(16Rr - 5r^2) - s^2(16Rr - 2r^2) + r^2(48R^2 + 16Rr + r^2)$   
 $= r^2(48R^2 + 16Rr + r^2 - 3s^2)$

$\stackrel{\text{Gerretsen}}{\geq} r^2(48R^2 + 16Rr + r^2 - 12R^2 - 12Rr - 9r^2)$   
 $= 4r^2(9R^2 + r(R - 2r)) \stackrel{\text{Euler}}{\geq} 36R^2r^2 > 0 \Rightarrow (i) \text{ is true } \therefore \frac{1}{4} + \frac{\sum a^3}{4abc} \geq \left( \frac{\sum a^2}{\sum ab} \right)^2$

applying which on a triangle with sides  $\frac{2m_a}{3}, \frac{2m_b}{3}, \frac{2m_c}{3}$ , we get

$$\frac{1}{4} + \frac{\frac{8}{27}(m_a^3 + m_b^3 + m_c^3)}{4\left(\frac{8}{27}\right)m_a m_b m_c} \geq \left( \frac{\frac{4}{9}(m_a^2 + m_b^2 + m_c^2)}{\frac{4}{9}(m_a m_b + m_b m_c + m_c m_a)} \right)^2$$

$$\Rightarrow \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \stackrel{(3)}{\geq} \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2 \therefore (1), (2), (3) \Rightarrow$$

$$\left( \frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c}$$

$$\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2 \text{ (Proved)}$$

### Solution 2 by Bogdan Fuștei-Romania

From  $\frac{1}{2}Rs^2 \geq m_a m_b m_c$  we have:  $\frac{R}{2r} \geq \frac{m_a m_b m_c}{r_a r_b r_c}$

$$4R + r \geq m_a + m_b + m_c$$

$$\frac{9}{2}R \geq 4R + r \Leftrightarrow R \geq 2r \text{ (Euler)} \Rightarrow \frac{R}{2r} \geq \frac{m_a + m_b + m_c}{9r}$$

$$\left( \frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9r r_a r_b r_c} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}; (1)$$

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We know that:  $\frac{R}{r} \geq \frac{abc+a^3+b^3+c^3}{2abc}$

$$\frac{x^3 + y^3 + z^3}{4xyz} + \frac{1}{4} \geq \left( \frac{x^2 + y^2 + z^2}{xy + yz + zx} \right)^2, \forall x, y, z > 0$$

Let the triangle  $m_a m_b m_c$  with  $R_m, r_m, S_m$

$$S_m = \frac{3}{4}S; (S \text{ --area of } \triangle ABC); abc = 4RS \Rightarrow m_a m_b m_c = 4R_m S_m$$

$$m_a m_b m_c = 4R_m \cdot \frac{3}{4}S \Rightarrow R_m = \frac{m_a m_b m_c}{3S}$$

$$S_m = s_m \cdot r_m; s_m = \frac{m_a + m_b + m_c}{3} \Rightarrow \frac{3}{4}S = \frac{(m_a + m_b + m_c) \cdot r_m}{2} \Rightarrow$$

$$r_m = \frac{3S}{2(m_a + m_b + m_c)} \Rightarrow 2r_m = \frac{3S}{m_a + m_b + m_c}$$

$$\frac{R_m}{2r_m} = \frac{m_a m_b m_c}{3S} \cdot \frac{m_a + m_b + m_c}{3S} = \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2}$$

$$\frac{R}{2r} \geq \frac{1}{4} + \frac{a^3 + b^3 + c^3}{4abc} \geq \left( \frac{a^2 + b^2 + c^2}{ab + bc + ca} \right)^2$$

$$\frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^3 + m_b^3 + m_c^3}{4m_a m_b m_c} \geq$$

$$\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2; (2)$$

$$\text{From (1),(2) we get: } \left( \frac{R}{2r} \right)^2 \geq \frac{m_a m_b m_c (m_a + m_b + m_c)}{9S^2} \geq \frac{1}{4} + \frac{m_a^2 + m_b^2 + m_c^2}{4m_a m_b m_c} \geq$$

$$\geq \left( \frac{m_a^2 + m_b^2 + m_c^2}{m_a m_b + m_b m_c + m_c m_a} \right)^2$$

**1697. In  $\triangle ABC$ ,  $I$  –incenter,  $N$  –ninepoint center. Prove that:**

$$NI \geq r \cdot \tan^2 \left( \frac{A - B}{2} \right)$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

*Solution by Marian Ursărescu-Romania*

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$$\tan^2\left(\frac{A-B}{2}\right) = \frac{\sin^2\left(\frac{A-B}{2}\right)}{\cos^2\left(\frac{A-B}{2}\right)} = \frac{1 - \cos^2\left(\frac{A-B}{2}\right)}{\cos^2\left(\frac{A-B}{2}\right)} = \frac{1}{\cos^2\left(\frac{A-B}{2}\right)} - 1$$

We must show that:  $NI \geq r \cdot \left(\frac{1}{\cos^2\left(\frac{A-B}{2}\right)} - 1\right)$ ; (1)

Lemma: In any  $\triangle ABC$  we have:  $\cos^2\left(\frac{A-B}{2}\right) \geq \frac{2r}{R}$ ; (2)

Proof:  $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

We must show that:  $\cos^2\left(\frac{A-B}{2}\right) \geq 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Leftrightarrow$

$$\cos^2\left(\frac{A-B}{2}\right) \geq 4 \sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \cos \frac{A+B}{2}\right) \Leftrightarrow$$

$$\cos^2\left(\frac{A-B}{2}\right) \geq 4 \sin \frac{C}{2} \left(\cos \frac{A-B}{2} - \sin \frac{C}{2}\right) \Leftrightarrow$$

$$4 \sin^2 \frac{C}{2} - 4 \sin \frac{C}{2} \cos\left(\frac{A-B}{2}\right) + \cos^2\left(\frac{A-B}{2}\right) \geq 0 \Leftrightarrow$$

$$\left(2 \sin \frac{C}{2} - \cos\left(\frac{A-B}{2}\right)\right)^2 \geq 0, \text{ true.}$$

From (2) we have:  $\frac{1}{\cos^2\left(\frac{A-B}{2}\right)} \leq \frac{R}{2r} \Leftrightarrow \frac{1}{\cos^2\left(\frac{A-B}{2}\right)} - 1 \leq \frac{R-2r}{2r}$ ; (3)

From (1),(3) we must show:  $NI \geq \frac{R-2r}{2r} \Leftrightarrow 4NI^2 \geq (R-2r)^2$ ; (4)

From median theorem, we have:

$$4NI^2 = 4 \left[ \frac{2(OI^2 + IH^2) - OH^2}{4} \right] =$$

$$= 2(R^2 - 2Rr + 4R^2 + 4Rr + 3r^2 - s^2) - 9R^2 + a^2 + b^2 + c^2 =$$

$$= 10R^2 + 4Rr + 6r^2 - 2s^2 - 9R^2 + 2s^2 - 2r^2 - 8Rr =$$

$$= R^2 - 4Rr + 4r^2 = (R-2r)^2 \Rightarrow (4) \text{ is true. Proved.}$$

1698. In  $\triangle ABC$  the following relationship holds:

$$R \left( 16R \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} + r \right) \leq 5R^2 + 9r^2$$

Proposed by Daniel Sitaru-Romania

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**Solution by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} &= \frac{s}{4R} \Rightarrow R \left( 16R \cos^2 \frac{A}{2} \cos^2 \frac{B}{2} \cos^2 \frac{C}{2} + r \right) = \\ &= R \left( 16R \left( \frac{s}{4R} \right)^2 + r \right) = R \left( \frac{s^2}{R} + r \right) = s^2 + Rr \stackrel{(*)}{\leq} 5R^2 + 9r^2 \end{aligned}$$

$$(*) \Leftrightarrow s^2 \leq 5R^2 - Rr + 9r^2$$

$$\text{But: } s^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)} \stackrel{(*)}{\leq} 5R^2 - Rr + 9r^2$$

$$(**) \Leftrightarrow 3R^2 + 11Rr - 10r^2 \geq 2(R - 2r)\sqrt{R(R - 2r)} \Leftrightarrow$$

$$(R - 2r)(3R - 5r) \geq 2(R - 2r)\sqrt{R(R - 2r)} \Leftrightarrow$$

$$(R - 2r) \left[ 3R - 5r - 2\sqrt{R(R - 2r)} \right] \geq 0$$

$$\text{Because } R \geq 2r \text{ (Euler)} \Rightarrow R - 2r \geq 0$$

$$\text{We just check: } 3R - 5r - 2\sqrt{R(R - 2r)} \geq 0 \stackrel{3R \geq 6r > 5r}{\Leftrightarrow}$$

$$(3R - 5r)^2 > 4(R^2 - 2Rr) \Leftrightarrow 9R^2 - 30Rr + 25r^2 > 4R^2 - 8Rr$$

$$5R^2 - 22Rr + 25r^2 > 0 \stackrel{t = \frac{R}{r} \geq 2}{\Leftrightarrow} 5t^2 - 22t + 25 > 0$$

$$5 \left( t - \frac{11}{5} \right)^2 + \frac{4}{5} > 0 \text{ true for } t \geq 2 \Rightarrow (**) \text{ is true} \Rightarrow (*) \text{ is true. Proved.}$$

**1699. In  $\triangle ABC$  the following relationship holds:**

$$\frac{m_a^2 + m_b^2 + m_c^2}{3\sqrt{3}S} \leq \frac{1}{9} + \frac{8}{9} \left( \frac{R}{2r} \right)^4$$

*Proposed by Adil Abdullayev-Baku-Azerbaijan*

**Solution 1 by George Florin Şerban-Romania**

$$\begin{aligned} \frac{m_a^2 + m_b^2 + m_c^2}{3\sqrt{3}S} &= \frac{\frac{3}{4}(a^2 + b^2 + c^2)}{3\sqrt{3}rs} = \frac{3 \cdot 2(s^2 - r^2 - 4Rr)}{4 \cdot 3\sqrt{3}rs} = \\ &= \frac{s^2 - r^2 - 4Rr}{2\sqrt{3}sr} \stackrel{\text{Gerretsen}}{\leq} \frac{4R^2 + 4Rr + 3r^2 - r^2 - 4Rr}{2\sqrt{3}sr} \stackrel{\text{Mitrinovic}}{\leq} \\ &\leq \frac{4R^2 + 2r^2}{2\sqrt{3}r \cdot 3\sqrt{3}r} = \frac{2R^2 + r^2}{9r^2} = \frac{2}{9} \left( \frac{R}{r} \right)^2 + \frac{1}{9} \stackrel{(?)}{\leq} \frac{1}{9} + \frac{8}{9} \left( \frac{R}{2r} \right)^4 = \frac{1}{9} + \frac{8}{9} \cdot \frac{1}{16} \left( \frac{R}{r} \right)^4 \end{aligned}$$

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$\stackrel{t=\frac{R}{r} \geq 2}{\iff} \frac{2}{9} t^2 \leq \frac{1}{18} t^4 \iff 9t^2(t^2 - 4) \geq 0 \iff 9t^2(t - 2)(t + 2) \geq 0$  true for  $t \geq 2$ . Proved.

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

$$\begin{aligned} \bullet \quad m_a^2 + m_b^2 + m_c^2 &= \frac{3}{4}(a^2 + b^2 + c^2) \stackrel{a^2+b^2+c^2 \leq 9R^2}{\leq} \frac{27}{4} R^2; \\ \bullet \quad 3\sqrt{3}S &= 3\sqrt{3} \cdot pr \stackrel{p \geq 3\sqrt{3}r}{\geq} 27r^2; \\ &\rightarrow \frac{m_a^2 + m_b^2 + m_c^2}{3\sqrt{3}S} \leq \frac{27}{4} R^2 \cdot \frac{1}{27r^2} = \left(\frac{R}{2r}\right)^2 \end{aligned}$$

Let  $t = \left(\frac{R}{2r}\right)^2 \geq 1$ . We must show that:

$$\frac{8}{9} t^2 + \frac{1}{9} \geq t \iff 8t^2 - 9t + 1 \geq 0 \iff (t - 1)(8t - 1) \geq 0$$

Which is clearly true because:  $t \geq 1 \rightarrow t - 1 \geq 0, 8t - 1 \geq 16 - 1 = 15$ .

**1700. In  $\triangle ABC$  the following relationship holds:**

$$\sum_{cyc} a(\sin 3B - \sin 3C) \leq 8s \sum_{cyc} \sin(B - C)$$

*Proposed by Daniel Sitaru-Romania*

**Solution 1 by Soumava Chakraborty-Kolkata-India**

$$\begin{aligned} \sum a(\sin 3B - \sin 3C) &= \sum a(3\sin B - 4\sin^3 B) - \sum a(3\sin C - 4\sin^3 C) \\ &= 3 \sum \frac{ab}{2R} - 3 \sum \frac{ac}{2R} - 4 \sum \frac{ab^3}{8R^3} + 4 \sum \frac{ac^3}{8R^3} \\ &= -\frac{\sum(ab^3 - ac^3)}{2R^3} = -\frac{ab^3 - ac^3 + bc^3 - ba^3 + ca^3 - cb^3}{2R^3} \\ &= -\frac{ab(b^2 - a^2) + c^3(b - a) - c(b - a)(b^2 + a^2 + ab)}{2R^3} \\ &= -\frac{(b - a)(a^2b + ab^2 + c^3 - cb^2 - ca^2 - abc)}{2R^3} \\ &= -\frac{(b - a)\{a^2(b - c) - c(b + c)(b - c) + ab(b - c)\}}{2R^3} \end{aligned}$$

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$$\begin{aligned}
 &= -\frac{(b-a)(b-c)(a^2-bc-c^2+ab)}{2R^3} = \frac{(b-c)(c-a)(b-a)(a+b+c)}{2R^3} \\
 &\therefore \sum a(\sin 3B - \sin 3C) \stackrel{(1)}{=} \frac{(b-c)(c-a)(b-a)(a+b+c)}{2R^3} \\
 &\quad \text{Again, } 2s \sum \sin(B-C) = 4s \sum \sin \frac{B-C}{2} \cos \frac{B+C}{2} \\
 &= 4s \sum \left\{ \left( \frac{b-c}{a} \right) \cos \frac{A}{2} \left( \frac{b+c}{a} \right) \sin \frac{A}{2} \right\} = 4s \sum \left\{ \left( \frac{b^2-c^2}{a^2} \right) \left( \frac{\sin A}{2} \right) \right\} \\
 &= \frac{4s}{4R} \sum \frac{b^2-c^2}{a} = \left( \frac{s}{R} \right) \frac{\sum bc(b^2-c^2)}{4Rrs} = \frac{bc(b^2-c^2) + ca(c^2-a^2) + ab(a^2-b^2)}{4R^2r} \\
 &= \frac{bc(b^2-c^2) + c^3a - ca^3 + a^3b - ab^3}{4R^2r} \\
 &= \frac{bc(b^2-c^2) + a^3(b-c) - a(b-c)(b^2+c^2+bc)}{4R^2r} \\
 &= \frac{(b-c)(b^2c + bc^2 + a^3 - ab^2 - ac^2 - abc)}{4R^2r} \\
 &= \frac{(b-c)\{b^2(c-a) - a(c+a)(c-a) + bc(c-a)\}}{4R^2r} \\
 &= \frac{(b-c)(c-a)(b^2-ac-a^2+bc)}{4R^2r} = \frac{(b-c)(c-a)(b-a)(a+b+c)}{4R^2r} \\
 &\therefore 2s \sum \sin(B-C) \stackrel{(2)}{=} \frac{(b-c)(c-a)(b-a)(a+b+c)}{4R^2r} \therefore (1), (2) \\
 &\Rightarrow \text{proposed inequality} \Leftrightarrow \\
 &\quad (b-c)(c-a)(b-a)(a+b+c) \left( \frac{1}{4R^2r} - \frac{1}{2R^3} \right) \geq 0 \\
 &\Leftrightarrow (b-c)(c-a)(b-a)(a+b+c) \left( \frac{R-2r}{4R^3r} \right) \geq 0 \rightarrow \text{true} \\
 &\therefore (b-c)(c-a)(b-a) \geq 0 \text{ as } a \geq b \geq c \text{ and } R-2r \stackrel{\text{Euler}}{\geq} 0 \therefore \sum a(\sin 3B - \sin 3C) \\
 &\leq 2s \sum \sin(B-C) \text{ (Proved)}
 \end{aligned}$$

**Solution 2 by Tran Hong-Dong Thap-Vietnam**

For all  $x, y \in \mathbb{R}$  we have identity:

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$$\sin x \sin 3y - \sin 3x \sin y = 4 \sin x \sin y \sin(x+y) \sin(x-y) \Rightarrow$$

$$\sum_{cyc} a(\sin 3B - \sin 3C) =$$

$$= 2R[\sin A(\sin 3B - \sin 3C) + \sin B(\sin 3C - \sin 3A) + \sin C(\sin 3A - \sin 3B)] =$$

$$= 2R[4 \sin A \sin B \sin(A+B) \sin(A-B) + 4 \sin C \sin A \sin(C+A) \sin(C-A) +$$

$$+ 4 \sin B \sin C \sin(B+C) \sin(B-C)]$$

$$= 8R \sin A \sin B \sin C [\sin(A-B) + \sin(B-C) + \sin(C-A)] =$$

$$= \frac{4sr}{R} [\sin(A-B) + \sin(B-C) + \sin(C-A)] = \Omega$$

$$0 < A, B, C < \pi \Rightarrow -\pi < A-B, B-C, C-A < \pi \Rightarrow$$

$$\sin(A-B) + \sin(B-C) + \sin(C-A) \geq 0 \stackrel{\frac{r}{R} \leq \frac{1}{2}}{\Rightarrow}$$

$$\Omega \leq \frac{4s}{2} [\sin(A-B) + \sin(B-C) + \sin(C-A)] \leq$$

$$\leq 8s [\sin(A-B) + \sin(B-C) + \sin(C-A)]$$

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*It's nice to be important but more important it's to be nice.*

*At this paper works a TEAM.*

*This is RMM TEAM.*

*To be continued!*

*Daniel Sitaru*