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ROMANIAN MATHEMATICAL MAGAZINE

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1501. In $\triangle A B C$ the following relationship holds:

$$
\frac{r_{a}}{m_{a} h_{a}}+\frac{r_{b}}{m_{b} h_{b}}+\frac{r_{c}}{m_{c} h_{c}} \geq \frac{4(R-r)}{R^{2}}
$$

Proposed by Marian Ursărescu-Romania
Solution 1 by Florică Anastase-Romania

$$
\begin{aligned}
& \text { 1) } \sum a^{2}=2\left(s^{2}-r^{2}-4 R r\right) \\
& \text { 2) } \sum a^{2} r_{a}=4 s^{2}(R-r) \\
& \text { 3) } m_{a} \leq \frac{a^{2}+b^{2}+c^{2}}{2 \sqrt{3} a} \\
& \text { 4) } 3 \sqrt{3} r \leq s \leq \frac{3 \sqrt{3}}{2} R \text {-Mitrinovic } \\
& \text { 5) } 16 R r-5 r^{2} \leq s^{2} \leq 4 R^{2}+4 R r+3 r^{2}-G e r r e t s e n ~ \\
& \text { 6) } R \geq 2 r-E u l e r \\
& \text { (3) } \rightarrow \frac{1}{m_{a}} \geq \frac{2 \sqrt{3} a}{a^{2}+b^{2}+c^{2}} \rightarrow \frac{r_{a}}{m_{a}} \geq \frac{2 \sqrt{3} a r_{a}}{a^{2}+b^{2}+c^{2}} \rightarrow \\
& \frac{r_{a}}{m_{a} h_{a}} \geq \frac{\sqrt{3} a^{2} r_{a}}{\left(a^{2}+b^{2}+c^{2}\right) S} \\
& \sum_{c y c} \frac{r_{a}}{m_{a} h_{a}} \geq \sum_{c y c} \frac{\sqrt{3} a^{2} r_{a}}{\left(a^{2}+b^{2}+c^{2}\right) S}=\frac{\sqrt{3}}{\left(a^{2}+b^{2}+c^{2}\right) S} \sum_{c y c} a^{2} r_{a} \stackrel{(2)}{=} \\
& =\frac{\sqrt{3}}{\left(a^{2}+b^{2}+c^{2}\right) S} \cdot 4 s^{2}(R-r)=\frac{\sqrt{3} s^{2}}{S} \cdot \frac{4(R-r)}{a^{2}+b^{2}+c^{2}} \stackrel{(1)}{=} \\
& =\frac{\sqrt{3} s^{2}}{s r} \cdot \frac{4(R-r)}{2\left(s^{2}-r^{2}-4 R r\right)}=\frac{\sqrt{3} s}{r} \cdot \frac{4(R-r)}{2\left(s^{2}-r^{2}-4 R r\right)} \stackrel{(4)}{\geq} \\
& \geq \frac{\sqrt{3}(3 \sqrt{3} r)}{r} \cdot \frac{4(R-r)}{2\left(s^{2}-r^{2}-4 R r\right)}=\frac{9 \cdot 4(R-r)}{2\left(s^{2}-r^{2}-4 R r\right)} \stackrel{(*)}{\geq} \frac{4(R-r)}{R^{2}} \\
& (*) \leftrightarrow 9 R^{2} \geq 2 s^{2}-2 r^{2}-8 R r \leftrightarrow 2 s^{2} \leq 9 R^{2}+2 r^{2}+8 R r \stackrel{(5)}{\leftrightarrow} \\
& 8 R^{2}+8 R r+6 r^{2} \leq 9 R^{2}+2 r^{2}+8 R r \leftrightarrow R^{2} \geq 4 r^{2} \leftrightarrow R \geq 2 r \text { true from Euler. Proved. }
\end{aligned}
$$



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Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{align*}
& r_{a} w_{a}=\left(\operatorname{stan} \frac{A}{2}\right)\left(\frac{2 b c \cos \frac{A}{2}}{b+c}\right)=\left(\operatorname{ssin} \frac{A}{2}\right)\left(\frac{2 b c}{b+c}\right)^{H M \leq G M}{\underset{\sim}{m}}_{\leq}^{\sin }\left(\sin \frac{A}{2}\right) \sqrt{b c} \\
& =s \sqrt{(s-b)(s-c)}=(\sqrt{s(s-b)} \sqrt{s(s-c)}) \\
& \leq m_{b} m_{c} \Rightarrow r_{a} w_{a} \stackrel{(1)}{\leq} m_{b} m_{c} \text { and analogs } \\
& \text { Now, } \sum \frac{r_{a}}{m_{a} h_{a}}=\sum \frac{r_{a}^{2}}{m_{a} r_{a} h_{a}} \stackrel{w_{a} \geq h_{a} \text { and analogs }}{\geq} \\
& \geq \sum \frac{r_{a}^{2}}{m_{a} r_{a} w_{a}} \stackrel{\text { using (1)and analogs }}{\geq} \sum \frac{r_{a}^{2}}{m_{a} m_{b} m_{c}} \Rightarrow \sum \frac{r_{a}}{m_{a} h_{a}} \stackrel{(2)}{\geq} \frac{1}{m_{a} m_{b} m_{c}} \sum r_{a}^{2} \\
& \text { Now, } m_{a}^{2} m_{b}^{2} m_{c}^{2}=\frac{1}{64}\left(2 b^{2}+2 c^{2}-a^{2}\right)\left(2 c^{2}+2 a^{2}-b^{2}\right)\left(2 a^{2}+2 b^{2}-c^{2}\right) \stackrel{(3)}{=} \\
& =\frac{1}{64}\left\{-4 \sum a^{6}+6\left(\sum a^{4} b^{2}+\sum a^{2} b^{4}\right)+3 a^{2} b^{2} c^{2}\right\} \\
& \text { Now, } \sum a^{6}=\left(\sum a^{2}\right)^{3}-3\left(a^{2}+b^{2}\right)\left(b^{2}+c^{2}\right)\left(c^{2}+a^{2}\right)= \\
& =\left(\sum a^{2}\right)^{3}-3\left(2 a^{2} b^{2} c^{2}+\sum a^{2} b^{2}\left(\sum a^{2}-c^{2}\right)\right) \\
& =\left(\sum a^{2}\right)^{3}+3 a^{2} b^{2} c^{2}-3\left(\sum a^{2} b^{2}\right) \sum a^{2} \therefore \sum a^{6} \xrightarrow{M}\left(\sum a^{2}\right)^{3}+3 a^{2} b^{2} c^{2}-3\left(\sum a^{2} b^{2}\right) \sum a^{2}  \tag{4}\\
& \text { Again, } \sum a^{4} b^{2}+\sum a^{2} b^{4}=\sum a^{2} b^{2}\left(\sum a^{2}-c^{2}\right) \stackrel{(5)}{\cong}\left(\sum a^{2} b^{2}\right) \sum a^{2}-3 a^{2} b^{2} c^{2} \\
& \therefore(3),(4),(5) \Rightarrow m_{a}^{2} m_{b}^{2} m_{c}^{2}= \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}-12 a^{2} b^{2} c^{2}+12\left(\sum a^{2} b^{2}\right) \sum a^{2}+6\left(\sum a^{2} b^{2}\right) \sum a^{2}-18 a^{2} b^{2} c^{2}+3 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}+18\left(\sum a^{2} b^{2}\right) \sum a^{2}-27 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}+18\left(\left(\sum a b\right)^{2}-2 a b c(2 s)\right)\left(\sum a^{2}\right)-27 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-32\left(s^{2}-4 R r-r^{2}\right)^{3}+36\left(s^{2}-4 R r-r^{2}\right)\left(s^{2}+4 R r+r^{2}\right)^{2}-576 R r s^{2}\left(s^{2}-4 R r-r^{2}\right)-432 R^{2} r^{2} s^{2}\right\} \\
& =\frac{1}{16}\left\{s^{6}-s^{4}\left(12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3}\right\} \leq \frac{R^{2} s^{4}}{4}
\end{align*}
$$



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$\Leftrightarrow s^{6}-s^{4}\left(4 R^{2}+12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3} \stackrel{(i)}{\leq} 0$

$$
\text { Now, LHS of }(i) \stackrel{\text { Gerretsen }}{\leftrightarrows}-s^{4}\left(8 R r-36 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3} \stackrel{\text { 己 }}{\leq} 0
$$

$\Leftrightarrow s^{4}(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+r^{2}(4 R+r)^{3}{\underset{\sim i i i}{n}}_{\stackrel{?}{n}} 20 r s^{4}$
Now, LHS of (ii) $\qquad$ $s^{2}\left(16 R r-5 r^{2}\right)(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+r^{2}(4 R+r)^{3}$
and RHS of (ii) $\underbrace{\substack{\text { Geretsen } \\ \leftrightarrows}}_{(b)} 20 r s^{2}\left(4 R^{2}+4 R r+3 r^{2}\right)$
(a), (b) $\Rightarrow$ in order to prove (ii), it suffices to prove :

$$
s^{2}\left(16 R r-5 r^{2}\right)(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+r^{2}(4 R+r)^{3}
$$

$$
\geq 20 r s^{2}\left(4 R^{2}+4 R r+3 r^{2}\right) \Leftrightarrow s^{2}\left(108 R^{2}-256 R r+53 r^{2}\right)+r(4 R+r)^{3} \geq 0
$$

$$
\Leftrightarrow s^{2}\left(108 R^{2}-256 R r+80 r^{2}\right)+r(4 R+r)^{3} \stackrel{n}{\geq} 27 r^{2} s^{2}
$$

Now, LHS of $(i i i i) \underbrace{\substack{\text { Gerretsen }}}_{(c)}\left(108 R^{2}-256 R r+80 r^{2}\right)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3}$
and RHS of (iii) $\underbrace{\substack{\text { Geretsen } \\ \sim}}_{(d)} 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right)$
(c), (d) $\Rightarrow$ in order to prove (iii), it suffices to prove :
$\left(108 R^{2}-256 R r+80 r^{2}\right)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3} \geq 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right)$

$$
\Leftrightarrow 224 t^{3}-587 t^{2}+308 t-60 \geq 0\left(\text { where } t=\frac{R}{r}\right)
$$

$$
\Leftrightarrow(t-2)\{(t-2)(224 t+309)+648\} \geq 0 \rightarrow \text { true } \because t \stackrel{\text { Euler }}{\geq} 2 \Rightarrow(i i i) \Rightarrow(i i)
$$

$$
\Rightarrow(i) i s \text { true } \Rightarrow m_{a}^{2} m_{b}^{2} m_{c}^{2} \leq \frac{R^{2} s^{4}}{4} \Rightarrow m_{a} m_{b} m_{c} \stackrel{(6)}{\check{m}} \frac{R s^{2}}{2}
$$

(2), (6) $\Rightarrow \sum \frac{r_{a}}{m_{a} h_{a}} \geq \frac{2\left((4 R+r)^{2}-2 s^{2}\right)}{R s^{2}} \stackrel{?}{\stackrel{4}{2}} \frac{4(R-r)}{R^{2}} \Leftrightarrow \frac{(4 R+r)^{2}}{s^{2}}-2 \stackrel{?}{\sim} \geq 2-\frac{2 r}{R}$


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$$
\begin{gathered}
\leq(4 R-2 r)\left(2 R^{2}+10 R r-r^{2}+2(R-2 r) \sqrt{R^{2}-2 R r}\right) \stackrel{?}{\leftrightarrows} R(4 R+r)^{2} \\
\Leftrightarrow R(4 R+r)^{2}-\left(2 R^{2}+10 R r-r^{2}\right)(4 R-2 r) \stackrel{?}{\sim} \underset{\sim}{\sum} 2(4 R-2 r)(R-2 r) \sqrt{R^{2}-2 R r} \\
\Leftrightarrow(R-2 r)\left(8 R^{2}-12 R r+r^{2}\right) \sum_{(v)}^{\sum_{\sim}^{2}} 2(4 R-2 r)(R-2 r) \sqrt{R^{2}-2 R r}
\end{gathered}
$$

$\because R-2 r \stackrel{\text { Euler }}{\geqq} \therefore$ in order to prove $(v)$,it suffices to prove :

$$
8 R^{2}-12 R r+r^{2}>2(4 R-2 r) \sqrt{R^{2}-2 R r}
$$

$$
\Leftrightarrow\left(8 R^{2}-12 R r+r^{2}\right)^{2}-4\left(R^{2}-2 R r\right)(4 R-2 r)^{2}>0 \Leftrightarrow r^{2}(4 R+r)^{2}>0
$$

$$
\rightarrow \text { true } \Rightarrow(v) \Rightarrow(i v) \text { is true (Proved) }
$$

Solution 3 by Rahim Shahbazov-Baku-Azerbaijan

$$
\begin{gather*}
\frac{r_{a}}{m_{a} h_{a}}+\frac{r_{b}}{m_{b} h_{b}}+\frac{r_{c}}{m_{c} h_{c}} \geq \frac{4(R-r)}{R^{2}} ;(1)  \tag{1}\\
\frac{m_{a}}{h_{a}} \leq \frac{R}{2 r}-\left(\text { Panaitopol's inequality } \rightarrow m_{a} h_{a} \leq \frac{R}{2 r} \cdot h_{a}^{2}\right. \\
\sum_{c y c} \frac{r_{a}}{m_{a} h_{a}} \geq \sum_{c y c} \frac{r_{a}}{\frac{R}{2 r} \cdot h_{a}^{2}}=\frac{2 r}{R} \cdot \sum_{c y c} \frac{r_{a}}{h_{a}^{2}} \geq \frac{4(R-r)}{R^{2}} \rightarrow \\
r \sum_{c y c} \frac{r_{a}}{h_{a}^{2}} \geq 2\left(1-\frac{r}{R}\right) ;(2) \\
\frac{r_{a}}{h_{a}^{2}}=\frac{\frac{S}{s-a}}{4 S^{2}} \cdot a^{2}=\frac{a^{2}}{4 S(s-a)}=\frac{a^{2}}{4 r s(s-a)} \xrightarrow{(2)} \\
\frac{1}{4 s}\left(\frac{a^{2}}{s-a}+\frac{b^{2}}{s-b}+\frac{c^{2}}{s-c}\right) \geq 2\left(1-\frac{r}{R}\right)
\end{gather*}
$$

$$
\text { Let: } a=x+y, b=y+z, c=z+x \text { (Ravi) }
$$

$$
\begin{gathered}
\frac{1}{4(x+y+z)}\left(\frac{(x+y)^{2}}{z}+\frac{(y+z)^{2}}{x}+\frac{(z+x)^{2}}{y}\right) \geq 2-\frac{8 x y z}{(x+y)(y+z)(z+x)} \\
(x+y)^{2} \geq 4 x y
\end{gathered}
$$

$L h s \geq \frac{1}{4(x+y+z)} \cdot 4\left(\frac{x y}{z}+\frac{y z}{x}+\frac{z x}{y}\right)=\frac{1}{(x+y+z)} \cdot\left(\frac{x y}{z}+\frac{y z}{x}+\frac{z x}{y}\right) \geq$

$$
\geq 2-\frac{8 x y z}{(x+y)(y+z)(z+x)} \rightarrow
$$



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$$
\begin{gathered}
\frac{1}{(x+y+z)} \cdot\left(\frac{x y}{z}+\frac{y z}{x}+\frac{z x}{y}\right)+\frac{8 x y z}{(x+y)(y+z)(z+x)} \geq 2 \\
\text { Let: } x=\frac{1}{m}, y=\frac{1}{n}, z=\frac{1}{p} \rightarrow \\
\frac{m^{2}+n^{2}+p^{2}}{m n+n p+m p}+\frac{8 m n p}{(m+n)(n+p)(p+m)} \geq 2
\end{gathered}
$$

## Solution 4 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\text { Suppose: } a \geq b \geq c \rightarrow r_{a} \geq r_{b} \geq r_{c} ; m_{a} \leq m_{b} \leq m_{c} ; h_{a} \leq h_{b} \leq h_{c} \rightarrow \\
\frac{1}{m_{a}} \geq \frac{1}{m_{b}} \geq \frac{1}{m_{c}} ; \frac{1}{h_{a}} \geq \frac{1}{h_{b}} \geq \frac{1}{h_{c}} \rightarrow \frac{1}{m_{a} h_{a}} \geq \frac{1}{m_{b} h_{b}} \geq \frac{1}{m_{c} h_{c}} \\
\begin{array}{c}
\text { Lhs }=\sum_{\text {cyc }} \frac{r_{a}}{m_{a} h_{a}} \stackrel{\text { Chebyshev's }}{\geq} \frac{1}{3}\left(r_{a}+r_{b}+r_{c}\right)\left(\frac{1}{m_{a} h_{a}}+\frac{1}{m_{b} h_{b}}+\frac{1}{m_{c} h_{c}}\right) \geq \\
\text { Chebyshev's } \frac{1}{3}(4 R+r) \frac{1}{3}\left(\frac{1}{m_{a}}+\frac{1}{m_{b}}+\frac{1}{m_{c}}\right)\left(\frac{1}{h_{a}}+\frac{1}{h_{b}}+\frac{1}{h_{c}}\right) \stackrel{\text { Schwartz }}{\geq} \\
\geq \frac{1}{3}(4 R+r) \frac{1}{3} \cdot \frac{9}{m_{a}+m_{b}+m_{c}} \cdot \frac{1}{r} \stackrel{m_{a}+m_{b}+m_{c} \leq 4 R+r}{\geq} \\
\geq \frac{1}{9}(4 R+r) \frac{9}{4 R+r} \cdot \frac{1}{r}=\frac{1}{r} \stackrel{(*)}{\geq} \frac{4(R-r)}{R^{2}}
\end{array}
\end{gathered}
$$

$(*) \leftrightarrow R^{2} \geq 4 r(R-r) \leftrightarrow R^{2}-4 R r+4 r^{2} \geq 0 \leftrightarrow(R-2 r)^{2} \geq 0$ true. Proved.

## Solution 5 by Bogdan Fuştei-Romania

> 1) $a^{2}=\left(r_{b}+r_{c}\right)\left(r_{a}-r\right)$ and analogs
> 2) $r_{b} r_{c}=\frac{h_{a}\left(r_{b}+r_{c}\right)}{2}$ and analogs.

$$
\begin{gathered}
\leq \frac{R}{2 r}-(\text { Panaitopol's inequality }) \rightarrow m_{a} h_{a} \leq \frac{R}{2 r} \cdot h_{a}^{2} \rightarrow \frac{r_{a}}{m_{a} h_{a}} \geq \frac{2 r}{R} \cdot \frac{r_{a}}{h_{a}^{2}} \\
2 S=a h_{a} \rightarrow a^{2} h_{a}^{2}=4 S^{2} ; S^{2}=\left(r_{a} r_{b} r_{c} r\right) \rightarrow \frac{r_{a}}{m_{a} h_{a}} \geq \frac{2 r}{R} \cdot \frac{a^{2} r_{a}}{4 S^{2}} \\
\rightarrow \frac{r_{a}}{m_{a} h_{a}} \geq \frac{2 r}{R} \cdot \frac{a^{2} r_{a}}{4 r r_{a} r_{b} r_{c}}=\frac{1}{2 R} \cdot \frac{a^{2}}{r_{b} r_{c}}=\frac{1}{2 R} \cdot \frac{\left(r_{b}+r_{c}\right)\left(r_{a}-r\right)}{\frac{h_{a}\left(r_{b}+r_{c}\right)}{2}} \rightarrow \\
\frac{r_{a}}{m_{a} h_{a}} \geq \frac{r_{a}-r}{R h_{a}} \text { and analogs. } \\
\frac{1}{h_{a}}+\frac{1}{h_{b}}+\frac{1}{h_{c}}=\frac{1}{R}
\end{gathered}
$$



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$$
\begin{gather*}
\sin ^{2} \frac{A}{2}=\frac{r_{a}-r}{4 R}=\frac{\text { www.ssmrmh.ro }}{2 R} \cdot \frac{r_{a}}{h_{a}} \rightarrow \frac{r_{a}}{h_{a}}=\frac{r_{a}-r}{2 r} \rightarrow \\
\sum_{c y c} \frac{r_{a}}{m_{a} h_{a}} \geq \frac{1}{R} \sum_{c y c}\left(\frac{r_{a}}{h_{a}}-\frac{R}{h_{a}}\right)=\frac{1}{R}\left(\frac{r_{a}-r+r_{b}-r+r_{c}-r}{2 r}-1\right)= \\
\stackrel{r_{a}+r_{b}+r_{c}=4 R+r}{\cong} \frac{1}{R}\left(\frac{2 R-r}{r}-1\right)=\frac{2(R-r)}{R r} ;(1)
\end{gather*}
$$

But: $\frac{2(R-r)}{R r} \geq \frac{4(R-r)}{R^{2}} \leftrightarrow \frac{1}{r} \geq \frac{2}{R} \leftrightarrow R \geq 2 r$ (Euler);
From (1), (2) the inequality is proved.
1502. In $\triangle A B C$ the following relationship holds:

$$
\frac{\mathbf{r}_{\mathbf{a}}-\mathbf{r}}{\sqrt{\mathbf{r}_{\mathrm{a}} \mathbf{W}_{\mathbf{a}}}}+\frac{\mathbf{r}_{\mathbf{b}}-\mathbf{r}}{\sqrt{\mathbf{r}_{\mathbf{b}} \mathbf{W}_{\mathbf{b}}}}+\frac{\mathbf{r}_{\mathbf{c}}-\mathbf{r}}{\sqrt{\mathbf{r}_{\mathbf{c}} \mathbf{W}_{\mathbf{c}}}} \geq \sqrt{\frac{2 \mathbf{R}}{\mathbf{r}}}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& b+c-a=4 R \cos \frac{A}{2} \cos \frac{B-C}{2}-4 R \sin \frac{A}{2} \cos \frac{A}{2}=4 R \cos \frac{A}{2}\left(\cos \frac{B-C}{2}-\cos \frac{B+C}{2}\right) \\
& =8 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Rightarrow s-a \stackrel{(1)}{=} 4 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
& \text { Also, } a \cos A+b \cos B+\cos C=R(\sin 2 A+\sin 2 B+\sin 2 C) \\
& =R\{2 \sin (A+B) \cos (A-B)+2 \sin C \cos C\} \\
& =2 R \operatorname{sinC}\{\cos (A-B)-\cos (A+B)\}=4 R \Pi \sin A=4 R\left(\frac{\mathbf{a b c}}{\mathbf{8 R}^{3}}\right)=\frac{4 R r s}{2 R^{2}}=\frac{\mathbf{2 r s}}{\mathbf{R}} \\
& \Rightarrow \sum \mathrm{acos} \mathrm{~A} \stackrel{(2)}{\cong} \frac{2 \mathrm{rs}}{\mathrm{R}} \\
& \text { LHS }=\sum \frac{r_{a}-\mathbf{r}}{\sqrt{r_{a} \mathbf{w}_{a}}}=\sum\left(\frac{\mathbf{r}_{a}-\mathbf{r}}{\mathbf{w}_{a}} \sqrt{\frac{h_{a}}{\mathbf{r}_{a}}} \sqrt{\frac{\mathbf{w}_{a}}{\mathbf{h}_{a}}}\right) \stackrel{\because w_{a} \geq h_{a} \text { and analogs }}{\sum} \sum\left(\frac{\mathbf{r}_{a}-r}{\mathbf{w}_{a}} \sqrt{\frac{h_{a}}{r_{a}}}\right) \\
& =\sum\left\{\frac{a\left(\frac{r s}{s-a}-\frac{r s}{s}\right)(b+c)}{2 a b c \cos \frac{A}{2}} \sqrt{\frac{2 r s}{4 R \operatorname{stan} \frac{A}{2} \sin \frac{A}{2} \cos \frac{A}{2}}}\right\}
\end{aligned}
$$



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$=\sqrt{\frac{r}{2 R}} \sum\left[\frac{4 \operatorname{Rsin} \frac{A}{2} \cos \frac{A}{2}\left\{\frac{\operatorname{ars}}{s(s-a)}\right\}(b+c)}{8 R r \cos \frac{A}{2} \sin \frac{A}{2}}\right] \stackrel{\operatorname{by}(1)}{=} \sqrt{\frac{r}{2 R}} \sum\left[\frac{4 \operatorname{Rin} \frac{A}{2} \cos \frac{A}{2}\left\{\frac{4 \operatorname{Rrsin} \frac{A}{2} \cos \frac{A}{2}}{4 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}\right\}(b+c)}{8 R \operatorname{rscos} \frac{A}{2} \sin \frac{A}{2}}\right]$
$=\left(\frac{1}{2 s}\right) \sqrt{\frac{r}{2 R}} \sum\left[\left\{\frac{4 R \sin ^{2} \frac{A}{2} \cos \frac{A}{2}}{4 R \cos \frac{A}{2} \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}\right\}(b+c)\right]$
$=\left(\frac{1}{2 s}\right) \sqrt{\frac{\mathrm{r}}{2 R}} \sum\left[\left\{\frac{4 \operatorname{Rin}^{2} \frac{A}{2}}{4 R\left(\frac{\mathrm{r}}{4 \mathrm{R}}\right)}\right\}(\mathrm{b}+\mathrm{c})\right]=\left(\frac{\mathbf{R}}{\mathrm{rs}}\right) \sqrt{\frac{\mathrm{r}}{2 R}} \sum\left\{2 \sin ^{2} \frac{\mathrm{~A}}{2}(\mathrm{~b}+\mathrm{c})\right\}$
$\left.=\left(\frac{\mathbf{R}}{\mathbf{r s}}\right) \sqrt{\frac{\mathbf{r}}{2 \mathbf{R}}} \sum\{1-\cos \mathbf{A})(\mathbf{b}+\mathbf{c})\right\}=\left(\frac{\mathbf{R}}{\mathbf{r s}}\right) \sqrt{\frac{\mathbf{r}}{2 \mathbf{R}}}\left[4 \mathbf{s}-\sum\{(2 s-\mathbf{a}) \cos \mathrm{A}\}\right]$
$=\left(\frac{R}{r s}\right) \sqrt{\frac{r}{2 R}}\left\{4 s-2 s\left(1+\frac{r}{R}\right)+\sum \mathrm{acos} A\right\}$

1503. In $\triangle A B C$ the following relationship holds:

$$
2\left(\frac{r_{a}}{r_{b}}+\frac{r_{b}}{r_{c}}+\frac{r_{c}}{r_{a}}\right) \geq \frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{r}}+3
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution by Marian Ursărescu-Romania

$$
\begin{equation*}
\frac{r_{a}}{r_{b}}=\frac{\frac{s}{s-a}}{\frac{s}{s-b}}=\frac{s-b}{s-a} \text { and } \frac{h_{a}}{r_{a}}=\frac{\frac{2 s}{a}}{\frac{s}{s-a}}=\frac{2(s-a)}{a} \tag{1}
\end{equation*}
$$

We must show that: $2\left(\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}\right) \geq 2\left(\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}\right)+3$

$$
\begin{equation*}
\text { Let: } s-a=x ; s-b=y ; s-c=z \tag{2}
\end{equation*}
$$

We must show that: $2\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right) \geq \mathbf{2}\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)+\mathbf{3} \Leftrightarrow$

$$
2\left(x\left(\frac{1}{z}-\frac{1}{y+z}\right)+y\left(\frac{1}{x}-\frac{1}{x+z}\right)+z\left(\frac{1}{y}+\frac{1}{x+y}\right)\right) \geq 3 \Leftrightarrow
$$



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$$
\begin{gather*}
2\left(\frac{x y}{z(y+z)}+\frac{\text { www.ssmrmh.ro } \left._{x x}^{x(x+z)}+\frac{z x}{y(x+y)}\right) \geq 3 \Leftrightarrow}{\frac{x y}{z(y+z)}+\frac{y z}{x(x+z)}+\frac{z x}{y(x+y)} \geq \frac{3}{2}}\right. \\
\frac{x^{2} y^{2}}{x y z(y+z)}+\frac{y^{2} z^{2}}{x y z(x+z)}+\frac{z^{2} x^{2}}{x y z(x+y)} \geq \frac{3}{2} \\
\frac{x^{2} y^{2}}{y+z}+\frac{y^{2} z^{2}}{x+z}+\frac{z^{2} x^{2}}{x+y} \geq \frac{3}{2} x y z
\end{gather*}
$$

From Bergstrom inequality we have: $\frac{x^{2} y^{2}}{y+z}+\frac{y^{2} z^{2}}{x+z}+\frac{z^{2} x^{2}}{x+y} \geq \frac{(x y+y z+z x)^{2}}{2(x+y+z)}$
From inequality $(m+n+p)^{2} \geq \mathbf{3}(m n+n p+p m) \Rightarrow$

$$
\begin{equation*}
(x y+y z+z x)^{2} \geq 3 x y z(x+y+z) \tag{5}
\end{equation*}
$$

From (4)+(5) we have: $\frac{x^{2} y^{2}}{y+z}+\frac{y^{2} z^{2}}{x+z}+\frac{z^{2} x^{2}}{x+y} \geq \frac{3}{2} x y z \Rightarrow$ (3) it's true.
1504. In $\triangle A B C, I$-incenter, $\boldsymbol{R}_{\boldsymbol{a}}, \boldsymbol{R}_{\boldsymbol{b}}, \boldsymbol{R}_{\boldsymbol{c}}$-circumradii in
$\triangle B I C, \triangle C I A$ - respectively $\triangle A I B$ the following relationship holds:

$$
\frac{\mathbf{R}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}}+\frac{\mathbf{R}_{\mathrm{b}}}{\mathbf{w}_{\mathrm{b}}}+\frac{\mathbf{R}_{\mathbf{c}}}{\mathbf{w}_{\mathbf{c}}} \geq 2\left(\frac{\mathbf{r}_{\mathrm{a}}+\mathbf{r}_{\mathrm{b}}+\mathbf{r}_{\mathrm{c}}+A I+B I+\mathbf{C I}}{\mathbf{m}_{\mathbf{a}}+\mathbf{m}_{b}+\mathbf{m}_{\mathrm{c}}+\mathbf{h}_{\mathrm{a}}+\mathbf{h}_{\mathrm{b}}+\mathbf{h}_{\mathrm{c}}-3 \mathbf{r}}\right)^{2}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{gather*}
r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\operatorname{s\operatorname {sin}(\frac {B+C}{2})\operatorname {cos}\frac {A}{2}}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{scos}^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
\therefore r_{b}+r_{c} \stackrel{(i)}{=} 4 R \cos ^{2} \frac{A}{2} \tag{i}
\end{gather*}
$$

Applying sine rule on $\triangle B I C, 2 R_{a} \sin \left(\pi-\frac{B+C}{2}\right)=a=4 R \sin \frac{A}{2} \cos \frac{A}{2} \Rightarrow$

$$
\begin{gathered}
2 R_{a} \cos \frac{A}{2}=4 R \sin \frac{A}{2} \cos \frac{A}{2} \\
\Rightarrow R_{a}=2 R \sin \frac{A}{2} \text { and analogs } \Rightarrow \sum \frac{R_{a}}{w_{a}}=\frac{2 R(b+c) \sin \frac{A}{2}}{2 b \cos \frac{A}{2}}=\frac{R}{s} \sum\left\{\operatorname{stan} \frac{A}{2}\left(\frac{1}{b}+\frac{1}{c}\right)\right\}
\end{gathered}
$$



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$$
=\frac{\mathbf{R}}{\mathbf{s}} \sum\left\{\left(\frac{1}{\mathbf{b}}+\frac{1}{c}\right) \mathbf{r}_{\mathrm{a}}\right\}=\frac{\mathbf{R}}{\mathbf{s}} \sum \frac{\left(\mathbf{r}_{\mathbf{b}}+\mathbf{r}_{\mathrm{c}}\right)}{\mathrm{a}}
$$

$\stackrel{\text { by (i) }}{\cong} \frac{R}{s} \sum \frac{4 R \cos ^{2} \frac{A}{2}}{4 R \sin \frac{A}{2} \cos \frac{A}{2}}=R \sum \frac{1}{\operatorname{stan} \frac{A}{2}}=R \sum \frac{1}{r_{a}}=\frac{R}{r} \Rightarrow \frac{R_{a}}{w_{a}}+\frac{R_{b}}{w_{b}}+\frac{R_{c}}{w_{c}} \stackrel{(i i)}{m} \frac{R}{r}$

$$
\begin{aligned}
& \text { Now, }(b+c)^{2} \geq 32 \operatorname{Rrcos}^{2} \frac{A^{b y(i)}}{2} \stackrel{n}{=} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right) \\
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a) \\
& \Leftrightarrow(b+c)^{2}+4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \text { true }
\end{aligned}
$$

$$
\therefore b+c \stackrel{\sim}{c}_{\geq}^{(\mathrm{iii})} 4 \sqrt{2 R r} \cos \frac{A}{2} \text { and analogs }
$$

$$
\text { Now, } \sum m_{a} \stackrel{\text { Ioscu }}{\geq} \sum \frac{b+c}{2} \cos \frac{A^{\text {by (iii)and analogs }}}{2} \stackrel{\sim}{\geq} \sqrt{2 R r} \sum\left(2 \cos ^{2} \frac{A}{2}\right)
$$

$$
=\sqrt{2 R r} \sum(1+\cos A)=\sqrt{2 R r}\left(\frac{4 R+r}{R}\right)=\sqrt{\frac{2 r}{R}}(4 R+r)
$$

$$
\Rightarrow \sqrt{\frac{\mathbf{R}}{2 r}} \sum \mathbf{m}_{\mathrm{a}} \stackrel{(\mathrm{iv})}{\stackrel{m}{\geq}} \sum \mathbf{r}_{\mathrm{a}}
$$

$$
\text { Again, }\left(s^{2}-2 R r+r^{2}\right)^{2} \geq 8 R r\left(s^{2}+4 R r+r^{2}\right)
$$

(a)

$$
\Leftrightarrow s^{4} \xrightarrow{m}\left(12 R r-2 r^{2}\right) s^{2}+8 R r\left(4 R r+r^{2}\right)-\left(2 R r-r^{2}\right)^{2}
$$

Now, LHS of (a) $\stackrel{\text { Gerretsen }}{\geq}\left(16 \operatorname{Rr}-5 r^{2}\right) s^{2} \stackrel{?}{\geq}\left(12 R r-2 r^{2}\right) s^{2}+8 R r\left(4 R r+r^{2}\right)-\left(2 R r-r^{2}\right)^{2}$

$$
\Leftrightarrow\left(4 R r-3 r^{2}\right) s^{2} \sum_{(b)}^{?} 8 R r\left(4 R r+r^{2}\right)-\left(2 R r-r^{2}\right)^{2}
$$

Again, LHS of $(b) \stackrel{\text { Gerretsen }}{\gtrless}\left(4 R r-3 r^{2}\right)\left(16 R r-5 r^{2}\right) \stackrel{?}{\underset{\sim}{\geq}} 8 \operatorname{Rr}\left(4 R r+r^{2}\right)-\left(2 R r-r^{2}\right)^{2}$

$$
\Leftrightarrow 9 \mathbf{R}^{2}-\mathbf{2 0 R r}+4 \mathbf{r}^{2} \stackrel{\stackrel{?}{n}}{\geq} \mathbf{0}
$$

$$
\Leftrightarrow(R-2 r)(9 R-2 r) \stackrel{?}{\text { in }} 0 \rightarrow \text { true } \because \mathbf{R} \stackrel{\text { Euler }}{\geqq} 2 r \Rightarrow(b) \Rightarrow(a) \text { is true }
$$

(v)
$\therefore \mathbf{s}^{2}-\mathbf{2 R r}+\mathbf{r}^{2} \geqq 2 \sqrt{2 R r} \sqrt{\mathbf{s}^{2}+4 \mathbf{R r}+\mathbf{r}^{2}}$


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$$
\begin{aligned}
& \text { Now, } \sum \mathbf{A I}=\mathbf{r} \sum \frac{\mathbf{1}}{\sin \frac{\mathbf{A}}{2}}=\mathbf{r} \sqrt{\frac{\mathbf{b w w}(\mathbf{s}-\mathbf{a})}{(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}} \stackrel{\text { css }}{\leftrightarrows} \frac{\mathbf{r}}{\mathbf{r} \sqrt{\mathbf{s}}} \sqrt{\sum \mathbf{b c}} \sqrt{\sum(\mathbf{s}-\mathbf{a})} \\
& =\sqrt{\mathbf{s}^{2}+4 \mathbf{R r}+\mathbf{r}^{2}} \stackrel{\text { by }(\mathrm{v})}{\leftrightharpoons} \frac{\mathbf{s}^{2}-2 \mathbf{R r}+\mathbf{r}^{2}}{2 \sqrt{2 R r}} \\
& \Rightarrow s^{2}-2 R r+r^{2} \stackrel{(v i)}{\geqq} 2 \sqrt{2 R r} \sum A I \\
& \text { Also, } h_{a}+h_{b}+h_{c}-3 r=\frac{s^{2}+4 R r+r^{2}}{2 R}-3 r=\frac{s^{2}-2 R r+r^{2}}{2 R} \stackrel{\text { by (vi) }}{\geqq} \frac{2 \sqrt{2 R r}}{2 R} \sum A I \\
& \Rightarrow \sqrt{\frac{\mathbf{R}}{2 r}}\left(\mathbf{h}_{\mathbf{a}}+\mathbf{h}_{\mathbf{b}}+\mathbf{h}_{\mathrm{c}}-3 \mathbf{r}\right) \stackrel{(\text { vii) }}{\geqq} \sum \mathbf{A I}
\end{aligned}
$$

$$
\begin{aligned}
& \text { (iv) }+(\text { vii }) \Rightarrow \sqrt{\frac{R}{2 r}}\left(m_{a}+m_{b}+m_{c}+h_{a}+h_{b}+h_{c}-3 r\right) \geq r_{a}+r_{b}+r_{c}+A I+B I+C I
\end{aligned}
$$

$$
\begin{aligned}
& \geq 2\left(\frac{r_{a}+r_{b}+r_{c}+\text { AI }+ \text { BI }+\mathbf{C I}}{m_{a}+m_{b}+m_{c}+h_{a}+h_{b}+h_{c}-3 r}\right)^{2} \text { (Proved) }
\end{aligned}
$$

1505. In $\triangle A B C$ the following relationship holds:

$$
(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right) \geq 12\left(r_{a}+r_{b}+r_{c}\right)
$$

Proposed by Bogdan Fuştei-Romania
Solution 1 by Marian Ursărescu-Romania

$$
\begin{gather*}
\text { We have: } \frac{1}{\operatorname{sinA}}=\frac{2 r}{a} \text { and } r_{a}+r_{b}+r_{c}=4 R+r \\
\text { We must show: }(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2} 2 R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 6(4 R+r) \tag{1}
\end{gather*}
$$

$$
\text { From Cauchy inequality: } \frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{1}{3}\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right)^{2}
$$

From (1) $+(2)$ we must show: $\frac{R}{3}\left((\sqrt{a}+\sqrt{b}+\sqrt{c})\left(\frac{1}{r_{a}}+\frac{1}{r_{b}}+\frac{1}{r_{c}}\right)\right)^{2} \geq 6(4 R+r)$


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$$
\begin{equation*}
\text { But: }(\sqrt{a}+\sqrt{b}+\sqrt{c})\left(\frac{1}{\sqrt{a}}+\frac{1}{\sqrt{b}}+\frac{1}{\sqrt{c}}\right) \geq 9 \tag{4}
\end{equation*}
$$

From (3)+(4) we must show: $27 R \geq 24 R+6 r \Leftrightarrow 3 R \geq 6 r \Leftrightarrow R \geq 2 r$ true (Euler) Solution 2 by Ertan Yldirim-Turkey

$$
\begin{gather*}
\text { Lema 1: } 4 R+r=r_{a}+r_{b}+r_{c} \\
\text { Lema } 2: \frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=2 R \\
L H S=(a+b+c+2 \sqrt{a b}+2 \sqrt{b c}+2 \sqrt{c a}) \cdot 2 R \cdot\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \\
\begin{array}{c}
A m-G m \\
\geq
\end{array}\left(a+b+c+3 \sqrt[3]{2 \sqrt{a b} \cdot 2 \sqrt{b c} \cdot 2 \sqrt{c a})} 2 R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right. \\
=(a+b+c+6 \sqrt[3]{a b c})\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) 2 R \\
=\left[(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+6 \sqrt[6]{a b c}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right] 2 R  \tag{*}\\
\text { Since Am } \geq H m: \frac{a+b+c}{3} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3 \\
\text { Since Am} \geq H m: \sqrt[3]{a b c} \geq \frac{3}{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}} \Rightarrow \sqrt[3]{a b c}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 3 \\
{\left[(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)+6 \sqrt[6]{a b c}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\right] 2 R \geq(9+6 \cdot 3) 2 R=27 \cdot 2 R}
\end{gather*}
$$

Then we will show that: $27 \cdot 2 R \geq 12\left(r_{a}+r_{b}+r_{c}\right)=12(4 R+r)$

$$
9 R \geq 8 R+2 r \Leftrightarrow R \geq 2 r \text { true (Euler) }
$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right)=2 R\left(\sum a+2 \sum \sqrt{a b}\right) \sum \frac{1}{a} \\
=\frac{2 R\left(2 s+2 \sum \sqrt{a b}\right)\left(s^{2}+4 R r+r^{2}\right)}{4 R r s} \\
\underset{\substack{G M-H M \\
\sum}}{\left(s+2 \sum \frac{a b}{a+b}\right)\left(s^{2}+4 R r+r^{2}\right)} \\
r s
\end{gathered} \frac{\left(s+\frac{2 \sum a b(b+c)(c+a)}{\prod(a+b)}\right)\left(s^{2}+4 R r+r^{2}\right)}{r s} .
$$



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$$
\begin{gathered}
=\frac{\left(s+\frac{\left(\sum a b\right)^{2}+8 R r s^{2}}{s\left(s^{2}+2 R r+r^{2}\right)}\right)\left(s^{2}+4 R r+r^{2}\right)}{r s} \\
=\frac{\left[\left(s^{2}+2 R r+r^{2}\right) s^{2}+\left(s^{2}+4 R r+r^{2}\right)^{2}+8 R r s^{2}\right]\left(s^{2}+4 R r+r^{2}\right)}{r\left(s^{2}+2 R r+r^{2}\right) s^{2}} \geq 12\left(r_{a}+r_{b}+r_{c}\right) \\
\Leftrightarrow\left[\left(s^{2}+2 R r+r^{2}\right) s^{2}+\left(s^{2}+4 R r+r^{2}\right)^{2}+8 R r s^{2}\right]\left(s^{2}+4 R r+r^{2}\right) \stackrel{\text { ? }}{\substack{~}} \\
\geq 12 r(4 R+r) s^{2}\left(s^{2}+2 R r+r^{2}\right) \\
\Leftrightarrow 2 s^{6}+r(4 R+r)^{3} \underset{(1)}{\sum_{(1)}^{2}} s^{4}\left(22 R r+7 r^{2}\right)+s^{2}\left(8 R^{2} r^{2}+34 R r^{3}+8 r^{4}\right)
\end{gathered}
$$

$$
\text { Now, LHS of }(1) \stackrel{\text { Gerretsen }}{\gtrless} s^{4}\left(32 R r-10 r^{2}\right)+r(4 R+r)^{3} \stackrel{?}{\Perp}
$$

$$
\geq s^{4}\left(22 R r+7 r^{2}\right)+s^{2}\left(8 R^{2} r^{2}+34 R r^{3}+8 r^{4}\right)
$$

$$
\Leftrightarrow s^{4}\left(10 R r-17 r^{2}\right)+r(4 R+r)^{3}{\underset{\sim}{2}}_{\stackrel{?}{2}} s^{2}\left(8 R^{2} r^{2}+34 R r^{3}+8 r^{4}\right)
$$

Now, LHS of (2) $\stackrel{\text { Gerretsen }}{\geq} s^{2}\left(16 R r-5 r^{2}\right)\left(10 R r-17 r^{2}\right)+r(4 R+r)^{3} \xrightarrow{\text { ² }}$ $\geq s^{2}\left(8 R^{2} r^{2}+34 R r^{3}+8 r^{4}\right)$

$$
\Leftrightarrow s^{2}\left(152 R^{2}-356 R r+77 r^{2}\right)+r(4 R+r)^{3} \stackrel{\text { ¿े }}{\geq} 0
$$

$$
\Leftrightarrow s^{2}\left(152 R^{2}-356 R r+104 r^{2}\right)+r(4 R+r)^{3} \stackrel{?}{\check{n}} 27 r^{2} s^{2}
$$

$$
\Leftrightarrow s^{2}(R-2 r)(152 R-52 r)+r(4 R+r)^{3}{\underset{(3)}{?} 27 r^{2} s^{2}, ~(3)}^{2}
$$

Now, $L H S$ of $(3) \underbrace{{\underset{L}{l}}_{\text {Gerretsen }}^{\sim}}_{(a)}(R-2 r)(152 R-52 r)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3}$ and RHS of $(3) \underbrace{\substack{\text { Gerretsen } \\ \leq}}_{(b)} 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right)$
$(a),(b) \Rightarrow$ in order to prove (3), it suffices to prove :

$$
\begin{gathered}
(R-2 r)(152 R-52 r)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3} \geq 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right) \\
\Leftrightarrow 624 t^{3}-1629 t^{2}+837 t-150 \geq 0\left(\text { where } t=\frac{R}{r}\right)
\end{gathered}
$$



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$$
\begin{aligned}
& \Leftrightarrow(t-2)(624 t(t-2)+867 t+75) \geq 0 \rightarrow \text { true } \Rightarrow(3) \Rightarrow(2) \Rightarrow(1) \text { is true } \\
& \quad \therefore(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right) \geq 12\left(r_{a}+r_{b}+r_{c}\right)(\text { Proved })
\end{aligned}
$$

Solution 4 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gather*}
\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}=\frac{s^{2}+r^{2}+4 R r}{2 s r} \\
r_{a}+r_{b}+r_{c}=4 R+r \\
L H S=(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2}\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\operatorname{sinC}}\right) \geq 9 \sqrt[3]{4 R r s}\left(\frac{s^{2}+r^{2}+4 R r}{2 s r}\right) \\
(\sqrt{a}+\sqrt{b}+\sqrt{c})^{2} \stackrel{A m-G m}{\geqq}(3 \sqrt[3]{\sqrt{a b c}})^{2}=9 \sqrt[3]{a b c}=9 \sqrt[3]{4 R r s} \\
\text { We need to prove: } 9 \sqrt[3]{4 R r s}\left(\frac{s^{2}+r^{2}+4 R r}{2 s r}\right) \geq 12(4 R+r) \Leftrightarrow \\
9 \sqrt[3]{4 R r s}\left(s^{2}+r^{2}+4 R r\right)^{3} \geq 2 \cdot 12 s r(4 R+r) \Leftrightarrow \\
27 R\left(s^{2}+r^{2}+4 R r\right)^{3} \geq 2 \cdot 4 \cdot 16(s r)^{2}(4 R+r)^{3} \quad(*) \tag{*}
\end{gather*}
$$

Because:
(1)

$$
9 R \geq 2(4 R+r) \Leftrightarrow R \geq 2 r \text { (Euler) }
$$

(2)

$$
\begin{equation*}
s^{2}+r^{2}+4 R r \geq 4 r(4 R+r) \Leftrightarrow s^{2} \geq 12 R r+3 r^{2} \tag{3}
\end{equation*}
$$

But: $s^{2} \geq 16 R r-5 r^{2} \xrightarrow{2} 12 R r+3 r^{2}$
(3) $\Leftrightarrow 4 R r \geq 8 r^{2} \Leftrightarrow R \geq 2 r$ (Euler) $\Rightarrow$ (3) true $\Rightarrow$ (2) true.

$$
\begin{gathered}
3\left(s^{2}+r^{2}+4 R r\right)^{2} \stackrel{(4)}{\geq} 16 s^{2}\left(4 R r+r^{2}\right) \Leftrightarrow \\
3\left[s^{4}+2 s^{2}\left(r^{2}+4 R r\right)+\left(r^{2}+4 R r\right)^{2}\right] \geq 16 s^{2}\left(4 R r+r^{2}\right) \Leftrightarrow \\
3 s^{4}-10\left(r^{2}+4 R r\right) s^{2}+3\left(r^{2}+4 R r\right)^{2} \geq 0 \Leftrightarrow \\
3\left[s^{2}-3\left(r^{2}+4 R r\right)\right]\left[s^{2}-\left(r^{2}+4 R r\right)\right] \geq 0 \\
s^{2} \geq 3 r^{2}+12 R r \text { true by (2) }
\end{gathered}
$$

$$
s^{2} \stackrel{(5)}{\geq} r^{2}+4 R r
$$



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But: $s^{2} \geq 16 R r-5 r^{2} \xrightarrow{\text { M }} r^{2}+4 R r$
(6) $\Leftrightarrow 12 R r \geq 6 r^{2} \Leftrightarrow R \geq \frac{r}{2} \Rightarrow$ (6) true $\Rightarrow$ (5) true $\Rightarrow$ (4)true $\Rightarrow$ (*) true. Proved.
1506. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c}\left(\frac{r_{a}}{h_{a}}-\frac{h_{a}}{r_{a}}\right) \geq \frac{(R-2 r)(2 R-3 r)}{2 R(R-r)}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& h_{a}=\frac{2 S}{a} ; r_{a}=\frac{S}{s-a} \Rightarrow \frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}=2\left(\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}\right) \\
& =2 \cdot \frac{b c(s-a)+c a(s-b)+a b(s-c)}{a b c}=\frac{2(s(a b+b c+c a)-3 a b c)}{a b c} \\
& =\frac{2\left(s\left(s^{2}+4 R r+r^{2}\right)-12 R r s\right)}{4 R r s}=\frac{s^{2}-8 R r+r^{2}}{2 R r} \Rightarrow \\
& \frac{r_{a}}{h_{a}}+\frac{r_{b}}{h_{b}}+\frac{r_{c}}{h_{c}}=\frac{1}{2}\left(\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-c}\right)=\frac{1}{2} \sum_{c y c}\left(\frac{\cot \frac{A}{2}+\cot \frac{B}{2}}{\cot \frac{C}{2}}\right)=\frac{2 R-r}{r} \\
& \text { Thus: } \\
& \sum_{c y c}\left(\frac{r_{a}}{h_{a}}-\frac{h_{a}}{r_{a}}\right)=\sum_{c y c} \frac{r_{a}}{h_{a}}-\sum_{c y c} \frac{h_{a}}{r_{a}}=\frac{2 R-r}{r}-\frac{s^{2}-8 R r+r^{2}}{2 R r}=\frac{4 R^{2}+6 R r-r^{2}-s^{2}}{2 R r} \\
& \stackrel{s^{2} \leq 4 R^{2}+4 R r+3 r^{2}}{\stackrel{2}{\geq}} \frac{2 R r-4 r^{2}}{2 R r}=\frac{R-2 r}{R} \stackrel{(*)}{\geq} \frac{(R-2 r)((2 R-3 r)}{2 R(R-r)}
\end{aligned}
$$

$(*) \Leftrightarrow(R-2 r)(2 R-2 r-2 R+3 r) \geq 0 \Leftrightarrow r(R-2 r) \geq 0$ true by $R \geq 2 r$ (Euler). Proved.
1507. In $\triangle A B C$ the following relationship holds:

$$
\sum \sqrt{\mathbf{r}_{\mathrm{a}} \mathbf{s}_{\mathrm{a}}} \geq \sqrt{\frac{\mathbf{r}}{2 \mathrm{R}}} \sum\left(\mathbf{w}_{\mathrm{a}}+\mathbf{r}_{\mathrm{a}}\right)
$$



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## Solution by Soumava Chakraborty-Kolkata-India

As perpendicular distance from a vertex to opposite side is least among all distances from that vertex to the opposite side,

$$
\begin{gathered}
\therefore \mathbf{h}_{\mathrm{a}} \leq \mathbf{s}_{\mathrm{a}} \text { and analogs } \Rightarrow \sum \sqrt{\mathrm{r}_{\mathrm{a}} \mathrm{~s}_{\mathrm{a}}} \geq \sum \sqrt{\mathrm{r}_{\mathrm{a}} \mathrm{~h}_{\mathrm{a}}}=\sum \sqrt{\operatorname{stan} \frac{A}{2} \cdot\left(\frac{2 \mathrm{rs}}{4 \operatorname{Rin} \frac{A}{2} \cos \frac{A}{2}}\right)} \\
=\sum \sqrt{\operatorname{stan} \frac{A}{2} \cdot\left(\frac{2 r s}{4 R \tan \frac{A}{2} \cos ^{2} \frac{A}{2}}\right)}=\mathrm{s} \sqrt{\frac{\mathrm{r}}{2 \mathrm{R}} \sum \sec \frac{A}{2}} \\
\Rightarrow \text { it suffices to prove }: \mathrm{s} \sqrt{\frac{\mathrm{r}}{2 \mathrm{R}} \sum \sec \frac{A}{2} \geq \sqrt{\frac{\mathrm{r}}{2 R}} \sum\left(\mathrm{w}_{\mathrm{a}}+\mathrm{r}_{\mathrm{a}}\right)}
\end{gathered}
$$

$$
\Leftrightarrow s \sum \sec \frac{\mathbf{A}_{2}^{(1)}}{2} \underset{\sim}{\sum} \sum\left(\mathbf{w}_{a}+r_{a}\right)
$$

$$
\text { Now, } \operatorname{ssec} \frac{A}{2} \geq w_{a}+r_{a}=\frac{2 b c}{b+c} \cos \frac{A}{2}+\operatorname{stan} \frac{A}{2} \Leftrightarrow
$$

$$
\Leftrightarrow \operatorname{ssec} \frac{A}{2} \cos \frac{A}{2} \geq \frac{2 b c}{b+c} \cos ^{2} \frac{A}{2}+\operatorname{stan} \frac{A}{2} \cos \frac{A}{2}
$$

$$
\Leftrightarrow s\left(1-\sin \frac{A}{2}\right) \geq \frac{2 b c}{b+c}\left(1-\sin \frac{A}{2}\right)\left(1+\sin \frac{A}{2}\right) \Leftrightarrow
$$

$$
\Leftrightarrow \operatorname{sa}(b+c) \geq 2 a b c\left(1+\sin \frac{A}{2}\right)\left(\because 0<\sin \frac{\mathbf{A}}{2}<1 \text { as } 0<\frac{A}{2}<\frac{\pi}{2}\right)
$$

$$
\Leftrightarrow s .4 R \sin \frac{A}{2} \cos \frac{A}{2} \cdot 4 R \cos \frac{A}{2} \cos \frac{B-C}{2} \geq 8 R(r) s\left(1+\sin \frac{A}{2}\right)
$$

$$
=8 R s .4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}\left(1+\sin \frac{A}{2}\right)
$$

$$
\Leftrightarrow \cos ^{2} \frac{A}{2} \cos \frac{B-C}{2} \geq 2 \sin \frac{B}{2} \sin \frac{C}{2}\left(1+\sin \frac{A}{2}\right)
$$

$$
\Leftrightarrow\left(1-\sin \frac{A}{2}\right)\left(1+\sin \frac{A}{2}\right) \cos \frac{B-C}{2}-\left(\cos \frac{B-C}{2}-\sin \frac{A}{2}\right)\left(1+\sin \frac{A}{2}\right) \geq 0
$$

$$
\Leftrightarrow\left(1+\sin \frac{A}{2}\right)\left\{\left(1-\sin \frac{A}{2}\right) \cos \frac{B-C}{2}-\cos \frac{B-C}{2}+\sin \frac{A}{2}\right\} \geq 0
$$

$$
\Leftrightarrow\left(1+\sin \frac{A}{2}\right)\left(-\sin \frac{A}{2} \cos \frac{B-C}{2}+\sin \frac{A}{2}\right) \geq 0
$$



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$$
\Leftrightarrow \sin \frac{A}{2}\left(1+\sin \frac{A}{2}\right)\left(1-\cos \frac{B-C}{2}\right) \geq 0 \rightarrow \text { true }
$$

$\because \mathbf{0}<\sin \frac{\mathbf{A}}{2}<\mathbf{1}\left(\right.$ as $\left.\mathbf{0}<\frac{\mathbf{A}}{2}<\frac{\pi}{2}\right)$ and $\mathbf{0}<\cos \frac{\mathbf{B}-\mathbf{C}}{2} \leq \mathbf{1}\left(\right.$ as $\left.-\frac{\boldsymbol{\pi}}{2}<\frac{\mathbf{B}-\mathbf{C}}{2}<\frac{\boldsymbol{\pi}}{2}\right)$
$\therefore \operatorname{ssec} \frac{A}{2} \geq w_{a}+r_{a}$ and analogs $\Rightarrow \mathbf{s} \sum \sec \frac{A}{2} \geq \sum\left(w_{a}+r_{a}\right) \Rightarrow$ (1)is true

$$
\therefore \sum \sqrt{\mathbf{r}_{\mathbf{a}} \mathbf{s}_{\mathbf{a}}} \geq \sqrt{\frac{\mathbf{r}}{2 \mathbf{R}}} \sum\left(\mathbf{w}_{\mathbf{a}}+\mathbf{r}_{\mathbf{a}}\right) \text { (Proved) }
$$

1508. In $\triangle A B C, \boldsymbol{n}_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\sum\left(R+w_{a}\right)\left(\frac{b}{c}+\frac{c}{b}\right) \geq 2\left(R+\sum m_{a}\right)+\sum \frac{n_{a} g_{a}}{h_{a}}-r
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
=\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \mathbf{a n d} \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})=\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{gathered}
$$

Adding the above two, we get :

$$
\text { By Tsintsifas, } 2\left(R+\sum \mathbf{m}_{\mathrm{a}}\right)+\sum \frac{\mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}}-\mathbf{r} \leq
$$

$$
\leq 2 R+\sum\left(\frac{b}{c}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathrm{a}}+\sum \frac{\mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}}-\mathbf{r} \stackrel{\mathrm{A}-\mathbf{G}}{\leftrightarrows} 2 R+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{a}+\sum \frac{\mathbf{n}_{a}^{2}+\mathbf{g}_{a}^{2}}{2 \mathbf{h}_{\mathrm{a}}}-\mathbf{r}
$$

$$
\stackrel{\text { by }(\mathbf{1})}{\cong} 2 \mathbf{R}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\sum \frac{\mathbf{a}\left\{4 \mathbf{m}_{\mathbf{a}}{ }^{2}-2 \mathbf{s}(\mathbf{s}-\mathbf{a})\right\}}{4 \mathbf{r s}}-\mathbf{r}
$$

$$
\begin{aligned}
& \left(b^{2}+\mathbf{c}^{2}\right)(2 s-b-c)=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a g}_{\mathrm{a}}{ }^{2}+2 \mathrm{a}(\mathrm{~s}-\mathrm{b})(\mathrm{s}-\mathrm{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(n_{a}^{2}+\mathrm{ga}^{2}\right)+\mathbf{a}(\mathbf{a}+\mathrm{b}-\mathbf{c})(\mathbf{c}+\mathbf{a}-\mathrm{b}) \Rightarrow \mathbf{2}\left(\mathrm{b}^{2}+\mathrm{c}^{2}\right) \\
& =2\left(n_{a}^{2}+g_{a}{ }^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+(\mathrm{b}-\mathrm{c})^{2}+4 \mathrm{r}_{\mathrm{b}} \mathrm{r}_{\mathrm{c}}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}_{\mathrm{a}}{ }^{2}\right)+4 \mathrm{r}_{\mathrm{b}} \mathrm{r}_{\mathrm{c}} \Rightarrow \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+(\mathrm{b}-\mathrm{c})^{2}+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a})=2\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2}\right)+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a}) \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+4 \mathrm{~m}_{\mathrm{a}}{ }^{2}=2\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}_{\mathrm{a}}{ }^{2}\right)+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a}) \Rightarrow \mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2} \stackrel{(1)}{\cong} 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}-\mathbf{2 s}(\mathrm{s}-\mathrm{a})
\end{aligned}
$$



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$$
\text { Again, } \sum\left(\mathbf{R}+\mathbf{w}_{\mathbf{a}}\right)\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right)=\left\{\mathbf{R} \sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right)-2 \mathbf{R}+\mathbf{r}\right\}+2 \mathbf{R}-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}
$$

$$
=2 \mathbf{R}-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\frac{\mathbf{R} \sum \mathbf{a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)}{4 \operatorname{Rrs}}-2 \mathbf{R}+\mathbf{r}
$$

$$
=2 \mathbf{R}-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\frac{\sum \mathbf{a b}(2 s-\mathbf{c})}{4 \mathbf{r s}}-2 \mathbf{R}+\mathbf{r}
$$

$$
=2 \mathbf{R}-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\frac{\mathbf{s}^{2}+4 \mathbf{R r}+\mathbf{r}^{2}-6 \mathbf{R r}}{2 \mathbf{r}}-2 \mathbf{R}+\mathbf{r}
$$

$$
=2 \mathbf{R}-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\frac{\mathbf{s}^{2}-2 \mathbf{R r}+\mathbf{r}^{2}-4 \mathbf{R r}+2 \mathbf{r}^{2}}{2 \mathbf{r}}
$$

$$
\Rightarrow \text { LHS }=2 R-r+\sum\left(\frac{b}{c}+\frac{c}{b}\right) w_{a}+\frac{s^{2}-6 R r+3 r^{2}}{2 r} \stackrel{\text { by }(\mathrm{i})}{\geq} \text { RHS (Proved) }
$$

1509. In $\triangle A B C$ the following relationship holds:

$$
\sum \frac{r_{b}+r_{c}}{a} \geq \frac{1}{2} \sum \frac{b+c}{\sqrt{\left(r_{b}-r\right)\left(r_{c}-r\right)}}
$$

$$
\begin{aligned}
& \text { www.ssmrmh.ro } \\
& =2 R-\mathbf{r}+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathbf{a}}+\sum \frac{\mathbf{a}\left\{\mathbf{2}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)-\mathbf{a}^{2}-2 \mathbf{s}(\mathbf{s}-\mathbf{a})\right\}}{4 \mathbf{r s}} \\
& =2 R-r+\sum\left(\frac{b}{c}+\frac{c}{b}\right) w_{a}+\frac{2 \sum \mathbf{a b}(2 s-c)-\sum a^{3}-2 s^{2}(2 s)+2 s \sum a^{2}}{4 r s} \\
& =2 R-r+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right) \mathbf{w}_{\mathrm{a}}+ \\
& +\frac{2\left(s^{2}+4 R r+r^{2}\right)-12 R r-\left(s^{2}-6 R r-3 r^{2}\right)-2 s^{2}+2\left(s^{2}-4 R r-r^{2}\right)}{2 r} \\
& =2 R-r+\sum\left(\frac{b}{c}+\frac{c}{b}\right) w_{a}+\frac{s^{2}-6 R r+3 r^{2}}{2 r} \\
& \Rightarrow \text { RHS } \stackrel{(\mathrm{i})}{\stackrel{\sim}{\leq}} 2 R-r+\sum\left(\frac{b}{c}+\frac{c}{b}\right) w_{a}+\frac{s^{2}-6 R r+3 r^{2}}{2 r}
\end{aligned}
$$



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Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{sos} \cos ^{2} \frac{A}{2}}{\left(\frac{S}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
& \Rightarrow \frac{r_{b}+r_{c}}{a}=\frac{4 R \cos ^{2} \frac{A}{2}}{4 R \sin \frac{A}{2} \cos \frac{A}{2}}=\frac{s}{\operatorname{stan} \frac{A}{2}}=\frac{s}{r_{a}} \\
& \Rightarrow \frac{r_{b}+r_{c}}{a}=\frac{s}{r_{a}} \text { and analogs } \Rightarrow \sum \frac{r_{b}+r_{c}}{a}=s \sum \frac{1}{r_{a}}=\frac{s}{r} \Rightarrow \text { LHS } \stackrel{(1)}{=} \frac{s}{r} \\
& \text { Now, } \sqrt{\left(\mathrm{r}_{\mathrm{b}}-\mathrm{r}\right)\left(\mathrm{r}_{\mathrm{c}}-\mathrm{r}\right)}=\mathrm{rs} \sqrt{\left(\frac{1}{s-\mathrm{b}}-\frac{1}{\mathrm{~s}}\right)\left(\frac{1}{\mathrm{~s}-\mathrm{c}}-\frac{1}{\mathrm{~s}}\right)}= \\
& =r \sqrt{\frac{b}{s(s-b)} \frac{c}{s(s-c)}}=r \sqrt{\frac{b c}{(s-b)(s-c)}} \\
& \Rightarrow \frac{b+c}{\sqrt{\left(r_{b}-r\right)\left(r_{c}-r\right)}}=\frac{1}{r}(b+c) \sqrt{\frac{(s-b)(s-c)}{b c}}=\frac{1}{r}\left(\frac{b+c}{\sqrt{b c}}\right) \sqrt{(s-b)(s-c)} \\
& =\frac{1}{r}\left(\sqrt{\frac{b}{c}}+\sqrt{\frac{\mathbf{c}}{\mathrm{b}}}\right) \sqrt{(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c})} \text { and analog } s \\
& \Rightarrow \frac{1}{2} \sum \frac{\mathbf{b}+\mathbf{c}}{\sqrt{\left(\mathbf{r}_{\mathbf{b}}-\mathbf{r}\right)\left(\mathbf{r}_{\mathbf{c}}-\mathbf{r}\right)}}=\frac{1}{2 \mathbf{r}} \sum\left\{\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right) \sqrt{(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}\right\} \stackrel{\text { CBS }}{\leftrightarrows} \frac{1}{2 \mathbf{r}} \sqrt{\sum\left(\sqrt{\frac{\mathbf{b}}{\mathbf{c}}}+\sqrt{\frac{\mathbf{c}}{\mathbf{b}}}\right)^{2}} \sqrt{\sum(\mathbf{s - b})(\mathbf{s}-\mathbf{c})} \\
& =\frac{1}{2 r} \sqrt{6+\sum\left(\frac{b}{c}+\frac{c}{b}\right)} \sqrt{\sum\left(s^{2}-s(b+c)+b c\right)} \\
& =\frac{1}{2 r} \sqrt{6+\frac{\sum b c(2 s-a)}{a b c}} \sqrt{3 s^{2}-s(4 s)+s^{2}+4 R r+r^{2}} \\
& =\frac{1}{2 r} \sqrt{\frac{2 s\left(s^{2}+4 R r+r^{2}\right)+12 R r s}{4 R r s}} \sqrt{4 R r+r^{2}}=\frac{1}{2 r} \sqrt{\frac{s^{2}+10 R r+r^{2}}{2 R r}} \sqrt{4 R r+r^{2}}
\end{aligned}
$$



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$$
\Rightarrow \text { RHS } \stackrel{(2)}{\sim} \frac{1}{2 r} \sqrt{\frac{(4 R+r)\left(s^{2}+10 R r+r^{2}\right)}{2 R}}
$$

$(1),(2) \Rightarrow$ it suffices to prove :

$$
\frac{s}{r} \geq \frac{1}{2 r} \sqrt{\frac{(4 R+r)\left(s^{2}+10 R r+r^{2}\right)}{2 R}} \Leftrightarrow 8 R s^{2} \geq(4 R+r)\left(s^{2}+10 R r+r^{2}\right)
$$

(i)

$$
\Leftrightarrow(4 R-r) s^{2} \geq\left(10 R r+r^{2}\right)(4 R+r)
$$

$$
\text { Now, }(4 R-r) s^{2} \stackrel{\text { Gerretsen }}{\gtrless}(4 R-r)\left(16 R r-5 r^{2}\right) \stackrel{?}{\geq}\left(10 R r+r^{2}\right)(4 R+r)
$$

$$
\Leftrightarrow 12 R^{2}-25 R r+2 r^{2} \stackrel{?}{\sim} \geq 0 \Leftrightarrow(R-2 r)(12 R-r) \stackrel{\stackrel{?}{n}}{\geq} 0
$$

Euler

$$
\rightarrow \text { true } \because \mathbf{R} \stackrel{\text { Euler }}{\geq} 2 \mathbf{2} \Rightarrow \text { (i)is true }
$$

$$
\therefore \sum \frac{\mathbf{r}_{\mathbf{b}}+\mathbf{r}_{\mathbf{c}}}{\mathbf{a}} \geq \frac{\mathbf{1}}{\mathbf{2}} \sum \frac{\mathbf{b}+\mathbf{c}}{\sqrt{\left(\mathbf{r}_{\mathbf{b}}-\mathbf{r}\right)\left(\mathbf{r}_{\mathbf{c}}-\mathbf{r}\right)}}(\text { Proved })
$$

1510. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{w_{a}}{h_{a}}\right)^{2}+\left(\frac{w_{b}}{h_{b}}\right)^{2}+\left(\frac{w_{c}}{h_{c}}\right)^{2} \leq \frac{3 m_{a} m_{b} m_{c}}{h_{a} h_{b} h_{c}}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution 1 by Bogdan Fuştei-Romania

$$
\begin{gathered}
\cos \frac{B-C}{2} \geq \sqrt{\frac{2 r}{R}} \text { and analogs } \\
\cos \frac{B-C}{2}=\frac{h_{a}}{w_{a}} \text { and analogs } \\
\frac{w_{a}}{h_{a}} \leq \sqrt{\frac{2 r}{R}} \Rightarrow\left(\frac{w_{a}}{h_{a}}\right)^{2} \leq \frac{R}{2 r} \text { and analogs, then: } \\
\sum_{c y c}\left(\frac{w_{a}}{h_{a}}\right)^{2} \leq \frac{3 R}{2 r} \\
m_{a} \geq \sqrt{s(s-a)}=\sqrt{r_{b} r_{c}} \Rightarrow m_{a} m_{b} m_{c} \geq r_{a} r_{b} r_{c}=S s
\end{gathered}
$$



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$$
\begin{gathered}
r_{a}=\frac{s}{s-a} \Rightarrow 2 r_{a}=\frac{2 s}{s-a}=\frac{\text { whw. }}{s-a} \Rightarrow \frac{2 r_{a}}{h_{a}}=\frac{a}{s-a} \text { and analogs. } \\
a b c=4 R S \Rightarrow \frac{8 r_{a} r_{b} r_{c}}{h_{a} h_{b} h_{c}}=\frac{a b c}{s(s-a)(s-b)(s-c)(\text { Heron })} \\
\left\{\begin{array}{l}
\frac{r_{a} r_{b} r_{c}}{h_{a} h_{b} h_{c}}=\frac{R}{2 r} \\
\frac{3 R}{2 r} \geq \sum_{c y c}\left(\frac{w_{a}}{h_{a}}\right)^{2} \Rightarrow\left(\frac{w_{a}}{h_{a}}\right)^{2}+\left(\frac{w_{b}}{h_{b}}\right)^{2}+\left(\frac{w_{c}}{h_{c}}\right)^{2} \leq \frac{3 R S s}{S^{2}}=\frac{4 R s}{S}=\frac{4 R s}{r s}=\frac{4 R}{r}
\end{array}\right.
\end{gathered}
$$

## Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\because w_{a}^{2} \leq s(s-a) \text { and analogs }: \sum\left(\frac{w_{a}}{h_{a}}\right)^{2} \leq \sum \frac{s a^{2}(s-a)}{4 r^{2} s^{2}}=\frac{1}{4 r^{2} s} \sum a^{2}(s-a) \\
=\frac{2 s\left(s^{2}-4 R r-r^{2}\right)-2 s\left(s^{2}-6 R r-3 r^{2}\right)}{4 r^{2} s} \\
=\frac{2 s\left(2 R r+2 r^{2}\right)}{4 r^{2} s} \Rightarrow L H S \stackrel{(i)}{\stackrel{(i)}{\leq} \frac{R+r}{r}}
\end{gathered}
$$

$$
\text { Also, } \because m_{a} \geq \sqrt{s(s-a)} \therefore \frac{3 m_{a} m_{b} m_{c}}{h_{a} h_{b} h_{c}} \geq \frac{3 s . r s .4 R r s}{8 r^{3} s^{3}}=\frac{3 R}{2 r} \Rightarrow R H S \stackrel{(i i)}{\stackrel{(i n}{2}} \frac{3 R}{2 r}
$$

(i), (ii) $\Rightarrow$ it suffices to prove $: \frac{3 R}{2 r} \geq \frac{R+r}{r} \Leftrightarrow 3 R \geq 2 R+2 r \Leftrightarrow R \geq 2 r$

$$
\rightarrow \text { true (Euler) (Proved) }
$$

1511. In $\triangle A B C$ the following relationship holds:

$$
1+\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}} \leq \frac{2 R\left(h_{a}+h_{b}+h_{c}\right)}{r\left(r_{a}+r_{b}+r_{c}\right)}
$$

Proposed by Bogdan Fuştei -Romania
Solution 1 by Rahim Shahbazov-Baku-Azerbaijan

$$
\begin{align*}
r_{a}+r_{b}+r_{c}= & 4 R+r \text { and } h_{a}+h_{b}+h_{c}=\frac{s^{2}+r^{2}+4 R r}{2 R} \\
& \frac{m_{a}}{l_{a}}+\frac{m_{b}}{l_{b}}+\frac{m_{c}}{l_{c}} \leq \frac{s^{2}}{4 R r+r^{2}} \text { (2) }  \tag{2}\\
& l_{a} \geq s_{a} \Rightarrow \frac{m_{a}}{l_{a}} \leq \frac{b^{2}+c^{2}}{2 b c}
\end{align*}
$$



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$$
\stackrel{(2)}{\Rightarrow} \frac{2 s^{2}}{4 R r+r^{2}} \geq \frac{a}{b}+\frac{b}{a}+\frac{a}{c}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b}
$$

Let: $a=x+y ; b=y+z ; c=x+z$ inequality becomes:

$$
\begin{gathered}
\frac{2(x+y+z)^{2}}{x y+y z+z x} \geq 2(x+y+z)\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right)-3 \\
\frac{1}{2}+\frac{x^{2}+y^{2}+z^{2}}{x y+y z+z x} \geq \frac{z}{x+y}+\frac{x}{y+z}+\frac{y}{z+x} \\
\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{2}{x+y}+\frac{2}{y+z}+\frac{2}{z+x} \text { true from } \frac{1}{x}+\frac{1}{y} \geq \frac{4}{x+y}
\end{gathered}
$$

## Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\text { ByTsintsifas, } L H S \leq 1+\frac{1}{2} \sum \frac{b^{2}+c^{2}}{b c}=1+\frac{\sum a b(2 s-c)}{8 R r s}=
$$

$$
=1+\frac{s^{2}-2 R r+r^{2}}{4 R r}=\frac{s^{2}+2 R r+r^{2}}{4 R r} \Rightarrow L H S \stackrel{(1)}{\stackrel{(1)}{\leq}} \frac{s^{2}+2 R r+r^{2}}{4 R r}
$$

$$
\text { Now, } R H S=\frac{2 R\left(\sum \frac{b c}{2 R}\right)}{r(4 R+r)} \stackrel{(2)}{=} \frac{s^{2}+4 R r+r^{2}}{r(4 R+r)} \therefore(1),(2) \Rightarrow
$$

$$
\text { it suffices to prove }: \frac{s^{2}+2 R r+r^{2}}{4 R r} \leq \frac{s^{2}+4 R r+r^{2}}{r(4 R+r)}
$$

$$
\Leftrightarrow r(4 R+r) s^{2}+r(4 R+r)\left(2 R r+r^{2}\right) \leq 4 R r s^{2}+4 R r\left(4 R r+r^{2}\right)
$$

$$
\Leftrightarrow r^{2} s^{2} \leq\left(4 R r+r^{2}\right)\left(2 R r-r^{2}\right) \Leftrightarrow s^{2} \widetilde{\leq} 8 R^{2}-2 R r-r^{2}
$$



$$
\begin{gathered}
\Leftrightarrow(R-2 r)(2 R+r) \stackrel{?}{\stackrel{?}{n}} 0 \rightarrow \text { true } \because R \stackrel{\text { Euler }}{\gtrless} 2 r \\
\Rightarrow(3) \text { is true }: ., 1+\frac{\boldsymbol{m}_{a}}{\boldsymbol{w}_{a}}+\frac{\boldsymbol{m}_{b}}{\boldsymbol{w}_{b}}+\frac{\boldsymbol{m}_{c}}{\boldsymbol{w}_{c}} \leq \frac{2 R\left(h_{a}+\boldsymbol{h}_{b}+h_{c}\right)}{r\left(r_{a}+r_{b}+r_{c}\right)} \text { (Proved) }
\end{gathered}
$$

1512. In $\triangle A B C$ the following relationship holds:

$$
\sqrt{\frac{b+c}{r_{b}+r_{c}}}+\sqrt{\frac{c+a}{r_{c}+r_{a}}}+\sqrt{\frac{a+b}{r_{a}+r_{b}}} \leq \sqrt{\frac{a+b+c}{r}}
$$

Proposed by Marin Chirciu-Romania


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Solution 1 by Avishek Mitra-West Bengal-India

$$
\begin{aligned}
& \sum_{c y c} \sqrt{\frac{b+c}{r_{b}+r_{c}}}=\sum_{c y c} \sqrt{\frac{b+c}{F\left(\frac{1}{s-b}+\frac{1}{s-c}\right)}}=\frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{(b+c)(s-b)(s-c)}{s-b+s-c}} \stackrel{A M-G M}{s}_{s}^{s}=\frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{b+c}{a}} \cdot \frac{a}{2}=\frac{1}{2 \sqrt{F}} \sum_{c y c} \sqrt{a(b+c)} \\
& \leq \frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{b+c}{a}} \cdot \frac{s-b+s-c}{2} \\
& \quad \leq \frac{1}{2 \sqrt{F}} \sqrt{\sum_{c y c} a \cdot \sum_{c y c}(b+c)}=\frac{1}{2 \sqrt{r s}} \sqrt{4 s(a+b+c)}=\sqrt{\frac{a+b+c}{r}}
\end{aligned}
$$

## Solution 2 by Marian Ursărescu-Romania

$$
\begin{equation*}
\text { We must show that: }\left(\sqrt{\frac{b+c}{r_{b}+r_{c}}}+\sqrt{\frac{c+a}{r_{c}+r_{a}}}+\sqrt{\frac{a+b}{r_{a}+r_{b}}}\right)^{2} \leq \frac{2 s}{r} \tag{1}
\end{equation*}
$$

## From Chauchy inequality we have:

$$
\begin{equation*}
\left(\sqrt{\frac{b+c}{r_{b}+r_{c}}}+\sqrt{\frac{c+a}{r_{c}+r_{a}}}+\sqrt{\frac{a+b}{r_{a}+r_{b}}}\right)^{2} \leq 3\left(\frac{b+c}{r_{b}+r_{c}}+\frac{c+a}{r_{c}+r_{a}}+\frac{a+b}{r_{a}+r_{b}}\right) \tag{2}
\end{equation*}
$$

$$
\begin{equation*}
\text { From (1)+(2) we must show: } \frac{b+c}{r_{b}+r_{c}}+\frac{c+a}{r_{c}+r_{a}}+\frac{a+b}{r_{a}+r_{b}} \leq \frac{2 s}{r} \tag{3}
\end{equation*}
$$

$$
\begin{gather*}
\text { But: } \frac{b+c}{r_{b}+r_{c}}=\frac{b+c}{\frac{S}{s-b}+\frac{s}{s-c}}=\frac{(b+c)(s-b)(s-c)}{S(s-b+s-c)}=\frac{(b+c)(s-b)(s-c)}{a S} \\
\sqrt{(s-b)(s-c)} \leq \frac{s-b+s-c}{2}=\frac{a}{2} \Rightarrow(s-b)(s-c) \leq \frac{a^{2}}{4} \text { then } \\
\frac{b+c}{r_{b}+r_{c}} \leq \frac{(b+c) a^{2}}{4 s r}=\frac{a(b+c)}{4 s r} \text { and similary, then } \frac{b+c}{r_{b}+r_{c}}+\frac{c+a}{r_{c}+r_{a}}+\frac{a+b}{r_{a}+r_{b}} \leq \frac{a b+b c+c a}{2 s r} \tag{4}
\end{gather*}
$$

From (3)+(4) we must show: $\frac{a b+b c+c a}{2 s r} \leq \frac{2 s}{r} \Leftrightarrow 3(a b+b c+c a) \leq(a+b+c)^{2}$ true.

## Solution 3 by Tran Hong-Dong Thap-Vietnam

Suppose: $a \geq b \geq c \Rightarrow b c \leq c a \leq a b ; \cos \frac{A}{2} \leq \cos \frac{B}{2} \leq \cos \frac{C}{2} \Rightarrow$

$$
\begin{gathered}
\cos ^{2} \frac{A}{2} \leq \cos ^{2} \frac{B}{2} \leq \cos ^{2} \frac{C}{2} \Rightarrow \frac{1}{\cos ^{2} \frac{A}{2}} \geq \frac{1}{\cos ^{2} \frac{B}{2}} \geq \frac{1}{\cos ^{2} \frac{C}{2}} \\
r_{b}+r_{c}=4 R \cos ^{2} \frac{A}{2} \text { and analogs }
\end{gathered}
$$



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$$
\begin{gathered}
\Rightarrow \sum_{c y c} \sqrt{\frac{b+c}{r_{b}+r_{c}} \stackrel{\text { www.ssmrmh.ro }}{\sim}} \sqrt{3 \cdot \sum_{c y c} \frac{b+c}{4 R \cos ^{2} \frac{A}{2}}} \\
\stackrel{\text { Cebyshev }}{\underset{\sim}{x}} \sqrt{3 \cdot \frac{1}{3}(b+c+c+a+a+b) \cdot \frac{1}{4 R} \cdot\left(\frac{1}{\cos ^{2} \frac{A}{2}}+\frac{1}{\cos ^{2} \frac{B}{2}}+\frac{1}{\cos ^{2} \frac{C}{2}}\right)} \\
=\sqrt{\frac{s}{R} \cdot \frac{s^{2}+(4 R+r)^{2}}{s^{2}}}=\sqrt{\frac{\left(s^{2}+(4 R+r)^{2}\right)}{R s}}
\end{gathered}
$$

We must show that: $\sqrt{\frac{\left(s^{2}+(4 R+r)^{2}\right)}{R s}} \leq \sqrt{\frac{a+b+c}{r}}=\sqrt{\frac{2 s}{r}} \Leftrightarrow$

$$
\begin{gather*}
r\left(s^{2}+(4 R+r)^{2}\right) \leq 2 s^{2} R \Leftrightarrow(2 R-r) s^{2} \geq r(4 R+r)^{2}  \tag{*}\\
s^{2} \geq 16 R r-5 r^{2} \xrightarrow{4 R-r \geq 7 r>0}
\end{gather*}
$$

$$
(2 R-r) s^{2} \geq(2 R-r)\left(16 R r-5 r^{2}\right)=r(2 R-2)(16 R-5 r)
$$

We must show that:

$$
\begin{aligned}
& (2 R-2)(16 R-5 r) \geq(4 R+r)^{2} \Leftrightarrow 8 R^{2}-17 R r+2 r^{2} \geq 0 \Leftrightarrow \\
& \quad(R-2 r)(8 R-r) \geq 0 \text { which is true because: } \\
& R \geq 2 r \Rightarrow R-2 r \geq 0 ; 8 R-r \geq 16 r-r=15 r>0 . \text { Proved. }
\end{aligned}
$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{sos} s^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
& \therefore r_{b}+r_{c} \stackrel{(i)}{=} 4 R \cos ^{2} \frac{A}{2} \\
& \text { Now, } \frac{b+c}{2}=\frac{4 R \cos \frac{A}{2} \cos \left(\frac{B-C}{2}\right)}{2} \leq 2 R \cos \frac{A}{2}\left(\because 0<\cos \frac{B-C}{2} \leq 1\right) \\
& =\sqrt{R} \sqrt{4 \operatorname{Rcos}^{2} \frac{A}{2}} \stackrel{b y(i)}{=} \sqrt{R\left(r_{b}+r_{c}\right)} \\
& \Rightarrow \sqrt{\frac{b+c}{r_{b}+r_{c}}} \leq \sqrt{R}\left(\frac{2 \sqrt{b+c}}{b+c}\right) \stackrel{A M \geq G M}{\stackrel{\tilde{m}}{\leq}} \sqrt{R}\left(\frac{2 \sqrt{b+c}}{2 \sqrt{b c}}\right)=\sqrt{R} \sqrt{\frac{1}{b}+\frac{1}{c}}
\end{aligned}
$$



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$$
\begin{gathered}
\therefore \sqrt{\frac{b+c}{r_{b}+r_{c}}} \leq \sqrt{R} \sqrt{\frac{1}{b}+\frac{1}{c}} \text { and analogs } \\
\Rightarrow \sum \sqrt{\frac{b+c}{r_{b}+r_{c}}} \leq \sqrt{R} \sum \sqrt{\frac{1}{b}+\frac{1}{c}} \stackrel{c B S}{\leftrightarrows} \sqrt{3 R} \sqrt{\sum\left(\frac{1}{b}+\frac{1}{c}\right)}=\sqrt{6 R} \sqrt{\sum \frac{1}{a}}=\sqrt{\frac{2 R}{4 R r s}} \sqrt{3 \sum a b} \\
=\sqrt{\frac{1}{2 r s}} \sqrt{(a+b+c)^{2}}=\sqrt{\frac{4 s^{2}}{2 r s}}=\sqrt{\frac{2 s}{r}}=\sqrt{\frac{a+b+c}{r}}(\text { Proved })
\end{gathered}
$$

Solution 5 by Bogdan Fuştei-Romania

$$
\begin{gathered}
b+c=\frac{\left(r_{a}+r\right)\left(r_{b}+r_{c}\right)}{a} \text { and analogs. } \\
\frac{b+c}{r_{b}+r_{c}}=\frac{r_{a}+r}{a} \text { and analogs. }
\end{gathered}
$$

We know that: $(x+y+z)^{2} \leq 3\left(x^{2}+y^{2}+z^{2}\right), \forall x, y, z \in \mathbb{R}$

$$
\begin{gathered}
\left(\sum_{c y c} \sqrt{\frac{b+c}{r_{b}+r_{c}}}\right)^{2} \leq 3\left(\frac{r_{a}+r}{a}+\frac{r_{b}+r}{b}+\frac{r_{c}+r}{c}\right) \\
(s-b)(s-c)=r r_{a} \text { and analogs. } \\
\sqrt{(s-b)(s-c)} \leq \frac{s-b+s-c}{2}=\frac{a}{2} \text { then } \\
\sqrt{r r_{a}} \leq \frac{a}{2} \Rightarrow r_{a} \leq \frac{a^{2}}{4 R} \Rightarrow \frac{r_{a}}{a} \leq \frac{a}{4 r} \text { and analogs. } \\
\frac{r_{a}}{a}+\frac{r_{b}}{b}+\frac{r_{c}}{c} \leq \frac{a+b+c}{4 r}=\frac{s}{2 r}
\end{gathered}
$$

We know that: $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \leq \frac{\sqrt{3}}{2 r} \Rightarrow \frac{r}{a}+\frac{r}{b}+\frac{r}{c} \leq \frac{\sqrt{3}}{2}$

$$
\sum_{c y c} \frac{r_{a}+r}{a} \leq \frac{\sqrt{3}}{2}+\frac{s}{2 r}=\frac{s+r \sqrt{3}}{2 r}
$$

We must show: $\frac{3 s+3 r \sqrt{3}}{2 r} \leq \frac{a+b+c}{r}=\frac{2 s}{r} \Rightarrow 3 s+3 \sqrt{3} r \leq 4 s \Rightarrow 3 \sqrt{3} r \leq s$ (Mitrinovic)

$$
\begin{equation*}
3 \sum_{c y c} \frac{r_{a}+r}{a} \leq \frac{a+b+c}{r} \tag{2}
\end{equation*}
$$

From (1)+(2) we get inequality.


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1513. In $\triangle A B C$ the following relationship holds:

$$
\left(3+\sum \frac{\mathbf{h}_{\mathrm{b}}+\mathbf{h}_{\mathrm{c}}}{\mathbf{h}_{\mathrm{a}}}\right) \sum \mathrm{AI} \geq \frac{2}{3}\left(\sum\left(\mathrm{~m}_{\mathrm{a}}+8 \mathbf{w}_{\mathrm{a}}\right)\right)
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
\sum\left\{(b+c) w_{a}\right\} & =\sum\left\{(b+c)\left(\frac{2 b \cos \frac{A}{2}}{b+c}\right)\right\}=\sum \frac{2 a b c \cos \frac{A}{2}}{a}=\sum \frac{8 \operatorname{Rrscos} \frac{A}{2}}{4 \operatorname{Ren} \frac{A}{2} \cos \frac{A}{2}} \\
& =2 s \sum \frac{r}{\sin \frac{A}{2}}=2 s \sum A I \Rightarrow \sum A I \stackrel{(1)}{\stackrel{m}{2}} \frac{1}{2 s} \sum\left\{(b+c) w_{a}\right\}
\end{aligned}
$$

WLOG, we may assume $\mathbf{a} \geq \mathbf{b} \geq \mathbf{c} \therefore \mathrm{w}_{\mathrm{a}} \leq \mathrm{w}_{\mathrm{b}} \leq \mathrm{w}_{\mathrm{c}}$ and $\mathrm{b}+\mathbf{c} \leq \mathbf{c}+\mathrm{a} \leq \mathrm{a}+\mathrm{b}$
$\therefore$ using Chebyshev, we get

$$
\begin{aligned}
& \frac{\mathbf{1}}{\mathbf{2 s}} \sum\left\{(\mathrm{b}+\mathbf{c}) \mathbf{w}_{\mathbf{a}}\right\} \geq \frac{\mathbf{1}}{\mathbf{6 s}}\left\{\sum(\mathrm{b}+\mathbf{c})\right\}\left(\sum \mathbf{w}_{\mathbf{a}}\right)=\frac{\mathbf{4 s}}{\mathbf{6 s}} \sum \mathbf{w}_{\mathbf{a}} \\
& =\frac{2}{3} \sum \mathrm{w}_{\mathrm{a}} \stackrel{\text { by }(1)}{\Rightarrow} \sum \mathrm{AI} \stackrel{(2)}{\underset{2}{2}} \frac{2}{3} \sum \mathrm{w}_{\mathrm{a}}
\end{aligned}
$$

Now, $3+\sum \frac{\mathbf{h}_{\mathbf{b}}+\mathbf{h}_{\mathbf{c}}}{\mathbf{h}_{\mathbf{a}}}=3+\sum \frac{\mathbf{c a}+\mathbf{a b}}{\mathbf{b c}}=3+\sum\left(\frac{\mathbf{a}}{\mathbf{b}}+\frac{\mathbf{a}}{\mathbf{c}}\right)=3+\sum\left(\frac{\mathbf{b}}{\mathbf{c}}+\frac{\mathbf{c}}{\mathbf{b}}\right)$

$$
\begin{aligned}
& =3+\sum \frac{\mathbf{b}^{2}+\mathbf{c}^{2}}{\mathbf{b c}} \stackrel{\text { Tsintsifas }}{\gtrless} 3+\sum \frac{2 \mathbf{m}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}} \\
& \therefore 3+\sum \frac{\mathbf{h}_{\mathbf{b}}+\mathbf{h}_{\mathbf{c}}^{(3)}}{\mathbf{h}_{\mathrm{a}}} \stackrel{(3)}{\geq} 3+\sum \frac{2 \mathbf{m}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}} \therefore \text { (2), (3) } \\
& \Rightarrow \text { LHS } \geq \frac{\mathbf{2}}{\mathbf{3}}\left(3+\sum \frac{\mathbf{2 m}}{\mathbf{w}_{\mathrm{a}}}\right) \sum \mathrm{w}_{\mathrm{a}}=\frac{?}{2} \frac{2}{3}\left(\sum\left(\mathrm{~m}_{\mathrm{a}}+8 \mathrm{w}_{\mathrm{a}}\right)\right) \\
& \Leftrightarrow 3 \sum \mathbf{w}_{\mathrm{a}}+\sum \mathbf{w}_{\mathrm{a}} \sum \frac{2 \mathrm{~m}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}} \stackrel{?}{\underset{\sim}{2}} \sum \mathrm{~m}_{\mathrm{a}}+\mathbf{8} \sum \mathbf{w}_{\mathrm{a}} \\
& \Leftrightarrow \sum \mathbf{w}_{\mathrm{a}} \sum \frac{2 \mathrm{~m}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}}-\mathbf{6} \sum \mathbf{w}_{\mathrm{a}} \stackrel{\stackrel{?}{2}}{\stackrel{1}{2}} \sum \mathrm{~m}_{\mathrm{a}}-\sum \mathbf{w}_{\mathrm{a}} \\
& \Leftrightarrow\left(\sum \mathbf{w}_{\mathrm{a}}\right) \sum\left(\frac{\mathbf{2} \mathbf{m}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}}-\mathbf{2}\right) \stackrel{?}{2} \sum\left(\mathbf{m}_{\mathrm{a}}-\mathbf{w}_{\mathrm{a}}\right)
\end{aligned}
$$



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$$
\begin{aligned}
& \Leftrightarrow\left(\sum \mathbf{w}_{a}\right)\left\{\frac{\mathbf{2}\left(m_{a}-\mathbf{w}_{a}\right)}{\mathbf{w}_{a}}+\frac{\mathbf{2}\left(m_{b}-\mathbf{w}_{b}\right)}{\mathbf{w}_{b}}+\frac{\mathbf{2}\left(\mathbf{m}_{\mathbf{c}}-\mathbf{w}_{c}\right)}{\mathbf{w}_{\mathbf{c}}}\right\} \xlongequal[\sim]{\geq}\left(\mathbf{m}_{a}-\mathbf{w}_{a}\right)+\left(\mathbf{m}_{b}-\mathbf{w}_{b}\right) \\
& +\left(\mathbf{m}_{\mathbf{c}}-\mathbf{w}_{\mathbf{c}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& \Leftrightarrow\left(\mathbf{m}_{\mathrm{a}}-\mathbf{w}_{\mathrm{a}}\right)\left(\frac{\mathbf{w}_{\mathrm{a}}+2 \mathbf{w}_{\mathrm{b}}+2 \mathbf{w}_{\mathbf{c}}}{\mathbf{w}_{\mathrm{a}}}\right)+\left(\mathbf{m}_{\mathrm{b}}-\mathbf{w}_{\mathrm{b}}\right)\left(\frac{\mathbf{w}_{\mathrm{b}}+2 \mathbf{w}_{\mathrm{c}}+2 \mathbf{w}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{b}}}\right) \\
& +\left(m_{c}-w_{c}\right)\left(\frac{\mathbf{w}_{\mathbf{c}}+2 \mathbf{w}_{\mathbf{a}}+2 \mathbf{w}_{\mathbf{b}}}{\mathbf{w}_{\mathbf{c}}}\right) \stackrel{?}{\geq} \mathbf{0} \rightarrow \text { true } \\
& \therefore\left(\mathbf{3}+\sum \frac{\mathbf{h}_{\mathbf{b}}+\mathbf{h}_{\mathbf{c}}}{\mathbf{h}_{\mathrm{a}}}\right) \sum \mathrm{AI} \geq \frac{\mathbf{2}}{\mathbf{3}}\left(\sum\left(\mathbf{m}_{\mathrm{a}}+\mathbf{8} \mathbf{w}_{\mathrm{a}}\right)\right) \text { (Proved) }
\end{aligned}
$$

1514. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right) \sqrt{4-\frac{2 r}{R}} \leq \frac{s}{r}
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right) \sqrt{4-\frac{2 r}{R} \stackrel{(1)}{\sim}} \frac{s}{r} \\
(1) \Leftrightarrow\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right)^{2}\left(\frac{4 R-2 r}{R}\right) \stackrel{(2)}{\sim} \frac{s^{2}}{r^{2}}
\end{gathered}
$$

(i)

We shall now prove : $R(4 R+r)^{2} \geq(4 R-2 r) s^{2}$
Now, RHS of (i) $\stackrel{\text { Rouche }}{\underset{\leq}{\leq}}(4 R-2 r)\left(2 R^{2}+10 R r-r^{2}+2(R-2 r) \sqrt{R^{2}-2 R r}\right) \stackrel{?}{\dot{m}} \mathbf{R}(4 R+r)^{2}$

$$
\begin{gathered}
\Leftrightarrow R(4 R+r)^{2}-\left(2 R^{2}+10 R r-r^{2}\right)(4 R-2 r) \stackrel{?}{\cong} 2(4 R-2 r)(R-2 r) \sqrt{R^{2}-2 R r} \\
\Leftrightarrow(R-2 r)\left(8 R^{2}-12 R r+r^{2}\right) \sum_{(i i)}^{?} 2(4 R-2 r)(R-2 r) \sqrt{R^{2}-2 R r}
\end{gathered}
$$

$\because \mathbf{R}-\mathbf{2 r} \stackrel{\text { Euler }}{\geqq} \therefore$ in order to prove (ii), it suffices to prove :


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$8 R^{2}-12 R r+r^{2}>2(4 R-2 r) \sqrt{R^{2}-2 R r}$
$\Leftrightarrow\left(8 R^{2}-12 R r+r^{2}\right)^{2}-4\left(R^{2}-2 R r\right)(4 R-2 r)^{2}>0 \Leftrightarrow r^{2}(4 R+r)^{2}>0 \rightarrow$ true
$\Rightarrow(i i) \Rightarrow(i)$ is true $: \frac{4 R-2 r}{R} \leq \frac{(4 R+r)^{2}}{s^{2}}$
$\Rightarrow\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right)^{2}\left(\frac{4 \mathbf{R}-2 \mathbf{r}}{\mathbf{R}}\right) \stackrel{(\underset{i i i}{ })}{\stackrel{(4 R}{\leq}} \frac{(4)^{2}}{s^{2}}\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right)^{2}$
$\therefore$ (iii) $\Rightarrow$ in order to prove (2), it suffices to prove : $\frac{(4 \mathrm{R}+\mathbf{r})^{2}}{\mathbf{s}^{2}}\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right)^{2} \leq \frac{\mathbf{s}^{2}}{\mathbf{r}^{2}}$

$$
\Leftrightarrow \frac{\mathbf{s}^{2}}{\mathbf{r}^{2}} \stackrel{(3)}{2}\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right)\left(1+\frac{4 \mathbf{R}}{\mathbf{r}}\right)
$$

Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathbf{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and $\mathrm{c}=\mathrm{x}+\mathrm{y}$

$$
\begin{aligned}
& \text { Now, }, \frac{\mathbf{s}^{2}}{\mathbf{r}^{2}}=\frac{\mathbf{s}^{4}}{\Delta^{2}}=\frac{s^{4}}{s(s-a)(s-b)(s-c)} \stackrel{(m)}{=} \frac{\left(\sum \mathrm{x}\right)^{3}}{\mathrm{xyz}} \text { and } \\
& 1+\frac{4 R}{r}=1+\frac{4 s a b c}{4 s(s-a)(s-b)(s-c)}=1+\frac{\Pi(y+z)}{\mathrm{xyz}} \stackrel{(n)}{=} \frac{\mathrm{xyz}+\Pi(\mathrm{y}+\mathrm{z})}{\mathrm{xyz}} \\
& \text { Also, } \sum \frac{\mathbf{a}}{\mathbf{b}}=\sum \frac{\mathrm{y}+\mathrm{z}}{\mathrm{z}+\mathrm{x}} \stackrel{(\mathrm{p})}{=} \frac{\sum(\mathrm{x}+\mathrm{y})(\mathrm{y}+\mathrm{z})^{2}}{\prod(\mathrm{y}+\mathrm{z})} \\
& \therefore(\mathbf{m}),(\mathbf{n}),(\mathbf{p}) \Rightarrow(3) \Leftrightarrow \frac{\left(\sum \mathbf{x}\right)^{3}}{\mathrm{xyz}} \geq\left[\frac{\mathrm{xyz}+\Pi(\mathbf{y}+\mathrm{z})}{\mathrm{xyz}}\right]\left[\frac{\sum(\mathbf{x}+\mathbf{y})(\mathbf{y}+\mathbf{z})^{2}}{\Pi(\mathbf{y}+\mathbf{z})}\right] \\
& \Leftrightarrow\{\Pi(\mathbf{y}+\mathbf{z})\}\left(\sum \mathbf{x}\right)^{\mathbf{3}} \geq\{\mathbf{x y z}+\Pi(\mathbf{y}+\mathbf{z})\} \sum(\mathbf{x}+\mathbf{y})(\mathbf{y}+\mathbf{z})^{2} \\
& \text { (4) } \\
& \Leftrightarrow \sum \mathbf{x}^{2} \mathbf{y}^{4}+\sum \mathbf{x}^{3} \mathbf{y}^{3} \xrightarrow{\sum} \mathrm{xyz}\left(\sum \mathrm{x}^{2} \mathbf{y}\right)+3 \mathrm{x}^{2} \mathbf{y}^{2} \mathbf{z}^{2} \\
& \text { Now, if } \mathbf{u}, \mathbf{v}, \mathbf{w}>0 \text {, then } \\
& : v^{3}+v^{3}+u^{3} \stackrel{A-G}{\geqq} 3 v^{2} u, w^{3}+w^{3}+v^{3} \stackrel{A-G}{\geq} 3 w^{2} v \text { and } u^{3}+u^{3} \\
& +w^{3} \stackrel{A-G}{\geqq} 3 u^{2} \mathbf{w} \text { and adding these three : } \\
& \sum \mathbf{u}^{3} \geq \sum \mathbf{u v}^{2} \text { and choosing } \mathbf{u}=\mathrm{xy}, \mathrm{v}=\mathrm{yz} \text { and } \mathbf{w}=\mathrm{zx} \text {, we get }:
\end{aligned}
$$

$$
\begin{aligned}
& \text { (a) } \\
& \text { A-G } \\
& \sum x^{3} y^{3} \geq x y z\left(\sum x^{2} y\right) \text { and } \sum x^{2} y^{4} \underbrace{\sum}_{(\text {b) }} 3 x^{2} y^{2} z^{2}
\end{aligned}
$$

$(a)+(b) \Rightarrow(4)$ is true $\Rightarrow(3) \Rightarrow(2) \Rightarrow(1)$ is true (Proved)


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1515. In $\triangle A B C$ the following relationship holds:

$$
\frac{\cos ^{2}\left(\frac{A-B}{2}\right)}{\tan \frac{C}{2}}+\frac{\cos ^{2}\left(\frac{B-C}{2}\right)}{\tan \frac{A}{2}}+\frac{\cos ^{2}\left(\frac{C-A}{2}\right)}{\tan \frac{B}{2}} \geq 6 \sqrt{3} \cdot \frac{r}{R}
$$

Proposed by George Apostolopoulos-Messolonghi-Greece

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& b+c=\frac{a \cdot \cos \left(\frac{B-C}{2}\right)}{\sin \left(\frac{A}{2}\right)} \Rightarrow \cos \left(\frac{B-C}{2}\right)=\frac{b+c}{a} \cdot \sin \left(\frac{A}{2}\right) \\
& L H S=\sum_{c y c} \frac{\cos ^{2}\left(\frac{B-C}{2}\right)}{\tan \left(\frac{A}{2}\right)} \stackrel{A m-G m}{\geq} 3 \cdot \sqrt[3]{\frac{\prod \cos ^{2}\left(\frac{B-C}{2}\right)}{\tan \left(\frac{A}{2}\right)}}=3 \cdot \sqrt[3]{\frac{\Pi\left(\frac{b+c}{a}\right)^{2} \sin ^{2}\left(\frac{A}{2}\right)}{\prod \tan \left(\frac{A}{2}\right)}} \\
& =3 \cdot \sqrt[3]{\prod\left(\frac{b+c}{a}\right)^{2} \prod \sin \left(\frac{A}{2}\right) \cos \left(\frac{A}{2}\right)}=3 \cdot \sqrt[3]{\prod\left(\frac{b+c}{a}\right)^{2} \prod\left(\frac{\sin A}{2}\right)} \\
& =3 \cdot \sqrt[3]{\left(\frac{(a+b)(b+c)(c+a)}{a b c}\right)^{2} \cdot \frac{\sin A \cdot \sin B \cdot \sin C}{8}} \\
& \stackrel{A m-G m}{\geq} 3 \cdot \sqrt[3]{8^{2} \cdot \frac{\sin A \cdot \sin B \cdot \sin C}{8}}=6 \cdot \sqrt[3]{\sin A \cdot \sin B \cdot \sin C} \\
& =6 \cdot \sqrt[3]{\frac{S r}{2 R^{2}}} \stackrel{\sim}{*}^{(*)} 6 \sqrt{3} \cdot \frac{r}{R} \\
& (*) \Leftrightarrow \frac{s r}{2 R^{2}} \geq 3 \sqrt{3} \cdot\left(\frac{r}{R}\right)^{3} \Leftrightarrow s R \geq 6 \sqrt{3} r^{2} \text { which is true because: } \\
& s \geq 3 \sqrt{3} r ; R \geq 2 r ; s R \geq 6 \sqrt{3} r^{2}
\end{aligned}
$$

1516. In $\triangle A B C$ the following relationship holds:

$$
\frac{r_{a}+r}{r_{a}-r}+\frac{r_{b}+r}{r_{b}-r}+\frac{r_{c}+r}{r_{c}-r} \leq \sqrt{\frac{2 R}{r}} \sum \frac{h_{a}}{s_{a}} \sqrt{\frac{h_{a}}{r_{a}}}
$$



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Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \sum \frac{r_{a}+r}{r_{a}-r}=\sum \frac{\frac{r s}{s-a}+\frac{r s}{s}}{\frac{\mathbf{r s}}{s-a}-\frac{\mathbf{r s}}{s}}=\sum \frac{\frac{b+c}{s(s-a)}}{\frac{(1)}{s(s-a)}} \stackrel{m}{=} \sum \frac{b+\mathbf{c}}{a} \\
& \text { Tsintsifas } \Rightarrow m_{a} \leq \frac{b^{2}+c^{2}}{2 b c} w_{a} \Rightarrow \frac{2 b c}{b^{2}+c^{2}} m_{a} \leq w_{a} \Rightarrow s_{a} \leq w_{a} \text { and analogs } \\
& \Rightarrow \sqrt{\frac{2 R}{r}} \sum \frac{h_{a}}{S_{a}} \sqrt{\frac{h_{a}}{r_{a}}} \\
& \geq \sqrt{\frac{2 R}{r}} \sum\left(\frac{2 r s}{a w_{a}} \sqrt{\frac{2 r s}{4 R \sin \frac{A}{2} \cos \frac{A}{2} \cdot \operatorname{stan} \frac{A}{2}}}\right)=\sum \frac{2 \operatorname{rs}(b+c)}{a \cdot 2 b \operatorname{csin} \frac{A}{2} \cos \frac{A}{2}}=\sum \frac{2 r s(b+c)}{4 R \operatorname{rssin} A} \\
& =\sum \frac{2 R \cdot 2 \operatorname{rs}(b+c)}{4 \operatorname{Rrsa}}=\sum \frac{b+c^{b y(1)}}{a} \stackrel{m}{\cong} \sum \frac{r_{a}+r}{r_{a}-r} \\
& \therefore \frac{\mathbf{r}_{\mathbf{a}}+\mathbf{r}}{\mathbf{r}_{\mathbf{a}}-\mathbf{r}}+\frac{\mathbf{r}_{\mathbf{b}}+\mathbf{r}}{\mathbf{r}_{\mathbf{b}}-\mathbf{r}}+\frac{\mathbf{r}_{\mathbf{c}}+\mathbf{r}}{\mathbf{r}_{\mathbf{c}}-\mathbf{r}} \leq \sqrt{\frac{2 \mathbf{R}}{\mathbf{r}}} \sum \frac{\mathbf{h}_{\mathbf{a}}}{\mathbf{s}_{\mathbf{a}}} \sqrt{\frac{\mathbf{h}_{\mathbf{a}}}{\mathbf{r}_{\mathbf{a}}}} \text { (Proved) }
\end{aligned}
$$

1517. In $\triangle A B C, n_{a}$-Nagel's cevian, the following relationship holds:

$$
2\left(\frac{6 \mathrm{R}}{\mathrm{r}}-\frac{3}{2}+\sum \frac{\mathrm{n}_{\mathrm{a}}^{2}}{\mathrm{ha}_{\mathrm{a}}^{2}}\right) \geq\left(1+\frac{4 \mathrm{R}}{\mathrm{r}}\right)\left(\sum \frac{\mathrm{a}}{\mathrm{~b}}\right)
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
2\left(\frac{6 R}{r}-\frac{3}{2}+\sum \frac{\mathbf{n}_{\mathrm{a}}^{2}}{\mathbf{h a}^{2}}\right) \stackrel{(i)}{\geq}\left(1+\frac{4 R}{r}\right)\left(\sum \frac{a}{b}\right)
$$

Stewart's theorem $\Rightarrow \mathbf{b}^{2}(s-c)+\mathbf{c}^{2}(s-b)=\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(s-b)(s-c)$

$$
\begin{aligned}
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=\mathbf{a n}_{a}{ }^{2}+a\left(s^{2}-s(2 s-a)+b c\right) \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c \\
& =\mathrm{an}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow \mathbf{s}\left(\mathbf{b}^{2}+\mathbf{c}^{2}-\mathbf{a}^{2}-2 \mathrm{bc}\right)=\mathrm{an}_{\mathrm{a}}{ }^{2}-\mathbf{a s}^{2} \Rightarrow \mathrm{an}_{\mathrm{a}}{ }^{2}=\mathbf{a s}{ }^{2}+\mathbf{s}(2 \mathrm{bccos} A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \operatorname{sbc}(s-b)(s-c)(s-a)}{b c(s-a)}
\end{aligned}
$$



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$$
\begin{gathered}
=\mathbf{a s}^{2}-\frac{4 \Delta^{2}}{s-a}=\mathbf{a s}^{2}-2 a\left(\frac{2 \Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=\mathbf{a s}^{2}-2 a_{a} r_{a} \\
\therefore \mathbf{n}_{\mathrm{a}}{ }^{2} \stackrel{(1)}{=} s^{2}-2 h_{a} r_{a} \text { and analogs }
\end{gathered}
$$

Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \cdot \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and $\mathrm{c}=\mathrm{x}+\mathrm{y}$

$$
\begin{aligned}
& \text { Now, } \frac{\mathbf{s}^{2}}{\mathbf{r}^{2}}=\frac{s^{4}}{\Delta^{2}}=\frac{s^{4}}{s(s-a)(s-b)(s-c)} \stackrel{(2)}{m} \frac{\left(\sum \mathrm{x}\right)^{3}}{\mathrm{xyz}} \text { and } 1+\frac{4 \mathrm{R}}{\mathrm{r}} \\
& =1+\frac{4 \mathrm{sabc}}{4 s(s-a)(s-b)(s-c)}=1+\frac{\Pi(\mathrm{y}+\mathrm{z})}{\mathrm{xyz}} \stackrel{(3)}{\cong} \frac{\mathrm{xyz}+\Pi(\mathrm{y}+\mathrm{z})}{\mathrm{xyz}}
\end{aligned}
$$

$$
\text { Also, } \sum \frac{\mathbf{a}}{\mathbf{b}}=\sum \frac{\mathrm{y}+\mathrm{z}}{\mathrm{z}+\mathbf{x}} \stackrel{(4)}{=} \frac{\sum(\mathrm{x}+\mathbf{y})(\mathrm{y}+\mathrm{z})^{2}}{\prod(\mathrm{y}+\mathrm{z})} \therefore \text { (2),(3),(4) } \Rightarrow \text { (ii) }
$$

$$
\Leftrightarrow \frac{\left(\sum x\right)^{3}}{x y z} \geq\left[\frac{x y z+\Pi(y+z)}{x y z}\right]\left[\frac{\sum(x+y)(y+z)^{2}}{\Pi(y+z)}\right]
$$

$$
\Leftrightarrow\{\Pi(\mathbf{y}+\mathbf{z})\}\left(\sum \mathbf{x}\right)^{3} \geq\{\mathbf{x y z}+\Pi(\mathbf{y}+\mathbf{z})\}(\mathbf{x}+\mathbf{y})(\mathbf{y}+\mathbf{z})^{2}
$$

(iii)

$$
\Leftrightarrow \sum \mathbf{x}^{2} \mathbf{y}^{4}+\sum \mathbf{x}^{3} \mathbf{y}^{3} \supseteq \mathrm{xyz}\left(\sum \mathbf{x}^{2} \mathbf{y}\right)+3 \mathrm{x}^{2} \mathbf{y}^{2} \mathbf{z}^{2}
$$

Now, if $\mathbf{u}, \mathbf{v}, \mathbf{w}>0$, then : $\mathbf{v}^{\mathbf{3}}+\mathbf{v}^{\mathbf{3}}+\mathbf{u}^{3} \stackrel{\text { A-G }}{2} 3 \mathbf{v}^{2} \mathbf{u}, \mathbf{w}^{3}+\mathbf{w}^{3}+\mathbf{v}^{3} \stackrel{\text { A-G }}{2}$

$$
\geq 3 w^{2} v \text { and } u^{3}+u^{3}+w^{3} \stackrel{A-G}{\unlhd} 3 u^{2} w \text { and adding these three : }
$$

$$
\begin{aligned}
& \text { Now, } \sum \frac{\mathbf{n}_{\mathrm{a}}{ }^{2} \text { by (1)and its analogs }}{\mathbf{h}_{\mathrm{a}}{ }^{2}} \stackrel{\mathrm{a}^{2}\left(\mathbf{s}^{2}-2 \mathbf{h}_{\mathrm{a}} \mathbf{r}_{\mathrm{a}}\right)}{4 \mathbf{r}^{2} \mathbf{s}^{2}}=\frac{\sum \mathrm{a}^{2}}{4 \mathbf{r}^{2}}-\frac{4 \mathbf{r}^{2} \mathbf{s}^{2}}{4 \mathbf{r}^{2} \mathbf{s}^{2}} \sum \frac{\mathbf{a}^{2}}{\mathbf{a}(\mathbf{s}-\mathbf{a})} \\
& =\frac{s^{2}-4 R r-r^{2}}{2 r^{2}}-\sum \frac{a-s+s}{s-a} \\
& =\frac{\mathbf{s}^{2}-4 R \mathbf{R r}-\mathbf{r}^{2}}{2 \mathbf{r}^{2}}+3-\frac{\mathbf{s} \sum(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{s r}^{2}}=\frac{\mathbf{s}^{2}-4 \mathbf{R r}-\mathbf{r}^{2}}{2 \mathbf{r}^{2}}+3-\frac{4 \mathbf{R r}+\mathbf{r}^{2}}{\mathbf{r}^{2}} \\
& =\frac{s^{2}-4 R r-r^{2}+6 r^{2}-2\left(4 R r+r^{2}\right)}{2 r^{2}}=\frac{s^{2}-12 R r+3 r^{2}}{2 r^{2}} \\
& \Rightarrow 2\left(\frac{6 R}{r}-\frac{3}{2}+\sum \frac{\mathbf{n}_{\mathrm{a}}{ }^{2}}{\mathrm{~h}_{\mathrm{a}}{ }^{2}}\right)=2\left(\frac{6 \mathrm{R}}{\mathrm{r}}-\frac{3}{2}+\frac{\mathrm{s}^{2}-12 \mathrm{Rr}+3 \mathrm{r}^{2}}{2 \mathrm{r}^{2}}\right) \\
& =\frac{\mathbf{s}^{2}-12 R r+3 r^{2}+12 R r-3 r^{2}}{\mathbf{r}^{2}}=\frac{\mathbf{s}^{2}}{\mathbf{r}^{2}} \therefore(\mathbf{i}) \Leftrightarrow \frac{\mathbf{s}^{2}}{\mathbf{r}^{2}} \stackrel{(i i)}{n}_{\geq}^{\left(1+\frac{4 R}{r}\right)\left(\sum \frac{\mathbf{a}}{\mathbf{b}}\right) ~(1)}
\end{aligned}
$$



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$\sum \mathbf{u}^{3} \geq \sum \mathbf{u v}^{2}$ and choosing $\mathbf{u}=x y, v=y z$ and $w=z x$, we get

$$
: \sum x^{3} y^{3} \stackrel{(a)}{\geq} x y z\left(\sum x^{2} y\right) \text { and } \sum x^{2} y^{4} \underbrace{\stackrel{A-G}{m}}_{(b)} 3 x^{2} y^{2} z^{2}
$$

(a) $+(\mathbf{b}) \Rightarrow$ (iii) is true $\Rightarrow(\mathrm{ii}) \Rightarrow$ (i) is true (Proved)
1518. In $\triangle A B C, n_{a}$-Nagel's cevian, the following relationship holds

$$
\frac{\sqrt{2}}{2} \sum \frac{n_{a}}{h_{a}}+\sqrt{\frac{2 r}{R}} \sum \sin \frac{A}{2} \leq \frac{s}{r}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\therefore \mathbf{n}_{\mathbf{a}} \stackrel{(1)}{=} \mathbf{s}^{2}-2 h_{\mathbf{a}} \mathbf{r}_{\mathrm{a}} \text { and analogs }
$$

$$
\text { Now, } \because \frac{2 \mathrm{r}}{\mathrm{R}} \stackrel{\text { Euler }}{\leq} \frac{\mathrm{R}}{2 \mathrm{r}} \therefore \frac{\sqrt{2}}{2} \frac{\mathrm{n}_{\mathrm{a}}}{\mathrm{~h}_{\mathrm{a}}}+\sqrt{\frac{2 \mathrm{r}}{\mathrm{R}}} \sin \frac{\mathrm{~A}}{2} \leq
$$

$$
\leq \frac{\sqrt{2}}{2} \frac{n_{a}}{h_{a}}+\sqrt{\frac{R}{2 r}} \sin \frac{A}{2} \stackrel{\text { CBS }}{\tilde{\leq}} \sqrt{2} \sqrt{\left(\frac{2}{4}\right) \frac{n_{a}^{2}}{\mathbf{h}_{a}^{2}}+\left(\frac{R}{2 r}\right) \sin ^{2} \frac{A}{2}}=
$$

$$
\stackrel{\text { by (1) }}{=} \sqrt{2} \sqrt{\frac{\mathbf{a}^{2}\left(s^{2}-2 h_{a} r_{a}\right)}{8 r^{2} s^{2}}+\left(\frac{R}{2 r}\right) \sin ^{2} \frac{A}{2}}
$$

$$
=\sqrt{2} \sqrt{\frac{a^{2}}{8 r^{2}}-\frac{h_{a} r_{a}}{h_{a}^{2}}+\left(\frac{R}{2 r}\right) \sin ^{2} \frac{A}{2}}=\sqrt{2} \sqrt{\frac{a^{2}}{8 r^{2}}-\frac{4 R \sin \frac{A}{2} \cos \frac{A}{2} \operatorname{stan} \frac{A}{2}}{2 r s}+\left(\frac{R}{2 r}\right) \sin ^{2} \frac{A}{2}}
$$

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-\mathbf{b c}(2 s-a)=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}\left(\mathbf{s}^{2}-\mathbf{s}(2 s-a)+b c\right) \Rightarrow \\
& \Rightarrow \mathbf{s}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)-2 \mathbf{s b c}=\mathrm{an}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathbf{a s}-\mathrm{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+c^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{\mathrm{a}}{ }^{2}-\mathrm{as}^{2} \Rightarrow \mathrm{an}_{\mathrm{a}}{ }^{2}=\mathrm{as}^{2}+\mathrm{s}(2 \mathrm{bccos} A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \operatorname{sbc}(s-\mathbf{b})(s-\mathbf{c})(s-\mathbf{a})}{\mathbf{b c}(s-a)} \\
& =a s^{2}-\frac{4 \Delta^{2}}{s-a}=a s^{2}-2 a\left(\frac{2 \Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=a s^{2}-2 a_{a} r_{a}
\end{aligned}
$$



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$$
\begin{aligned}
& =\sqrt{2} \sqrt{\frac{\mathbf{a}^{2}}{8 \mathbf{r}^{2}}-\left(\frac{2 R}{r}\right) \sin ^{2} \frac{A}{2}+\left(\frac{R}{2 r}\right) \sin ^{2} \frac{\mathbf{A}}{2}} \\
& \stackrel{1}{2}<2 \\
& <\sqrt{2} \sqrt{\frac{\mathbf{a}^{2}}{8 \mathbf{r}^{2}}-\left(\frac{2 R}{r}\right) \sin ^{2} \frac{A}{2}+\left(\frac{2 R}{r}\right) \sin ^{2} \frac{\mathbf{A}}{2}}=\frac{\mathbf{a}}{2 \mathbf{r}}
\end{aligned}
$$

$$
\Rightarrow \frac{\sqrt{2} n_{\mathrm{a}}}{2} \frac{\sqrt{h_{\mathrm{a}}}}{\frac{2 \mathrm{r}}{\mathrm{R}}} \sin \frac{\mathrm{~A}}{2}<\frac{\mathrm{a}}{2 \mathrm{r}} \text { and analogs and } \therefore \text { summing up, we obtain : }
$$

$$
\frac{\sqrt{2}}{2} \sum \frac{\mathbf{n}_{a}}{\mathbf{h}_{\mathrm{a}}}+\sqrt{\frac{2 \mathrm{r}}{\mathrm{R}}} \sum \sin \frac{\mathrm{~A}}{2}<\sum \frac{\mathrm{a}}{2 \mathrm{r}}=\frac{2 \mathrm{~s}}{2 \mathrm{r}}=\frac{\mathrm{s}}{\mathbf{r}^{\prime}}
$$

$$
\text { or, in other words, } \frac{\sqrt{2}}{2} \sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathrm{~h}_{\mathrm{a}}}+\sqrt{\frac{2 \mathrm{r}}{\mathrm{R}}} \sum \sin \frac{\mathrm{~A}}{2} \leq \frac{\mathrm{s}}{\mathrm{r}}(\text { Proved })
$$

1519. In $\triangle A B C, I_{a}, I_{b}, I_{c}$-excenters, $I D \perp B C, I E \perp C A, I F \perp A B, h_{1}, h_{2}, h_{3}-$ altitudes in $\triangle D E F, D \in(B C), E \in(C A), F \in(A B), I$ - incenter. Prove that:

$$
\frac{h_{1}}{A I_{a}}+\frac{h_{2}}{B I_{b}}+\frac{h_{c}}{C I_{c}} \leq \frac{3 R}{8 r}
$$

Proposed by Radu Diaconu - Romania
Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
A I_{a}=\frac{s}{\cos \frac{A}{2}} ; B I_{a}=\frac{s}{\cos \frac{B}{2}} ; C I_{a}=\frac{s}{\cos \frac{C}{2}} \\
{[E F D]=\frac{1}{2} \cdot I D \cdot I F \cdot \sin B+\frac{1}{2} \cdot I E \cdot I F \cdot \sin A+\frac{1}{2} \cdot I E \cdot I D \cdot \sin C} \\
=\frac{1}{2} \cdot r^{2} \cdot(\sin A+\sin B+\sin C)=\frac{1}{2} \cdot r^{2} \cdot \frac{s}{R} \\
E F^{2}=(s-a)^{2}+(s-a)^{2}-2(s-a)(s-a) \cos A \\
=2(s-a)^{2}(1-\cos A)=4(s-a)^{2} \sin ^{2} \frac{A}{2} \Rightarrow E F=2(s-a) \sin \frac{A}{2} \\
\text { Similarly:ED }=2(s-c) \sin \frac{C}{2} ; F D=2(s-b) \sin \frac{B}{2}
\end{gathered}
$$



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$$
\begin{gathered}
\Rightarrow h_{1}=\frac{2[E F D]}{E F}=\frac{\frac{w w w . s s m r m h . r o}{R}}{2(s-a) \sin \frac{A}{2}}=\frac{s r^{2}}{2 R(s-a) \sin \frac{A}{2}} \\
\Rightarrow \frac{h_{1}}{A I_{a}}=\frac{s r^{2}}{2 R(s-a) \sin \frac{A}{2}} \cdot \frac{\cos \frac{A}{2}}{s}=\frac{r^{2}}{2 R} \cdot \frac{1}{(s-a) \tan \frac{A}{2}}=\frac{r^{2}}{2 R r}=\frac{r}{2 R}
\end{gathered}
$$

Similarly: $\frac{h_{2}}{B I_{b}}=\frac{h_{3}}{C I_{c}}=\frac{r}{2 R} \Rightarrow \frac{h_{1}}{A I_{a}}+\frac{h_{2}}{B I_{b}}+\frac{h_{3}}{C I_{c}}=\frac{3 r}{2 R} \stackrel{(*)}{\leq} \frac{3 R}{8 r}$
(*) $\Leftrightarrow \mathbf{2 4 r} r^{2} \leq 6 R^{2} \Leftrightarrow 4 r^{2} \leq R^{2} \Leftrightarrow 2 r \leq R($ Euler) $\Rightarrow$ (*) is true $\Rightarrow$ proved Solution 2 by Thanasis Gakopoulos-Larisa-Greece


$$
s=\frac{a+b+c}{2}
$$

Plagiogonal system $B C \equiv B x, B A \equiv B y, \cos B=\frac{a^{2}+c^{2}-b^{2}}{2 a c}$

$$
\begin{gathered}
B(0,0), D\left(d_{1}, 0\right), F\left(0, f_{2}\right), d_{1}=f_{2}=s-b, I_{b}\left(i_{b}, i_{b}\right), i_{b}=\frac{a c}{a-b+c} \\
E\left(e_{1} e_{2}\right), e_{1}=\frac{a}{b}(s-a), e_{2}=\frac{c}{b}(s-c) \\
B I_{b}^{2}=2 \cdot i_{b}^{2}(1+\cos B) \rightarrow B I_{b}^{2}=\frac{a c(a+b+c)}{a-b+c} \\
F O: x+y-(s-b)=0
\end{gathered}
$$



$$
\left.\begin{array}{c}
\text { ROMANIAN MATHEMATICAL MAGAZINE } \\
\boldsymbol{h}_{2}^{2}=\frac{\left[e_{1}+e_{2}-(s-b)\right]^{2} \cdot \sin ^{2} B}{1+1-2 \cdot \cos B} \rightarrow h_{2}^{2}=\frac{(-a+b+c)^{2}(a+b-c)^{2}(a-b+c)(a+b+c)}{16 a b^{2} c^{2}} \\
\frac{h_{s}^{2}}{B I_{b}^{2}}=\frac{(-a+b+c)^{2}(-b+c)^{2}(a+b-c)^{2}}{16 a^{2} b^{2} c^{2}} \rightarrow \frac{h_{2}}{B I_{b}} \rightarrow \frac{(-a+b+c)(a-b+c)(a+b-c)}{4 a b c}=\frac{r}{2 R} \\
\text { Similarly } \frac{h_{1}}{A I_{a}}=\frac{h_{2}}{B I_{b}}=\frac{h_{3}}{C I_{c}}=\frac{r}{2 R} \\
\frac{h_{1}}{A I_{a}}+\frac{h_{2}}{B I_{b}}+\frac{h_{3}}{C I_{b}}=\frac{3 r}{2 R} \\
2 r \leq R \rightarrow 4 r^{2} \leq R^{2} \rightarrow 24 r^{2} \leq 6 R^{2} \rightarrow \frac{3 r}{2 R} \leq \frac{3 R}{8 r}
\end{array}\right\} \rightarrow \frac{h_{1}}{A I_{a}}+\frac{h_{2}}{B I_{b}}+\frac{h_{3}}{C I_{c}} \leq \frac{3 R}{8 r}
$$

1520. In $\triangle A B C$ the following relationship holds:

$$
\left(\sum_{c y c} r_{a}\right)\left(\sum_{c y c} \frac{1}{r_{a}}\right)+\frac{2 \mu r}{R} \geq \mu+9, \mu \leq 8
$$

## Proposed by Marin Chirciu-Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\sum r_{a} \cdot \sum \frac{1}{r_{a}}+\frac{16 r}{R} \geq 17 \Leftrightarrow \frac{4 R+r}{r}+\frac{16 r}{R} \geq 17 \Leftrightarrow \frac{R(4 R+r)+16 r^{2}}{R r} \geq 17 \\
\Leftrightarrow 4 R^{2}-16 R r+16 r^{2} \geq 0 \Leftrightarrow 4(R-2 r)^{2} \geq 0 \\
\rightarrow \text { true } \Rightarrow: \sum r_{a} \cdot \sum \frac{1}{r_{a}}+\frac{16 r}{R}-17 \geq 0 \therefore \text { it suffices to prove }: \\
\sum r_{a} \cdot \sum \frac{1}{r_{a}}+\frac{2 \mu r}{R}-\mu-9 \geq \sum r_{a} \cdot \sum \frac{1}{r_{a}}+\frac{16 r}{R}-17 \\
\Leftrightarrow \frac{2 r}{R}(\mu-8) \geq \mu-8 \Leftrightarrow(\mu-8)\left(\frac{2 r}{R}-1\right) \geq 0 \rightarrow \text { true } \\
\because \mu-8 \leq 0 \text { and } \frac{2 r}{R}-1 \stackrel{\text { Euler }}{\sim} 0 \text { (Proved) }
\end{gathered}
$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
r_{a}+r_{b}+r_{c}=4 R+r
$$



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$$
\begin{gathered}
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\frac{1}{r_{a}}+\frac{1}{r_{b}}+\frac{1}{r_{c}}=\frac{1}{r} \\
\left(\frac{1}{r_{a}}+\frac{1}{r_{b}}+\frac{1}{r_{c}}\right)\left(\frac{1}{r_{a}}+\frac{1}{r_{b}}+\frac{1}{r_{c}}\right)=\frac{4 R+r}{r}=\frac{4 R}{r}+1
\end{gathered}
$$

Let: $x=\frac{R}{r} \geq 2$. We just check: $4 x+1+\frac{2 \mu}{x} \geq \mu+9 \Leftrightarrow$

$$
\begin{aligned}
4 x+\frac{2 \mu}{x} & \geq \mu+8 ; 4 x^{2}+2 \mu \\
4 x^{2}-(\mu+8) x+2 \mu & \geq 0
\end{aligned}
$$

$$
(x-2)(4 x-\mu) \geq 0 \text { true for } x \geq 0 \text { and } \mu \leq 8
$$

1521. In $\triangle \mathrm{ABC}$ the following relationship holds:

$$
\frac{m_{a}^{2}+m_{b}^{2}+m_{c}^{2}}{h_{a}^{2}+h_{b}^{2}+h_{c}^{2}} \leq \frac{R}{2 r}
$$

## Proposed by Rahim Shahbazov-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
m_{a}^{2}+m_{b}^{2}+m_{c}^{2}=\frac{3}{4}\left(a^{2}+b^{2}+c^{2}\right) \\
h_{a}^{2}+h_{b}^{2}+h_{c}^{2}=\frac{4 S^{2}}{a^{2}}+\frac{4 S^{2}}{b^{2}}+\frac{4 S^{2}}{c^{2}}=4 S^{2}\left(\frac{a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}}{a^{2} b^{2} c^{2}}\right) \\
\frac{m_{a}^{2}+m_{b}^{2}+m_{c}^{2}}{h_{a}^{2}+h_{b}^{2}+h_{c}^{2}}=\frac{3 a^{2} b^{2} c^{2}\left(a^{2}+b^{2}+c^{2}\right)}{16 S^{2}\left(a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}\right)} \\
\frac{R}{2 r}=\frac{\frac{a b c}{4 S}}{\frac{2 S}{S}}=\frac{a b c}{4 S} \cdot \frac{s}{2 S}=\frac{a b c(a+b+c)}{16 S^{2}}
\end{gathered}
$$

We must show that:

$$
\frac{3 a^{2} b^{2} c^{2}\left(a^{2}+b^{2}+c^{2}\right)}{16 S^{2}\left(a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}\right)} \leq \frac{a b c(a+b+c)}{16 S^{2}} \Leftrightarrow
$$

$$
\begin{equation*}
3 a^{2} b^{2} c^{2}\left(a^{2}+b^{2}+c^{2}\right) \leq(a+b+c)\left(a^{2} b^{2}+b^{2} c^{2}+c^{2} a^{2}\right) \ldots \tag{*}
\end{equation*}
$$

$$
(a-b)^{2}(b+c-a)^{3}+(b-c)^{2}(a+c-b)^{3}+(a-c)^{2}(a+b-c)^{3} \geq 0
$$



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Which is true, because in $\triangle A B C$ we have

$$
\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{a}>0 ; a+b-c>0 ; c+a-b>0
$$

## Proved. Equality for $a=b=c$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\text { Now, } F(x, y, z)=\sum x^{5}+x y z \sum x y-\sum x^{3} y^{2}-\sum x^{2} y^{3}
$$

is a homogeneous and symmetric polynomial

$$
\begin{aligned}
F(x, y, 0) & =x^{5}+y^{5}-x^{3} y^{2}-x^{2} y^{3} \stackrel{\text { Chebyshev }}{\geq} \frac{1}{2}\left(x^{3}+y^{3}\right)\left(x^{2}+y^{2}\right)-x^{3} y^{2} \\
& -x^{2} y^{3} \stackrel{A}{2}_{\geq}{ }^{A-G} x\left(x^{3}+y^{3}\right)-x^{3} y^{2}-x^{2} y^{3} \\
& \geq x^{2} y^{2}(x+y)-x^{3} y^{2}-x^{2} y^{3}=0 \Rightarrow F(x, y, 0) \geq 0
\end{aligned}
$$

$$
\begin{aligned}
F(x, 1,1)= & x^{5}+1+1+x(x+1+x)-x^{3}-1-x^{2}-x^{2}-1-x^{3} \\
& =x^{5}-2 x^{3}+x=x\left(x^{2}-1\right)^{2} \geq 0 \Rightarrow F(x, 1,1) \geq 0
\end{aligned}
$$

$\therefore$ by SD5 theorem, $F(x, y, z) \geq 0 \Rightarrow(1)$ is true $\therefore \frac{m_{a}^{2}+m_{b}^{2}+m_{c}^{2}}{h_{a}^{2}+h_{b}^{2}+h_{c}^{2}} \leq \frac{R}{2 r}$

$$
\begin{aligned}
& \frac{\sum m_{a}^{2}}{\sum h_{a}^{2}}=\frac{\frac{3 \sum a^{2}}{4}}{\sum\left(\frac{b^{2} c^{2}}{4 R^{2}}\right)}=\frac{3 R^{2} \sum a^{2}}{\sum a^{2} b^{2}} \stackrel{?}{\dot{\sim}} \leq \frac{R}{2 r} \Leftrightarrow \sum a^{2} b^{2}{\underset{\sim}{\sim}}_{\stackrel{?}{n}}^{\sim} 6 R r \sum a^{2} \\
& \text { Let } s-a=x, s-b=y \text { and } s-c=z \therefore 3 s-2 s=\sum x \Rightarrow \\
& a=y+z, b=z+x \text { and } c=x+y \\
& \therefore(1) \Leftrightarrow \sum(y+z)^{2}(z+x)^{2} \geq 6 \frac{\Pi(y+z)}{4 F} \cdot \frac{F}{\sum x} \cdot \sum(y+z)^{2} \\
& \Leftrightarrow \mathbf{2}\left(\sum x\right)\left\{\sum(y+z)^{2}(z+x)^{2}\right\} \geq \mathbf{3}\left\{\prod(y+z)\right\}\left\{\sum(y+z)^{2}\right\} \\
& \Leftrightarrow \sum x^{5}+x y z \sum x y-\sum x^{3} y^{2}-\sum x^{2} y^{3} \stackrel{(2)}{\geq} 0
\end{aligned}
$$



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1522. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{r_{a}}{h_{a}}\right)^{2}+\left(\frac{r_{b}}{h_{b}}\right)^{2}+\left(\frac{r_{c}}{h_{c}}\right)^{2}+\frac{2 \mu r}{R} \geq \mu+3, \mu \leq 12
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\sum_{c y c}\left(\frac{r_{a}}{h_{a}}\right)^{2}=\frac{1}{4} \sum_{c y c} \frac{a^{2}}{(s-a)^{2}}=\frac{1}{4} \sum_{c y c} \frac{(a-s+s)^{2}}{(s-a)^{2}}=\frac{1}{4} \sum_{c y c} \frac{(s-a)^{2}-2 s(s-a)+s^{2}}{(s-a)^{2}} \\
=\frac{1}{4}\left[3-\frac{2}{r} \sum_{c y c} \frac{r s}{s-a}+\frac{1}{r^{2}} \sum_{c y c} \frac{r^{2} s^{2}}{(s-a)^{2}}\right] \\
=\frac{1}{4}\left[3-\frac{2}{r} \sum r_{a}+\frac{1}{r^{2}} \sum r_{a}^{2}\right]=\frac{1}{4}\left[3-\frac{2(4 R+r)}{r}+\frac{(4 R+r)^{2}-2 s^{2}}{r^{2}}\right] \\
=\frac{(4 R+r)^{2}-2 s^{2}-2 r(4 R+r)+3 r^{2}}{4 r^{2}}=\frac{8 R^{2}+r^{2}-s^{2}}{2 r^{2}} \\
\Rightarrow \sum_{c y c}\left(\frac{r_{a}}{h_{a}}\right)^{2}+\frac{24 r}{R}-15=\frac{8 R^{2}+r^{2}-s^{2}}{2 r^{2}}+\frac{24 r}{R}-15 \\
=\frac{R\left(8 R^{2}+r^{2}-s^{2}\right)+48 r^{3}-30 R r^{2}}{2 R r^{2}}
\end{gathered}
$$

$$
\underset{>}{\text { Gerretsen }} \frac{R\left(8 R^{2}+r^{2}\right)+48 r^{3}-30 R r^{2}-R\left(4 R^{2}+4 R r+3 r^{2}\right)}{2 R r^{2}}
$$

$$
=\frac{2(R+3 r)(R-2 r)^{2}}{R r^{2}} \geq 0 \Rightarrow \sum_{c y c}\left(\frac{r_{a}}{h_{a}}\right)^{2}+\frac{24 r}{R}-15 \geq 0
$$

$\therefore$ it suffices to prove $: \sum_{c y c}\left(\frac{r_{a}}{h_{a}}\right)^{2}+\frac{2 \mu r}{R}-\mu-3 \geq \sum_{c y c}\left(\frac{r_{a}}{h_{a}}\right)^{2}+\frac{24 r}{R}-15$

$$
\Leftrightarrow \frac{2 r}{R}(\mu-12) \geq \mu-12 \Leftrightarrow(\mu-12)\left(\frac{2 r}{R}-1\right) \geq 0 \rightarrow \text { true }
$$

$\because \mu \leq 12$ and $\frac{2 r}{R}{ }_{\text {Euler }}^{\text {en }} 1 \therefore\left(\frac{r_{a}}{h_{a}}\right)^{2}+\left(\frac{r_{b}}{h_{b}}\right)^{2}+\left(\frac{r_{c}}{h_{c}}\right)^{2}+\frac{2 \mu r}{R} \geq \mu+3 \forall \mu \leq 12$ (Proved)


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Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\Omega=\frac{a b}{(s-a)(s-b)}+\frac{b c}{(s-b)(s-c)}+\frac{a c}{(s-a)(s-c)} \\
=\frac{a b(s-c)+b c(s-a)+a c(s-b)}{(s-a)(s-b)(s-c)}=\frac{s(a b+b c+c a)-3 a b c}{s r^{2}} \\
=\frac{s\left(s^{2}+4 R r+r^{2}\right)-3 \cdot 4 R r s}{s r^{2}}=\frac{s^{2}-8 R r+r^{2}}{r^{2}} \\
=\frac{\Phi=\frac{a}{s-a}+\frac{b}{s-b}+\frac{c}{s-a}=\frac{4 R-2 r}{r}}{=\left(\frac{a}{s-a}\right)^{2}+\left(\frac{b}{s-b}\right)^{2}+\left(\frac{c}{s-c}\right)^{2}=\Phi^{2}-2 \Omega} \\
r^{2}-2 \cdot \frac{s^{2}-8 R r+r^{2}}{r^{2}}=\frac{16 R^{2}-2 s^{2}+2 r^{2}}{r^{2}}=\frac{2\left(8 R^{2}-s^{2}+r^{2}\right)}{r^{2}} \\
\left(\frac{r_{a}}{h_{a}}\right)^{2}+\left(\frac{r_{b}}{h_{b}}\right)^{2}+\left(\frac{r_{c}}{h_{c}}\right)^{2}=\left(\frac{S}{\frac{p-a}{a}}\right)^{2}+\left(\frac{\frac{S}{p-b}}{\frac{2 S}{b}}\right)^{2}+\left(\frac{\frac{S}{p-c}}{\frac{2 S}{c}}\right)^{2} \\
=\frac{1}{4}\left[\left(\frac{a}{s-a}\right)^{2}+\left(\frac{b}{s-b}\right)^{2}+\left(\frac{c}{s-c}\right)^{2}\right] \\
=\frac{8 R^{2}-s^{2}+r^{2} s^{2} \leq 4 R^{2}+4 R r+3 r^{2}}{\frac{2 r^{2}}{\geq}} \frac{4 R^{2}-4 R r-2 r^{2}}{2 r^{2}} \\
=\frac{2 R^{2}-2 R r-r^{2}}{r^{2}}=\frac{2 R^{2}}{r^{2}}-\frac{2 R}{r}-1
\end{gathered}
$$

Let: $t=\frac{R}{r} \geq 2$. We need to prove: $2 t^{2}-2 t-1+\frac{2 \mu}{t} \geq \mu+3$

$$
\begin{equation*}
2 t^{3}-2 t^{2}-(\mu+4) t+2 \mu \geq 0 \tag{*}
\end{equation*}
$$

Let: $\varphi(t)=2 t^{3}-2 t^{2}-(\mu+4) t+2 \mu, t \geq 2$

$$
\begin{gathered}
\varphi^{\prime}(t)=6 t^{2}-\mathbf{4 t - ( \mu + 4 )} \\
\varphi^{\prime \prime}(t)=\mathbf{1 2 t}-\mathbf{4} \\
\varphi^{\prime \prime}(t)=0 \Leftrightarrow t=\frac{\mathbf{1}}{3}<2 \\
\Rightarrow \varphi^{\prime}(t) \geq \varphi^{\prime}(2)=\mathbf{1 2}-\mu \geq \mathbf{0}
\end{gathered}
$$



## ROMANIAN MATHEMATICAL MAGAZINE <br> $\Rightarrow \varphi(t) \geq \varphi(2)=8-2(\mu+4)+2 \mu=0 \Rightarrow$ * $\left.^{*}\right)$ true.

1523. In $\triangle A B C$ the following relationship holds:

$$
\frac{r_{a}-h_{a}}{r_{a}+h_{a}}+\frac{r_{b}-h_{b}}{r_{b}+h_{b}}+\frac{r_{c}-h_{c}}{r_{c}+h_{c}}+\frac{3 r}{R+r} \leq 1
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Avishek Mitra-West Bengal-India

$$
\begin{gathered}
\frac{r_{a}-h_{a}}{r_{a}+h_{a}}+\frac{r_{b}-h_{b}}{r_{b}+h_{b}}+\frac{r_{c}-h_{c}}{r_{c}+h_{c}}+\frac{3 r}{R+r} \leq 1 \\
\Omega=\sum_{c y c} \frac{r_{a}-h_{a}}{r_{a}+h_{a}}=\sum_{c y c} \frac{\frac{S}{s-a}+\frac{2 S}{a}}{\frac{S}{s-a}+\frac{2 S}{a}}=\sum_{c y c} \frac{2 a-(b+c)}{b+c}=2 \sum_{c y c} \frac{a}{b+c}-3 \\
\sum_{c y c} \frac{a}{b+c}=\sum_{c y c} \frac{a+b+c}{b+c}-3=2 s\left(\sum_{c y c} \frac{1}{b+c}\right)-3= \\
=2 s \cdot \frac{\sum(b+c)(c+a)}{\prod(a+b)}-3=2 s \cdot \frac{\sum\left(a b+b c+c a+c^{2}\right)}{\sum a b(a+b)+2 a b c}-3= \\
=2 s \cdot \frac{\sum c^{2}+3 \sum a b}{a b c+\sum a b(2 s-c)}-3= \\
=2 s \cdot \frac{2\left(s^{2}-4 R r-r^{2}\right)+3\left(s^{2}+4 R r+r^{2}\right)}{2 s\left(\sum a b\right)-a b c}-3= \\
=2 s \cdot \frac{5 s^{2}+4 R r+r^{2}}{2 s\left(s^{2}+r^{2}+4 R r\right)-4 R r s}-3= \\
=2 s \cdot \frac{5 s^{2}+4 R r+r^{2}}{2 s\left(s^{2}+r^{2}+2 R r\right)}-3=\frac{2 s^{2}-2 R r-2 r^{2}}{s^{2}+r^{2}+2 R r} \\
\Omega=\frac{4 s^{2}-4 R r-4 r^{2}}{s^{2}+r^{2}+2 R r}-3=\frac{s^{2}-10 R r-7 r^{2}}{s^{2}+r^{2}+2 R r}
\end{gathered}
$$

$$
\text { Need to show: } \Omega \leq 1-\frac{3 r}{R+r}=\frac{R-2 r}{R+r} \Leftrightarrow
$$

$$
\frac{s^{2}-10 R r-7 r^{2}}{s^{2}+r^{2}+2 R r} \leq \frac{R-2 r}{R+r} \Leftrightarrow
$$

$$
\left(s^{2}-10 R r-7 r^{2}\right)(R+r) \leq(R-2 r)\left(s^{2}+r^{2}+2 R r\right) \Leftrightarrow
$$

$$
R s^{2}-10 R^{2} r-7 R r^{2}+s^{2} r-10 R r^{2}-7 r^{3} \leq R s^{2}+R r^{2}+2 R^{2} r-2 s^{2} r-2 r^{3}-
$$



$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \text { www.ssmrmh.ro } \\
& -4 R r^{2} \Leftrightarrow 3 s^{2} r \leq 12 R^{2} r+14 R r^{2}+5 r^{3} \\
& 3 s^{2} \leq 12 R^{2}+14 R r+5 r^{2} \text {. But: } s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \text { (Gerretsen) } \\
& \text { Need to show: } 12 R^{2}+12 R r+9 r^{2} \leq 12 R^{2}+14 R r+5 r^{2} \Leftrightarrow \\
& R \geq 2 r \text { (Euler). Proved. }
\end{aligned}
$$

## Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& \frac{r_{a}-h_{a}}{r_{a}+h_{a}}+\frac{r_{b}-h_{b}}{r_{b}+h_{b}}+\frac{r_{c}-h_{c}}{r_{c}+h_{c}}=\sum_{c y c} \frac{\frac{r_{a}}{h_{a}}-1}{\frac{r_{a}}{h_{a}}+1}=\sum_{c y c} \frac{\frac{a}{2(s-a)}-1}{\frac{a}{2(s-a)}+1}= \\
& =\sum_{c y c} \frac{\frac{a}{b+c-a}-1}{a} \\
& \frac{R}{r}=\frac{2 a b c}{(a+b-c)(b+c-a)(a+c-b)} \Rightarrow \frac{3 r}{R+r}=\frac{3}{\frac{R}{r}+1}= \\
& =\frac{3}{\frac{2 a b c}{(a+b-c)(b+c-a)(a+c-b)}+1} \\
& \text { Let: } \boldsymbol{x}=\frac{\boldsymbol{b}+\boldsymbol{c}-\boldsymbol{a}}{\boldsymbol{a}}, \boldsymbol{y}=\frac{\boldsymbol{a + c - b}}{\boldsymbol{b}}, z=\frac{\boldsymbol{a + b}-\boldsymbol{c}}{\boldsymbol{c}} \Rightarrow \mathbf{0}<x y z \leq 1, x+y+z \geq 3 \\
& x y z=1 \Leftrightarrow a=b=c \Leftrightarrow x+y+z=3 \\
& \Rightarrow \frac{1}{x+2}+\frac{1}{y+2}+\frac{1}{z+2}=\frac{1}{\frac{b+c-a}{a}+2}+\frac{1}{\frac{a+c-b}{b}+2}+\frac{1}{\frac{a+b-c}{c}+2} \\
& =\frac{a}{a+b+c}+\frac{b}{a+b+c}+\frac{c}{a+b+c}=1 \Leftrightarrow x y z+x y+y z+z x=4 \\
& \text { More, }(x y+y z+z x)^{2} \geq 3 x y z(x+y+z) \Rightarrow \\
& x+y+z \leq \frac{(x y+y z+z x)^{2}}{3 x y z}=\frac{(4-x y z)^{2}}{3 x y z} \\
& \text { Inequality becomes as: } \frac{1-x}{1+x}+\frac{1-y}{1+y}+\frac{1-z}{1+z}+\frac{3 x y z}{2+x y z} \leq 1 \\
& -(x y z)^{2}+x y z[x y+y z+z x+3(x+y+z-1)]-4(x y+y z+z x)+4 \leq 0 \\
& -(x y z)^{2}+x y z[4-x y z+3(x+y+z-1)]-4(4-x y z)+4 \leq 0 \\
& -2(x y z)^{2}+3 x y z(x+y+z)+5 x y z-12 \leq 0
\end{aligned}
$$



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\begin{gathered}
-2(x y z)^{2}+x y z[3(x+y+z)+5]-12 \stackrel{(*)}{\leq} 0 \\
\text { Let }: t=x y z, 0<t \leq 1
\end{gathered}
$$

$$
(*) \Leftrightarrow-2 t^{2}+[3(x+y+z)+5] t-12 \leq 0
$$

Put: $\varphi(t)=\Leftrightarrow-2 t^{2}+[3(x+y+z)+5] t-12,0<t \leq 1$

$$
\begin{gathered}
\varphi^{\prime}(t)=-4 t+3(x+y+z)+5 \\
\varphi^{\prime}(t)=0 \Leftrightarrow t=\frac{3(x+y+z)+5}{4}>1 \\
\Rightarrow \forall t \in(0,1], \varphi(t) \leq \varphi(1)=3(x+y+z)-9=0 \Rightarrow(*) \text { is true.Proved. }
\end{gathered}
$$

1524. In $\triangle A B C$ the following relationship holds:

$$
3\left(a^{3}+b^{3}+8 m_{c}^{3}+6 a b m_{c}\right) \leq 2\left(a+b+2 m_{c}\right)\left(3 a^{2}+3 b^{2}-c^{2}\right)
$$

When equality holds?

## Proposed by Daniel Sitaru-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
3 a^{2}+3 b^{2}-c^{2}=\left(a^{2}+b^{2}\right)+\left(2 a^{2}+2 b^{2}-c^{2}\right)=\left(a^{2}+b^{2}\right)+4 m_{c}^{2} \\
R H S=2\left(a+b+2 m_{c}\right)\left(a^{2}+b^{2}+4 m_{c}^{2}\right)
\end{gathered}
$$

Now, we just check

$$
\begin{gathered}
2\left(a+b+2 m_{c}\right)\left(a^{2}+b^{2}+4 m_{c}^{2}\right) \geq 3\left(a^{3}+b^{3}+8 m_{c}^{3}+6 a b m_{c}\right) \\
\text { In } \triangle A C C^{\prime}: C C^{\prime}=2 m_{a} ; A C=b ; A C^{\prime}=a . \\
\text { Let: } a=\alpha+\beta ; b=\beta+\gamma ; 2 m_{a}=\gamma+\alpha \quad(\alpha, \beta, \gamma>0) \\
\Leftrightarrow 2\left(\alpha^{3}+\beta^{3}+\gamma^{3}+3 \alpha \beta \gamma\right) \geq 2(\alpha \beta(\alpha+\beta)+\beta \gamma(\beta+\gamma)+\gamma \alpha(\gamma+\alpha)) \\
\alpha^{3}+\beta^{3}+\gamma^{3}+3 \alpha \beta \gamma \leq \alpha \beta(\alpha+\beta)+\beta \gamma(\beta+\gamma)+\gamma \alpha(\gamma+\alpha)
\end{gathered}
$$

Which is true Schur's inequality.

$$
\left.\begin{array}{rl}
\text { Equality } \Leftrightarrow a=b & =2 m_{c} \Leftrightarrow a^{2}=b^{2}=4 m_{c}^{2} \\
a^{2}=b^{2} & =\mathbf{2} a^{2}+2 b^{2}-c^{2}
\end{array}\right\} \begin{gathered}
a=b \\
\Leftrightarrow\left\{\begin{array}{c}
a=c \\
4 a^{2}-c^{2}=a^{2}
\end{array} \Leftrightarrow a=b=c \sqrt{3}\right.
\end{gathered}
$$



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1525. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{r_{a}}{s_{a}}\right)^{2}+\left(\frac{r_{b}}{s_{b}}\right)^{2}+\left(\frac{r_{c}}{s_{c}}\right)^{2}+\frac{2 \mu r}{R} \geq \mu+3, \mu \leq 5
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& s_{a}=\frac{2 b c m_{a}}{b^{2}+c^{2}} \stackrel{A-G}{\varsigma} m_{a} \Rightarrow\left(\frac{r_{a}}{s_{a}}\right)^{2} \geq\left(\frac{r_{a}}{m_{a}}\right)^{2} \text { and analogs } \\
& \Rightarrow\left(\frac{r_{a}}{s_{a}}\right)^{2}+\left(\frac{r_{b}}{s_{b}}\right)^{2}+\left(\frac{r_{c}}{s_{c}}\right)^{2} \stackrel{(1)}{\geq}\left(\frac{r_{a}}{m_{a}}\right)^{2}+\left(\frac{r_{b}}{m_{b}}\right)^{2}+\left(\frac{r_{c}}{m_{c}}\right)^{2} \\
& \because m_{a} \leq \frac{R h_{a}}{2 r}=\frac{R s}{a} \therefore \sum\left(\frac{r_{a}}{m_{a}}\right)^{2} \geq \sum \frac{a^{2} r^{2} s^{2}}{R^{2} s^{2}(s-a)^{2}}=\frac{r^{2}}{R^{2}} \sum \frac{(a-s+s)^{2}}{(s-a)^{2}} \\
& =\frac{r^{2}}{R^{2}} \sum\left[\frac{(s-a)^{2}-2 s(s-a)+s^{2}}{(s-a)^{2}}\right] \\
& =\frac{r^{2}}{R^{2}}\left[3-\frac{2 s}{s r^{2}} \sum(s-b)(s-c)+\frac{1}{r^{2}} \sum \frac{r^{2} s^{2}}{(s-a)^{2}}\right] \\
& =\frac{r^{2}}{R^{2}}\left[3-\frac{2\left(4 R r+r^{2}\right)}{r^{2}}+\frac{1}{r^{2}} \sum r_{a}^{2}\right]=\frac{r^{2}}{R^{2}}\left\{\frac{(4 R+r)^{2}-2 s^{2}+3 r^{2}-2\left(4 R r+r^{2}\right)}{r^{2}}\right\} \\
& =\frac{16 R^{2}+2 r^{2}-2 s^{2}}{R^{2}} \Rightarrow \sum\left(\frac{r_{a}}{m_{a}}\right)^{2}+\frac{10 r}{R}-8 \geq \frac{16 R^{2}+2 r^{2}-2 s^{2}}{R^{2}}+\frac{10 r}{R}-8 \\
& =\frac{16 R^{2}+2 r^{2}-2 s^{2}+10 R r-8 R^{2}}{R^{2}} \\
& \underset{\geq}{\text { Gerretsen }} \frac{8 R^{2}+10 R r+2 r^{2}-2\left(4 R^{2}+4 R r+3 r^{2}\right)}{R^{2}}=\frac{2 R r-4 r^{2}}{R^{2}} \\
& =\frac{2 r(R-2 r)}{R^{2}} \stackrel{\text { Euler }}{\geq} 0 . \therefore \sum\left(\frac{r_{a}}{m_{a}}\right)^{2}+\frac{10 r}{R}-8 \geq 0
\end{aligned}
$$



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$$
\begin{gathered}
\Rightarrow \sum\left(\frac{r_{a}}{m_{a}}\right)^{2} \geq 3+5\left(1-\frac{2 r}{R}\right)^{\stackrel{5 \geq \mu}{\geq}} 3+\mu\left(1-\frac{2 r}{R}\right) \Rightarrow \sum\left(\frac{r_{a}}{m_{a}}\right)^{2} \stackrel{(2)}{\geq} \mu+3-\frac{2 \mu r}{R} \\
\therefore(1),(2) \Rightarrow \sum\left(\frac{r_{a}}{s_{a}}\right)^{2} \geq \mu+3-\frac{2 \mu r}{R} \text { (Proved) }
\end{gathered}
$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\Omega=\frac{a b}{(s-a)(s-b)}+\frac{b c}{(s-b)(s-c)}+\frac{a c}{(s-a)(s-c)} \\
=\frac{a b(s-c)+b c(s-a)+a c(s-b)}{(s-a)(s-b)(s-c)}=\frac{s(a b+b c+c a)-3 a b c}{s r^{2}} \\
=\frac{s\left(s^{2}+4 R r+r^{2}\right)-3 \cdot 4 R r s}{s r^{2}}=\frac{s^{2}-8 R r+r^{2}}{r^{2}} \\
=\frac{4(2 R-r)^{2}}{r^{2}}-2 \cdot \frac{a}{s-a}+\frac{b}{s-b}+\frac{s^{2}-8 R r+r^{2}}{r^{2}}=\frac{16 R^{2}-2 s^{2}+2 r^{2}}{r^{2}}=\frac{2\left(8 R^{2}-s^{2}+r^{2}\right)}{r^{2}} \\
\Rightarrow\left(\frac{a}{s-a}\right)^{2}+\left(\frac{b}{s-b}\right)^{2}+\left(\frac{c}{s-c}\right)^{2}=\Phi^{2}-2 \Omega \\
\therefore s_{a} \leq m_{a} \leq \frac{R}{2 r} \cdot h_{a}(e t c) \\
=\left(\frac{r_{a}}{s_{a}}\right)^{2}+\left(\frac{r_{b}}{s_{b}}\right)^{2}+\left(\frac{r_{c}}{s_{c}}\right)^{2}=\left(\frac{2 r}{R}\right)^{2}\left[\left(\frac{r_{a}}{h_{a}}\right)^{2}+\left(\frac{r_{b}}{h_{b}}\right)^{2}+\left(\frac{r_{c}}{h_{c}}\right)^{2}\right] \\
=\left(\frac{2 r}{R}\right)^{2} \cdot \frac{1}{4} \cdot\left[\left(\frac{a}{s-a}\right)^{2}+\left(\frac{b}{s-b}\right)^{2}+\left(\frac{c}{s-c}\right)^{2}\right] \\
=\left(\frac{r}{R}\right)^{2} \cdot \frac{2\left(8 R^{2}-s^{2}+r^{2}\right)^{s^{2} \leq 4 R^{2}+4 R r+3 r^{2}}}{r^{2}} \frac{2\left(4 R^{2}-4 R r-2 r^{2}\right)}{R^{2}} \\
=\frac{4\left(2 R^{2}-2 R r-r^{2}\right)}{R^{2}}=4\left[2-\left(\frac{r}{R}\right)^{2}-2\left(\frac{r}{R}\right)\right] \\
L e t: t=\frac{r}{R}\left(0<t \leq \frac{1}{2}\right)
\end{gathered}
$$

We need to prove:

$$
\begin{equation*}
4\left(2-t^{2}-2 t\right)+2 \mu t \geq \mu+3 \Leftrightarrow(2 t-1)(2 t+(5-\mu)) \leq 0 \tag{*}
\end{equation*}
$$



## ROMANIAN MATHEMATICAL MAGAZINE www.ssmrmh.ro <br> $0<t \leq \frac{1}{2} \Rightarrow 2 t-1 \leq 0 ; \mu-5 \leq 0 \Rightarrow 2 t+5-\mu>0$ <br> $\Rightarrow$ (*) true. Proved.

1526. In $\triangle A B C$ the following relationship holds:

$$
\frac{1}{9}\left(\frac{7 R-2 r}{2 R}\right)^{3} \leq \frac{m_{a}^{3}}{h_{a}^{3}}+\frac{\mathbf{m}_{b}^{3}}{h_{b}^{3}}+\frac{\mathbf{m}_{\mathrm{c}}^{3}}{\mathbf{h}_{\mathrm{c}}^{3}} \leq \frac{243}{32}\left(\frac{R}{r}\right)^{3}-\frac{231 r}{2 R}
$$

Proposed by Marin Chirciu-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\text { Now, Rouche } \Rightarrow \mathbf{s}^{2}-(\mathbf{m}-\mathbf{n}) \geq \mathbf{0} \text { and } \mathbf{s}^{2}-(\mathbf{m}+\mathbf{n}) \leq \mathbf{0} \text {, where } \mathbf{m}
$$

$$
=2 R^{2}+10 R r-r^{2} \text { and } n=2(R-2 r) \sqrt{R^{2}-2 R r}
$$

$$
\therefore\left(\mathbf{s}^{2}-(\mathbf{m}+\mathbf{n})\right)\left(\mathbf{s}^{2}-(\mathbf{m}-\mathbf{n})\right) \leq \mathbf{0} \Rightarrow \mathbf{s}^{4}-\mathbf{s}^{2}(2 \mathbf{m})+\mathbf{m}^{2}-\mathbf{n}^{2} \leq \mathbf{0}
$$

$$
\Rightarrow s^{4}-s^{2}\left(4 R^{2}+20 R r-2 r^{2}\right)+r(4 R+r)^{3} \leq 0
$$

$$
\Rightarrow 2 R^{2} s^{4}-s^{2}\left(8 R^{4}+40 R^{3} r-4 R^{2} r^{2}\right)+2 R^{2} r(4 R+r)^{3} \stackrel{(m)}{\leftrightarrows} 0
$$

$$
(m) \Rightarrow \text { in order to prove }(i) \text {, it suffices to prove : }
$$

$$
2 R^{2} s^{4}-s^{2}\left(243 R^{4}+16 R^{3} r-20 R^{2} r^{2}-3696 r^{4}\right)+2 R^{2} r^{2}(4 R+r)^{2}
$$

$$
\leq 2 R^{2} s^{4}-s^{2}\left(8 R^{4}+40 R^{3} r-4 R^{2} r^{2}\right)+2 R^{2} r(4 R+r)^{3}
$$

$$
\Leftrightarrow s^{2}\left(235 R^{4}-24 R^{3} r-16 R^{2} r^{2}-3696 r^{4}\right)+2 R^{2} r(4 R+r)^{3}-2 R^{2} r^{2}(4 R+r)^{2} \geq 0
$$

$$
\begin{aligned}
& \text { Panaitopol } \Rightarrow \frac{\mathbf{m}_{a}}{\mathbf{h}_{\mathrm{a}}} \leq \frac{\mathrm{R}}{2 \mathrm{r}} \Rightarrow \frac{\mathbf{m}_{a}^{3}}{\mathbf{h}_{a}^{3}} \leq\left(\frac{\mathbf{R}}{2 \mathrm{r}}\right) \frac{\mathbf{m}_{a}^{2}}{\mathbf{h a}_{a}^{2}} \text { and analogs } \Rightarrow \sum \frac{\mathbf{m}_{a}^{3}}{\mathbf{h}_{a}^{3}} \leq\left(\frac{\mathbf{R}}{2 \mathrm{r}}\right) \sum \frac{\mathbf{m}_{a}^{2}}{\mathbf{h}_{a}^{2}} \\
& =\left(\frac{\mathbf{R}}{2 \mathbf{r}}\right) \frac{\sum\left\{\mathbf{a}^{2}\left(2 \mathbf{b}^{2}+2 \mathbf{c}^{2}-\mathbf{a}^{2}\right)\right\}}{16 \mathbf{r}^{2} \mathbf{s}^{2}} \\
& =\left(\frac{\mathbf{R}}{32 \mathbf{r}^{3} \mathbf{s}^{2}}\right)\left(2 \sum \mathbf{a}^{2} \mathbf{b}^{2}+2 \sum \mathbf{a}^{2} \mathbf{b}^{2}-\sum \mathrm{a}^{4}\right) \\
& =\left(\frac{R}{32 r^{3} s^{2}}\right)\left[2\left\{\left(s^{2}+4 R r+r^{2}\right)^{2}-16 R \operatorname{Rr}^{2}\right\}+16 r^{2} s^{2}\right] \stackrel{?}{\dot{m}} \frac{243}{32}\left(\frac{R}{r}\right)^{3}-\frac{231 r}{2 R} \\
& \Leftrightarrow R^{2}\left[2\left\{\left(s^{2}+4 R r+r^{2}\right)^{2}-16 \operatorname{Rrs}^{2}\right\}+16 r^{2} s^{2}\right] \stackrel{\text { थे }}{\leq} s^{2}\left(243 R^{4}-3696 r^{4}\right) \\
& \Leftrightarrow 2 R^{2} s^{4}-s^{2}\left(243 R^{4}+16 R^{3} r-20 R^{2} r^{2}-3696 r^{4}\right)+2 R^{2} r^{2}(4 R+r)^{2} \underset{(i)}{\stackrel{?}{\underset{\sim}{i}}} 0
\end{aligned}
$$



## ROMANIAN MATHEMATICAL MAGAZINE www.ssmrmh.ro <br> $\Leftrightarrow s^{2}(R-2 r)\left(235 R^{3}+446 R^{2} r+876 R r^{2}+1752 r^{3}\right)+2 R^{2} r(4 R+r)^{3}$

(ii)

$$
-2 R^{2} r^{2}(4 R+r)^{2} \xrightarrow{m} 192 r^{4} s^{2}
$$

Gerretsen
Now, LHS of (ii) $\qquad$ $\left(16 R r-5 r^{2}\right)(R-2 r)\left(235 R^{3}+446 R^{2} r+876 R r^{2}\right.$
(a)
$\left.+1752 r^{3}\right)+2 R^{2} r(4 R+r)^{3}-2 R^{2} r^{2}(4 R+r)^{2}$
Gerretsen
and RHS of (ii) $\qquad$ $192 \mathrm{r}^{4}\left(4 \mathrm{R}^{2}+4 \mathrm{Rr}+3 \mathrm{r}^{2}\right) \therefore(\mathrm{a}),(\mathrm{b})$
(b)
$\Rightarrow$ in order to prove (ii), it suffices to prove :
$\left(16 R r-5 r^{2}\right)(R-2 r)\left(235 R^{3}+446 R^{2} r+876 R r^{2}+1752 r^{3}\right)+2 R^{2} r(4 R+r)^{3}$
$-2 R^{2} r^{2}(4 R+r)^{2} \geq 192 r^{4}\left(4 R^{2}+4 R r+3 r^{2}\right)$
$\Leftrightarrow 3888 t^{5}-1495 t^{4}-128 t^{3}-688 t^{2}-56832 t+16944 \geq 0\left(\right.$ where $\left.t=\frac{R}{r}\right)$
$\Leftrightarrow(t-2)\left\{3888 t^{4}+6281 t^{3}+12434 t^{2}+19944 t+4236(t-2)\right\} \geq 0 \rightarrow$ true

$$
\begin{gathered}
\because \mathbf{t} \stackrel{\text { Euler }}{\stackrel{m}{s}} 2 \Rightarrow(\mathrm{ii}) \Rightarrow(\mathbf{i}) \text { is true } \\
\Rightarrow \frac{\mathbf{m}_{\mathrm{a}}^{3}}{\mathbf{h}_{\mathrm{a}}^{3}}+\frac{\mathbf{m}_{\mathrm{b}}^{3}}{\mathbf{h}_{\mathrm{b}}^{3}}+\frac{\mathbf{m}_{\mathbf{c}}^{3}}{\mathbf{h}_{\mathbf{c}}^{3}} \stackrel{(1)}{\leq} \frac{243}{32}\left(\frac{\mathbf{R}}{\mathbf{r}}\right)^{3}-\frac{231 \mathrm{r}}{2 \mathrm{R}}
\end{gathered}
$$


Now, $\sum \frac{\mathbf{m}_{\mathrm{a}}}{\mathrm{h}_{\mathrm{a}}} \stackrel{\text { Tereshin }}{\gtrless} \sum \frac{\mathbf{b}^{2}+\mathrm{c}^{2}}{2 \mathrm{bc}}=\frac{\sum \mathrm{ab}(2 \mathrm{~s}-\mathrm{c})}{2 \mathrm{abc}}=\frac{2 \mathrm{~s}\left(\mathrm{~s}^{2}+4 \mathrm{Rr}+\mathrm{r}^{2}\right)-12 \mathrm{Rrs}}{8 \mathrm{Rrs}}$

$$
=\frac{s^{2}-2 R r+r^{2}}{4 R r} \underset{\sim}{\gtrless} \frac{7 R-2 r}{2 R} \Leftrightarrow s^{2} \stackrel{?}{乌} 16 R r-5 r^{2}
$$

$\rightarrow$ true, by Gerretsen $\Rightarrow$ (iii) is true $\Rightarrow \frac{\mathbf{m}_{a}^{3}}{\mathbf{h}_{\mathrm{a}}^{3}}+\frac{\mathrm{m}_{\mathrm{b}}^{3}}{\mathbf{h}_{\mathrm{b}}^{3}}+\frac{\mathbf{m}_{\mathrm{c}}^{3}}{\mathbf{h}_{\mathrm{c}}^{3}} \stackrel{(2)}{\geq} \frac{1}{9}\left(\frac{7 \mathrm{R}-2 \mathrm{r}}{2 \mathrm{R}}\right)^{3}$
(1), (2) $\Rightarrow \frac{1}{9}\left(\frac{7 R-2 r}{2 R}\right)^{3} \leq \frac{m_{a}^{3}}{h_{a}^{3}}+\frac{\mathbf{m}_{b}^{3}}{h_{b}^{3}}+\frac{\mathbf{m}_{\mathrm{c}}^{3}}{\mathbf{h}_{\mathrm{c}}^{3}} \leq \frac{243}{32}\left(\frac{\mathrm{R}}{\mathrm{r}}\right)^{3}-\frac{231 \mathrm{r}}{2 \mathrm{R}}$ (Proved)


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1527. In $\triangle A B C$ the following relationship holds:

$$
2 \prod \frac{\mathrm{~m}_{\mathrm{a}}}{\mathrm{r}_{\mathrm{a}}} \leq 1+\frac{1}{4} \sum\left(\frac{\mathrm{a}}{\mathrm{w}_{\mathrm{a}}}\right)^{2}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{align*}
& m_{a}^{2} m_{b}^{2} m_{c}^{2}=\frac{1}{64}\left(2 b^{2}+2 c^{2}-2 a^{2}\right)\left(2 c^{2}+2 a^{2}-2 b^{2}\right)\left(2 a^{2}+2 b^{2}-2 c^{2}\right) \stackrel{(1)}{=} \\
& =\frac{1}{64}\left\{-4 \sum a^{6}+6\left(\sum a^{4} b^{2}+\sum a^{2} b^{4}\right)+3 a^{2} b^{2} c^{2}\right\} \\
& \text { Now, } \sum \mathbf{a}^{6}=\left(\sum \mathbf{a}^{2}\right)^{3}-\mathbf{3}\left(\mathbf{a}^{2}+b^{2}\right)\left(b^{2}+c^{2}\right)\left(\mathbf{c}^{2}+\mathbf{a}^{2}\right) \\
& =\left(\sum \mathbf{a}^{2}\right)^{3}-\mathbf{3}\left(2 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}+\sum \mathbf{a}^{2} \mathbf{b}^{2}\left(\sum \mathbf{a}^{2}-\mathbf{c}^{2}\right)\right) \\
& =\left(\sum \mathbf{a}^{2}\right)^{3}+3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}-\mathbf{3}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2} \therefore \sum \mathbf{a}^{6} \xlongequal{m}\left(\sum \mathbf{a}^{2}\right)^{3}+3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}-\mathbf{3}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}  \tag{2}\\
& \text { Again, } \sum \mathbf{a}^{4} \mathbf{b}^{2}+\sum \mathbf{a}^{2} \mathbf{b}^{4}=\sum \mathbf{a}^{2} \mathbf{b}^{2}\left(\sum \mathbf{a}^{2}-\mathbf{c}^{2}\right) \stackrel{(3)}{\cong}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}-3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2} \\
& \therefore(1),(2),(3) \Rightarrow \mathbf{m}_{\mathbf{a}}^{2} \mathbf{m}_{\mathrm{b}}^{2} \mathbf{m}_{\mathbf{c}}^{2} \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}-12 a^{2} b^{2} c^{2}+12\left(\sum a^{2} b^{2}\right) \sum a^{2}+6\left(\sum a^{2} b^{2}\right) \sum a^{2}\right. \\
& \left.-18 a^{2} b^{2} c^{2}+3 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}+18\left(\sum a^{2} b^{2}\right) \sum a^{2}-27 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-4\left(\sum a^{2}\right)^{3}+18\left(\left(\sum a b\right)^{2}-2 a b c(2 s)\right)\left(\sum a^{2}\right)-27 a^{2} b^{2} c^{2}\right\} \\
& =\frac{1}{64}\left\{-32\left(s^{2}-4 \operatorname{Rr}-r^{2}\right)^{3}+36\left(s^{2}-4 \operatorname{Rr}-r^{2}\right)\left(s^{2}+4 R r+r^{2}\right)^{2}\right. \\
& \left.-576 \operatorname{Rrs}^{2}\left(s^{2}-4 R r-r^{2}\right)-432 R^{2} \mathbf{r}^{2} s^{2}\right\} \\
& =\frac{1}{16}\left\{s^{6}-s^{4}\left(12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3}\right\} \leq \frac{R^{2} s^{4}}{4} \\
& \Leftrightarrow s^{6}-s^{4}\left(4 R^{2}+12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3} \underset{\leq}{ } 0
\end{align*}
$$



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Gerretsen
Now, LHS of (i) $\widetilde{\leftrightarrows} \quad-s^{4}\left(8 R r-36 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 \mathrm{Rr}^{3}+33 \mathrm{r}^{4}\right)$

$$
\begin{aligned}
& -\mathbf{r}^{3}(4 R+r)^{3} \stackrel{?}{\underline{\sim}} 0 \\
\Leftrightarrow & s^{4}(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+r^{2}(4 R+r)^{3} \sum_{(i i)}^{?} 20 r s^{4}
\end{aligned}
$$

Now, LHS of (ii) $\underbrace{\substack{\text { Gerretsen }}}_{\text {(a) }} s^{2}\left(16 R r-5 r^{2}\right)(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R^{2}+33 r^{3}\right)$

$$
+\mathbf{r}^{2}(4 \mathbf{R}+\mathbf{r})^{3}
$$


(a), (b) $\Rightarrow$ in order to prove (ii), it suffices to prove:

$$
\begin{gathered}
\mathbf{s}^{2}\left(16 R r-5 r^{2}\right)(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+\mathbf{r}^{2}(4 R+r)^{3} \\
\geq 20 \mathbf{r s}^{2}\left(4 R^{2}+4 R r+3 r^{2}\right) \Leftrightarrow s^{2}\left(108 R^{2}-256 R r+53 r^{2}\right)+r(4 R+r)^{3} \geq 0 \\
\Leftrightarrow s^{2}\left(108 R^{2}-256 R r+80 r^{2}\right)+r(4 R+r)^{3} \xrightarrow{(i i i)} \geq 27 r^{2} s^{2}
\end{gathered}
$$



Now, LHS of (iii)

$+\mathbf{r}(4 \mathrm{R}+\mathbf{r})^{3}$ and RHS of (iii) $\underbrace{\substack{\text { Geretsen } \\ \int}}_{\text {(d) }} 27 \mathbf{r}^{2}\left(4 \mathbf{R}^{2}+4 \mathrm{Rr}+3 \mathbf{r}^{2}\right)$
(c), (d) $\Rightarrow$ in order to prove (iii), it suffices to prove :
$\left(108 R^{2}-256 R r+80 r^{2}\right)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3} \geq 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right)$
$\Leftrightarrow 224 \mathrm{t}^{3}-587 \mathrm{t}^{2}+308 \mathrm{t}-\mathbf{6 0} \geq 0\left(\right.$ where $\left.\mathrm{t}=\frac{\mathrm{R}}{\mathrm{r}}\right)$

$$
\begin{aligned}
& \Leftrightarrow(t-2)\{(\mathbf{t}-2)(224 \mathbf{t}+309)+648\} \geq 0 \rightarrow \text { true } \because \mathbf{t}{ }^{\text {Euler }} 2 \Rightarrow(\text { iii }) \Rightarrow(\text { ii) } \\
& \Rightarrow(i) \text { is true } \Rightarrow m_{a}^{2} m_{b}^{2} m_{c}^{2} \leq \frac{R^{2} s^{4}}{4} \Rightarrow m_{a} m_{b} m_{c} \leq \frac{\text { Rs }^{2}}{2}
\end{aligned}
$$


1528. In $\triangle A B C$ the following relationship holds:

$$
\frac{w_{a}+w_{b}+w_{c}}{s^{2}} \leq \frac{R}{2} \sum \frac{1}{m_{a} m_{b}}
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
m_{a}^{2} m_{b}^{2} m_{c}^{2}= & \frac{1}{64}\left(2 b^{2}+2 c^{2}-2 a^{2}\right)\left(2 c^{2}+2 a^{2}-2 b^{2}\right)\left(2 a^{2}+2 b^{2}-2 c^{2}\right) \stackrel{(1)}{\cong} \frac{1}{64}\left\{-4 \sum a^{6}\right. \\
& \left.+6\left(\sum \mathbf{a}^{4} b^{2}+\sum a^{2} b^{4}\right)+3 a^{2} b^{2} c^{2}\right\}
\end{aligned}
$$

$$
\text { Now, } \sum \mathbf{a}^{6}=\left(\sum \mathbf{a}^{2}\right)^{3}-\mathbf{3}\left(\mathbf{a}^{2}+\mathbf{b}^{2}\right)\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)\left(\mathbf{c}^{2}+\mathbf{a}^{2}\right)
$$

$$
=\left(\sum \mathbf{a}^{2}\right)^{3}-\mathbf{3}\left(2 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}+\sum \mathbf{a}^{2} \mathbf{b}^{2}\left(\sum \mathbf{a}^{2}-\mathbf{c}^{2}\right)\right)
$$

$=\left(\sum \mathbf{a}^{2}\right)^{3}+3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}-\mathbf{3}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2} \therefore \sum \mathbf{a}^{6} \stackrel{\cong}{=}\left(\sum \mathbf{a}^{2}\right)^{3}+3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}-\mathbf{3}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}$
Again, $\sum \mathbf{a}^{4} \mathbf{b}^{2}+\sum \mathbf{a}^{2} \mathbf{b}^{4}=\sum \mathbf{a}^{2} \mathbf{b}^{2}\left(\sum \mathbf{a}^{2}-\mathbf{c}^{2}\right) \xlongequal{\cong}\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}-\mathbf{3 a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}$
$\therefore(\mathbf{1}),(2),(3) \Rightarrow \mathbf{m}_{\mathbf{a}}^{2} \mathbf{m}_{\mathbf{b}}^{2} \mathbf{m}_{\mathbf{c}}^{2}$

$$
\begin{aligned}
&= \frac{1}{64}\left\{-4\left(\sum \mathbf{a}^{2}\right)^{3}-12 \mathbf{a}^{2} b^{2} \mathbf{c}^{2}+12\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}+6\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}\right. \\
&\left.-18 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}+3 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}\right\} \\
&=\frac{1}{64}\left\{-4\left(\sum \mathbf{a}^{2}\right)^{3}+18\left(\sum \mathbf{a}^{2} \mathbf{b}^{2}\right) \sum \mathbf{a}^{2}-27 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}\right\} \\
&= \frac{1}{64}\left\{-4\left(\sum \mathbf{a}^{2}\right)^{3}+18\left(\left(\sum \mathbf{a b}\right)^{2}-2 \mathbf{a b c}(2 s)\right)\left(\sum \mathbf{a}^{2}\right)-27 \mathbf{a}^{2} \mathbf{b}^{2} \mathbf{c}^{2}\right\}
\end{aligned}
$$

$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \text { www.ssmrmh.ro } \\
& \text { Now, } 1+\frac{1}{4} \sum\left(\frac{\mathrm{a}}{\mathrm{w}_{\mathrm{a}}}\right)^{2} \geq 1+\frac{1}{4} \sum \frac{\mathrm{a}^{2}}{\mathrm{~s}(\mathrm{~s}-\mathrm{a})}=1+\frac{1}{4 \mathrm{~s}} \sum \frac{(\mathrm{a}-\mathrm{s}+\mathrm{s})^{2}}{\mathrm{~s}-\mathrm{a}} \\
& =1+\frac{1}{4 s} \sum\left\{\frac{(s-a)^{2}-2 s(s-a)+s^{2}}{s-a}\right\} \\
& =1+\frac{1}{4 s}\left\{\sum(s-a)-6 s+\frac{s^{2}\left(4 R r+r^{2}\right)}{s r^{2}}\right\}=1+\frac{1}{4 s}\left(-5 s+\frac{s(4 R+r)}{r}\right) \\
& =1+\frac{4(R-r)}{4 r}=\frac{R^{\text {by }}(4)}{r} \stackrel{\text { m }}{s} 2 \prod \frac{m_{a}}{r_{a}} \text { (Proved) }
\end{aligned}
$$



> ROMANIAN MATHEMATICAL MAGAZINE $\begin{aligned} & \text { www.ssmrmh.ro } \\ & \frac{\mathbf{1}}{\mathbf{6 4}}\left\{-\mathbf{3 2}\left(\mathbf{s}^{2}-\mathbf{4 R r}-\mathbf{r}^{2}\right)^{3}+\mathbf{3 6}\left(\mathbf{s}^{2}-\mathbf{4 R r}-\mathbf{r}^{2}\right)\left(\mathbf{s}^{2}+\mathbf{4 R r}+\mathbf{r}^{2}\right)^{2}\right. \\ & \\ & \left.\quad-\mathbf{5 7 6} \mathbf{R r s}^{2}\left(\mathbf{s}^{2}-\mathbf{4 R r}-\mathbf{r}^{2}\right)-\mathbf{4 3 2} \mathbf{R}^{2} \mathbf{r}^{2} \mathbf{s}^{2}\right\}\end{aligned}$
$=\frac{1}{16}\left\{s^{6}-s^{4}\left(12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3}\right\} \leq \frac{R^{2} s^{4}}{4}$
$\Leftrightarrow s^{6}-s^{4}\left(4 R^{2}+12 R r-33 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)-r^{3}(4 R+r)^{3} \underset{\leq}{ } 0$ Gerretsen
Now, LHS of (i) $\quad \underset{\leq}{y} \quad-s^{4}\left(8 R r-36 r^{2}\right)-s^{2}\left(60 R^{2} r^{2}+120 R r^{3}+33 r^{4}\right)$ $-r^{3}(4 R+r)^{3} \stackrel{\text { 르́n }}{\leq} 0$
$\Leftrightarrow s^{4}(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)+r^{2}(4 R+r)^{3} \underset{(i i)}{\sum_{i}} \mathbf{2 0 r s ^ { 4 }}$


$$
+\mathbf{r}^{2}(4 \mathbf{R}+\mathbf{r})^{3}
$$

and RHS of (ii) $\underbrace{\substack{\text { Geretsen } \\ \leq}}_{(\mathrm{b})} 20 \mathrm{rs}^{2}\left(4 \mathrm{R}^{2}+4 \mathrm{Rr}+3 \mathrm{r}^{2}\right)$
(a), (b) $\Rightarrow$ in order to prove (ii), it suffices to prove
$: s^{2}\left(16 R r-5 r^{2}\right)(8 R-16 r)+s^{2}\left(60 R^{2} r+120 R r^{2}+33 r^{3}\right)$ $+\mathbf{r}^{2}(4 R+r)^{3}$
$\geq 20 \mathrm{rs}^{2}\left(4 R^{2}+4 R r+3 r^{2}\right) \Leftrightarrow s^{2}\left(108 R^{2}-256 R r+53 r^{2}\right)+r(4 R+r)^{3} \geq 0$
$\Leftrightarrow s^{2}\left(108 R^{2}-256 R r+80 r^{2}\right)+r(4 R+r)^{3} \stackrel{\text { (iii) }}{\stackrel{i n}{\sum}} 27 r^{2} s^{2}$
Now, LHS of (iii) $\underbrace{\sum_{\text {Gerretsen }}}_{(c)}\left(108 R^{2}-256 R r+80 r^{2}\right)\left(16 R r-5 r^{2}\right)$
$+r(4 R+r)^{3}$ and RHS of $(\mathrm{iii}) \underbrace{\substack{\text { Geretsen }}}_{\text {(d) }} \mathbf{L r}^{2}\left(4 \mathrm{R}^{2}+4 R r+3 r^{2}\right)$
(c), (d) $\Rightarrow$ in order to prove (iii), it suffices to prove :


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$$
\begin{gathered}
\left(108 R^{2}-256 R r+80 r^{2}\right)\left(16 R r-5 r^{2}\right)+r(4 R+r)^{3} \geq 27 r^{2}\left(4 R^{2}+4 R r+3 r^{2}\right) \\
\Leftrightarrow 224 t^{3}-587 t^{2}+308 t-60 \geq 0\left(\text { where } t=\frac{R}{r}\right)
\end{gathered}
$$

$$
\Leftrightarrow(t-2)\{(t-2)(224 t+309)+648\} \geq 0 \rightarrow \text { true } \because \mathbf{t} \stackrel{\text { Euler }}{\geq} 2 \Rightarrow(\mathrm{iii}) \Rightarrow(\mathrm{ii})
$$

$$
\Rightarrow(i) \text { is true } \Rightarrow m_{a}^{2} m_{b}^{2} m_{c}^{2} \leq \frac{\mathbf{R}^{2} s^{4}}{4} \Rightarrow m_{a} m_{b} m_{c} \leq \frac{\mathbf{R s}^{2}}{2}
$$

$$
\Rightarrow \frac{\mathbf{R s}^{2}}{2} \sum \frac{1}{\mathbf{m}_{a} m_{b}} \geq m_{a} m_{b} m_{c} \sum \frac{1}{m_{a} m_{b}}=\sum m_{a} \geq \sum w_{a} \Rightarrow \frac{w_{a}+w_{b}+w_{c}}{s^{2}}
$$

$$
\leq \frac{\mathbf{R}}{2} \sum \frac{1}{\mathrm{~m}_{\mathrm{a}} \mathrm{~m}_{\mathrm{b}}}(\text { Proved })
$$

1529. In $\triangle A B C$ the following relationship holds:

$$
\left(m_{a}+m_{b}+m_{c}\right) \sqrt{\frac{2 R}{r}} \geq \frac{a^{2}}{r_{a}-r}+\frac{b^{2}}{r_{b}-r}+\frac{c^{2}}{r_{c}-r}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \sum \frac{a^{2}}{r_{a}-r}= \sum \frac{a^{2}}{r s\left(\frac{1}{s-a}-\frac{1}{s}\right)}=\sum \frac{s(s-a) a^{2}}{r s a}=\frac{\sum a(s-a)}{r} \\
&= \frac{s(2 s)-2\left(s^{2}-4 R r-r^{2}\right)}{r}=2(4 R+r) \Rightarrow R H S \stackrel{(1)}{=} 2(4 R+r) \\
& r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{scos}^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2}
\end{aligned}
$$

$$
\therefore r_{b}+r_{c} \stackrel{(i)}{\cong} 4 R \cos ^{2} \frac{A}{2}
$$

$$
\text { Now, } \begin{aligned}
(b+c)^{2} & \geq 32 \operatorname{Rrcos}{ }^{2} \frac{A^{b y}(i)}{2} \stackrel{(i)}{=} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right) \\
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a)
\end{aligned}
$$



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\Leftrightarrow(b+c)^{2}+4 \mathbf{a}^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \text { true }
$$

$$
\therefore b+c \stackrel{(\text { iii) }}{\geq} 4 \sqrt{2 \operatorname{Rr}} \cos \frac{A}{2} \text { and analogs }
$$

Now, $\sum \mathrm{m}_{\mathrm{a}} \stackrel{\text { Ioscu }}{\geq} \sum \frac{\mathrm{b}+\mathrm{c}}{2} \cos \frac{\mathrm{~A}^{\text {by (iii) and analogs }}}{2} \stackrel{\sim}{\geq} \sqrt{2 \mathrm{Rr}} \sum\left(2 \cos ^{2} \frac{\mathrm{~A}}{2}\right)$

$$
=\sqrt{2 R r} \sum(1+\cos A)=\sqrt{2 R r}\left(\frac{4 R+r}{R}\right)=\sqrt{\frac{2 r}{R}}(4 R+r)
$$

$$
\Rightarrow\left(m_{a}+m_{b}+m_{c}\right) \sqrt{\frac{2 R}{r}} \geq 2(4 R+r) \stackrel{\text { by (1) }}{\cong} \frac{a^{2}}{r_{a}-r}+\frac{b^{2}}{r_{b}-r}+\frac{c^{2}}{r_{c}-r}(\text { Proved })
$$

1530. In $\triangle A B C, M \in \operatorname{Int}(\triangle A B C), x=d(M, B C), y=d(M, C A)$,

$$
z=d(M, A B)
$$

Prove that:

$$
\frac{108 r^{4}}{a^{2}+b^{2}+c^{2}} \leq \min \left(x^{2}+y^{2}+z^{2}\right) \leq \frac{27 R^{2} r^{2}}{a^{2}+b^{2}+c^{2}}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& S_{1}=\frac{1}{2} \cdot M P \cdot A C=\frac{1}{2} \cdot y \cdot b \\
& S_{2}=\frac{1}{2} \cdot M Q \cdot A B=\frac{1}{2} \cdot z \cdot C \\
& S_{3}=\frac{1}{2} \cdot M N \cdot B C=\frac{1}{2} \cdot x \cdot a \\
& S_{\triangle A B C}=S_{1}+S_{2}+S_{3} \Leftrightarrow 2 S_{\triangle A B C}=a x+b y+c z \stackrel{B C S}{\sim} \sqrt{a^{2}+b^{2}+c^{2}} \cdot \sqrt{x^{2}+y^{2}+z^{2}} \\
& \Leftrightarrow 4 S_{\triangle A B C}^{2}=\left(a^{2}+b^{2}+c^{2}\right)\left(x^{2}+y^{2}+z^{2}\right) \\
& \Leftrightarrow \frac{4 s^{2} r^{2}}{a^{2}+b^{2}+c^{2}} \leq x^{2}+y^{2}+z^{2} \Rightarrow \min \left\{x^{2}+y^{2}+z^{2}\right\}=\frac{4 s^{2} r^{2}}{a^{2}+b^{2}+c^{2}} \\
& \text { Now, } \\
& \frac{4 s^{2} r^{2}}{a^{2}+b^{2}+c^{2}} \geq \frac{108 r^{4}}{a^{2}+b^{2}+c^{2}} \Leftrightarrow s^{2} \geq 27 r^{2} \Leftrightarrow s \geq 3 \sqrt{3} r \quad \text { (true) }
\end{aligned}
$$



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\frac{4 s^{2} r^{2}}{a^{2}+b^{2}+c^{2}} \leq \frac{27 R^{2} r^{2}}{a^{2}+b^{2}+c^{2}} \Leftrightarrow s^{2} \leq \frac{27}{4} R^{2} \Leftrightarrow s \leq \frac{3 \sqrt{3}}{2} R \quad \text { (true) }
$$

Proved.
1531. In any $\triangle A B C$ the following relationship holds:

$$
\left(\frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}\right)^{2}+\frac{2 \mu r}{R} \geq 9+\mu, \mu \leq 6
$$

Proposed by Marin Chirciu-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
h_{a}=\frac{2 s}{a} ; r_{a}=\frac{s}{s-a} \text { and analogs. } \\
\Rightarrow \frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}=2\left(\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}\right) \\
=\frac{2(b c(s-a)+a c(s-b)+a b(s-c))}{a b c}=\frac{2(s(a b+b c+c a)-3 a b c)}{a b c} \\
=\frac{2\left(s\left(s^{2}+4 R r+r^{2}\right)-12 R r s\right)}{4 R r s}=\frac{s^{2}-8 R r+r^{2}}{2 R r} \Rightarrow \\
L H S=\left(\frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}\right)^{2}+\frac{2 \mu r}{R}=\left(\frac{s^{2}-8 R r+r^{2}}{2 R r}\right)^{2}+\frac{2 \mu r}{R} \stackrel{y}{4}_{(*)}^{2} 9+\mu \\
(*) \Leftrightarrow \frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}} \geq 9+\mu-\frac{2 \mu r}{R} \Leftrightarrow \\
\left(s^{2}-8 R r+r^{2}\right)^{2} \geq 4 R r^{2}((9+\mu) R-2 \mu r) \\
\therefore s^{2} \geq 16 R r-5 r^{2} \Rightarrow s^{2}-8 R r+r^{2} \geq 8 R r-4 r^{2} \Rightarrow \\
\left(s^{2}-8 R r+r^{2}\right)^{2} \geq 16 r^{2}(2 R-r)^{2} \geq 4 R r^{2}((9+\mu) R-2 \mu r) \\
(* *) \Leftrightarrow 4(2 R-r)^{2} \geq R((9+\mu) R-2 \mu r) \Leftrightarrow 4\left(4 R^{2}-4 R r+r^{2}\right) \geq(9+\mu) R^{2}-2 \mu R r \\
\Leftrightarrow(7-\mu) R^{2}-2(8-\mu) R r+4 r^{2} \geq 0 \Leftrightarrow(R-2 r)((7-\mu) R-2 r) \geq 0 \\
W h i c h \text { is true because: }
\end{gathered}
$$

$R \geq 2 r$ and $\mu \leq 6<7 \Rightarrow(7-\mu) R-2 r \geq(7-\mu) 2 r-2 r=2(6-\mu) r \geq 0$. Proved.


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1532. In $\triangle A B C$ the following relationship holds:

$$
\frac{r_{a}}{r_{b}}+\frac{r_{b}}{r_{c}}+\frac{r_{c}}{r_{a}} \geq \frac{3}{2}+\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& a^{3}+a^{3}+b^{3} \stackrel{A-G}{\geq} 3 a^{2} b, b^{3}+b^{3}+c^{3} \stackrel{A-G}{\geq} 3 b^{2} c \text { and } \\
& c^{3}+c^{3}+a^{3} \stackrel{A-G}{\geq} 3 c^{2} \text { a and summing up, } \sum a^{3} \stackrel{(1)}{\geq} \sum a^{2} b \\
& \text { Now, } \frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}=\frac{a\left(\sum a b+c^{2}\right)+b\left(\sum a b+a^{2}\right)+c\left(\sum a b+b^{2}\right)}{(a+b)(b+c)(c+a)} \\
& =\frac{2 s\left(s^{2}+4 R r+r^{2}\right)+\sum a^{2} b}{2 s\left(s^{2}+2 R r+r^{2}\right)} \\
& \stackrel{\text { by }(1)}{\sum} \frac{2 s\left(s^{2}+4 R r+r^{2}\right)+2 s\left(s^{2}-6 R r-3 r^{2}\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)}=\frac{2 s^{2}-2 R r-2 r^{2}}{s^{2}+2 R r+r^{2}} \\
& \Rightarrow \frac{3}{2}+\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}=\frac{3}{2}+\frac{2 s^{2}-2 R r-2 r^{2}}{s^{2}+2 R r+r^{2}} \\
& =\frac{3\left(s^{2}+2 R r+r^{2}\right)+2\left(2 s^{2}-2 R r-2 r^{2}\right)}{2\left(s^{2}+2 R r+r^{2}\right)}=\frac{7 s^{2}+2 R r-r^{2}}{2\left(s^{2}+2 R r+r^{2}\right)} \\
& S \stackrel{(2)}{\underset{\leq}{s}} \frac{7 s^{2}+2 R r-r^{2}}{2\left(s^{2}+2 R r+r^{2}\right)} \\
& \text { Now, LHS }=\sum \frac{r_{a}^{2}}{r_{a} r_{b}} \underbrace{\sum_{(3)}^{\text {Bergstrom }}}_{(3)} \frac{(4 R+r)^{2}}{s^{2}} \\
& \text { (2), (3) } \Rightarrow \text { it suffices to prove : } \frac{(4 R+r)^{2}}{s^{2}} \geq \frac{7 s^{2}+2 R r-r^{2}}{2\left(s^{2}+2 R r+r^{2}\right)} \\
& \Leftrightarrow 2\left(s^{2}+2 R r+r^{2}\right)(4 R+r)^{2} \geq s^{2}\left(7 s^{2}+2 R r-r^{2}\right) \\
& \Leftrightarrow 7 s^{4}-s^{2}\left(32 R^{2}+14 R r+3 r^{2}\right)-\left(2 R r+r^{2}\right)(4 R+r)^{2} \stackrel{(\imath)}{\leq} 0 \\
& \text { Now, Rouche } \Rightarrow s^{2}-(m-n) \geq 0 \text { and } s^{2}-(m+n) \leq 0 \text {, where } m \\
& =2 R^{2}+10 R r-r^{2} \text { and } n=2(R-2 r) \sqrt{R^{2}-2 R r}
\end{aligned}
$$



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$$
\begin{aligned}
& \begin{aligned}
& \therefore\left(s^{2}-(m+n)\right)\left(s^{2}-(m-n)\right) \leq 0 \Rightarrow s^{4}-s^{2}(2 m)+m^{2}-n^{2} \leq 0 \\
& \Rightarrow 7 s^{4}-s^{2}\left(28 R^{2}+140 R r-14 r^{2}\right)+7 r(4 R+r)^{3} \stackrel{(4)}{\leq} 0 \\
&(4) \Rightarrow \text { in order to prove }(i), \text { it suffices to prove : } \\
& 7 s^{4}-s^{2}\left(32 R^{2}+14 R r+3 r^{2}\right)-\left(2 R r+r^{2}\right)(4 R+r)^{2} \\
& \leq 7 s^{4}-s^{2}\left(28 R^{2}+140 R r-14 r^{2}\right)+7 r(4 R+r)^{3} \\
& \Leftrightarrow s^{2}\left(4 R^{2}-126 R r+17 r^{2}\right)+7 r(4 R+r)^{3}+\left(2 R r+r^{2}\right)(4 R+r)^{2} \geq 0
\end{aligned}
\end{aligned}
$$

$$
\begin{aligned}
& \Leftrightarrow s^{2}\left(4 R^{2}-16 R r+16 r^{2}\right)+7 r(4 R+r)^{3}+\left(2 R r+r^{2}\right)(4 R+r)^{2} \stackrel{(i i)}{\geq}\left(110 R r-r^{2}\right) s^{2} \\
& \text { Now, LHS of }(i i) \underbrace{\sum_{\sim}^{\text {Gerretsen }}}_{(a)} \sum^{s}\left(16 R r-5 r^{2}\right)(R-2 r)^{2}+7 r(4 R+r)^{3} \\
&+\left(2 R r+r^{2}\right)(4 R+r)^{2}
\end{aligned}
$$

and RHS of $(i i) \underbrace{\substack{\text { Gerretsen } \\ \sim_{s}^{c}}}_{(b)}\left(4 R^{2}+4 R r+3 r^{2}\right)\left(110 R r-r^{2}\right) \therefore(a),(b)$ $\Rightarrow$ in order to prove (ii), it suffices to prove :

$$
\begin{aligned}
& 4\left(16 R r-5 r^{2}\right)(R-2 r)^{2}+7 r(4 R+r)^{3}+\left(2 R r+r^{2}\right)(4 R+r)^{2} \\
& \quad \geq\left(4 R^{2}+4 R r+3 r^{2}\right)\left(110 R r-r^{2}\right)
\end{aligned}
$$

$$
\Leftrightarrow 68 t^{3}-156 t^{2}+57 t-34 \geq 0\left(\text { where } t=\frac{R}{r}\right) \Leftrightarrow(t-2)\left\{10 t(t-2)+58 t^{2}+17\right\}
$$

$$
\geq 0 \rightarrow \text { true } \because t \stackrel{\text { Euler }}{\geq} 2 \Rightarrow(i i) \Rightarrow(i) \text { is true }
$$

$$
\therefore \frac{r_{a}}{r_{b}}+\frac{r_{b}}{r_{c}}+\frac{r_{c}}{r_{a}} \geq \frac{3}{2}+\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+a}(\text { Proved })
$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& r_{a}=\frac{S}{s-a} ; r_{b}=\frac{S}{s-b} ; r_{c}=\frac{S}{s-c} \\
\Rightarrow & \frac{r_{a}}{r_{b}}+\frac{r_{b}}{r_{c}}+\frac{r_{c}}{r_{a}}=\frac{s-b}{s-a}+\frac{s-c}{s-b}+\frac{s-a}{s-c}
\end{aligned}
$$

Let: $x=s-a ; y=s-b ; z=s-c \Rightarrow x+y+z=p \Rightarrow a=y+z ; b=x+z ; c=x+y$

$$
\text { Suppose: } \min \{x ; z\} \leq y \leq \max \{x ; z\}
$$



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We must show that

$$
\begin{array}{r}
\frac{y}{x}+\frac{z}{y}+\frac{x}{z} \geq \frac{3}{2}+\frac{y+z}{x+y+2 z}+\frac{x+z}{y+z+2 x}+\frac{x+y}{x+z+2 y} \\
\frac{y}{x}+\frac{z}{y}+\frac{x}{z}=\frac{y^{2}}{x y}+\frac{z^{2}}{y z}+\frac{x^{2}}{z} \quad \geq \cdot \frac{B . S .}{}(x+y+z)^{2} \\
x y+y z+z x \\
\\
\Rightarrow \frac{y+z}{x+y+2 z}+\frac{x+z}{y+z+2 x}+\frac{x+y}{x+z+2 y} \leq \frac{3}{2} \quad(*) \\
\Leftrightarrow x^{2} y+y^{2} z+z^{2} x-x y^{2}-y z^{2}-z x^{2} \geq 0 \Leftrightarrow(x-y)(y-z)(z-x) \geq 0
\end{array}
$$

Which is true because we suppose: $\min \{x ; z\} \leq y \leq \max \{x ; z\} \Rightarrow$ (*)is true.

$$
\begin{gathered}
\frac{3}{2}+\frac{y+z}{x+y+2 z}+\frac{x+z}{y+z+2 x}+\frac{x+y}{x+z+2 y} \leq 3 \leq \frac{y}{x}+\frac{z}{y}+\frac{x}{z} \text { Proved. } \\
\text { Equality } \Leftrightarrow x=y=x \Leftrightarrow a=b=c
\end{gathered}
$$

1533. In $\triangle A B C$ the following relationship holds:

$$
\frac{s_{a}}{s_{a}+m_{a}}+\frac{s_{b}}{s_{b}+m_{b}}+\frac{s_{c}}{s_{c}+m_{c}} \leq \frac{1}{2}+\frac{w_{a} w_{b} w_{c}}{r_{a} r_{b} r_{c}}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gather*}
\frac{w_{a} w_{b} w_{c}}{r_{a} r_{b} r_{c}}=\frac{8 a b c}{(a+b)(b+c)(c+a)} \\
s_{a}=\frac{2 b c}{b^{2}+c^{2}} \cdot m_{a} \text { and analogs. } \\
\frac{s_{a}}{s_{a}+m_{a}}+\frac{s_{b}}{s_{b}+m_{b}}+\frac{s_{c}}{s_{c}+m_{c}} \leq \frac{1}{2}+\frac{w_{a} w_{b} w_{c}}{r_{a} r_{b} r_{c}} \\
\Leftrightarrow \sum_{c y c} \frac{\frac{2 b c}{b^{2}+c^{2}} \cdot m_{a}}{\frac{2 b c}{b^{2}+c^{2}} \cdot m_{a}+m_{a}} \leq \frac{1}{2}+\frac{8 a b c}{(a+b)(b+c)(c+a)} \\
\Leftrightarrow \sum_{c y c} \frac{2 b c}{(b+c)^{2}} \leq \frac{1}{2}+\frac{8 a b c}{(a+b)(b+c)(c+a)} \tag{*}
\end{gather*}
$$

We prove (*) true for all $x, y, z>0$

$$
\frac{2 x y}{(x+y)^{2}}+\frac{2 x z}{(x+z)^{2}}+\frac{2 y z}{(y+z)^{2}} \leq \frac{1}{2}+\frac{8 x y z}{(x+y)(y+z)(z+x)}
$$



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$$
\begin{gathered}
\Leftrightarrow \frac{-4 x y}{(x+y)^{2}}+\frac{-4 x z}{(x+z)^{2}}+\frac{-4 y z}{(y+z)^{2}} \geq \frac{\text { www.ssmrmh.ro }}{(x+y)(y+z)(z+x)}-1 \\
\Leftrightarrow 1-\frac{4 x y}{(x+y)^{2}}+1-\frac{4 x z}{(x+z)^{2}}+1-\frac{4 y z}{(y+z)^{2}}+\frac{16 x y z-2(x+y)(y+z)(z+x)}{(x+y)(y+z)(z+x)} \geq 0 \\
\Leftrightarrow \frac{(x-y)^{2}}{(x+y)^{2}}+\frac{(x-z)^{2}}{(x+z)^{2}}+\frac{(y-z)^{2}}{(y+z)^{2}}-\frac{2 x(y-z)^{2}+2 y(x-z)^{2}+2 z(x-y)^{2}}{(x+y)(y+z)(z+x)} \geq 0 \\
\Leftrightarrow \sum_{c y c} \frac{(y-z)^{2}}{(y+z)^{2}}\left(\frac{1}{y+z}-\frac{2 x}{(x+y)(x+z)}\right) \geq 0 \\
\Leftrightarrow \sum_{c y c}\left(\frac{(y-z)^{2}}{(y+z)^{2}} \cdot \frac{(x-y)(x-z)}{(x+y)(y+z)(z+x)}\right) \geq 0 \\
\Leftrightarrow-\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)} \cdot \frac{(x-y)(y-z)(x-z)}{(x+y)(y+z)(z+x)} \geq 0 \\
\Leftrightarrow\left(\frac{(x-y)(y-z)(z-x)}{(x+y)(y+z)(z+x)}\right)^{2} \geq 0
\end{gathered}
$$

## Which is true.Equality $\Leftrightarrow x=y=z$

Let: $x=a ; y=b ; z=c \Rightarrow(*)$ is true.Proved.
Solution 2 by Soumva Chakraborty-Kolkata-India

$$
\begin{gathered}
\because s_{a}=\frac{2 b c}{b^{2}+c^{2}} m_{a} \text { and analogs } \therefore \text { LHS }=\sum \frac{\frac{2 b c}{b^{2}+c^{2}}}{\frac{2 b c}{b^{2}+c^{2}}+1}=\sum \frac{2 b c}{(b+c)^{2}} \\
=\frac{\sum\left\{2 b c(c+a)^{2}(a+b)^{2}\right\}}{\prod(b+c)^{2}}=\frac{\sum\left\{2 b c\left(\sum a b+a^{2}\right)^{2}\right\}}{\prod(b+c)^{2}} \\
=\frac{\sum\left[2 b c\left\{\left(\sum a b\right)^{2}+2 a^{2} \sum a b+a^{4}\right\}\right]}{\prod(b+c)^{2}}=\frac{2\left(\sum a b\right)^{3}+4 a b c(2 s) \sum a b+2 a b c \sum a^{3}}{\prod(b+c)^{2}} \\
\begin{array}{c}
\stackrel{(1)}{\cong} \frac{2\left(s^{2}+4 R r+r^{2}\right)^{3}+32 R r s^{2}\left(s^{2}+4 R r+r^{2}\right)+16 R r s^{2}\left(s^{2}-6 R r-3 r^{2}\right)}{\prod(b+c)^{2}} \\
R H S=\frac{1}{2}+\left\{\prod\left(\frac{2 b c c o s \frac{A}{2}}{b+c}\right)\right\}\left(\frac{1}{r s^{2}}\right)=\frac{1}{2}+\frac{128 R^{2} r^{2} s^{2}\left(\frac{s}{4 R}\right)}{r s^{2} \prod(b+c)}=\frac{1}{2}+\frac{32 R r s}{\prod(b+c)} \\
=\frac{s\left(s^{2}+2 R r+r^{2}\right)+32 R r s}{\prod(b+c)} \stackrel{(2)}{m} \frac{s\left(s^{2}+34 R r+r^{2}\right)}{\prod(b+c)}
\end{array}
\end{gathered}
$$



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$$
\begin{align*}
& (1),(2) \Rightarrow(a) \\
& \Leftrightarrow \frac{2\left(s^{2}+4 R r+r^{2}\right)^{3}+32 R r s^{2}\left(s^{2}+4 R r+r^{2}\right)+16 R r s^{2}\left(s^{2}-6 R r-3 r^{2}\right)}{\prod(b+c)^{2}} \\
& \leq \frac{s\left(s^{2}+34 R r+r^{2}\right)}{\prod(b+c)} \\
& \begin{array}{r}
\qquad \frac{2\left(s^{2}+4 R r+r^{2}\right)^{3}+32 R r s^{2}\left(s^{2}+4 R r+r^{2}\right)+16 R r s^{2}\left(s^{2}-6 R r-3 r^{2}\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)} \\
\leq s\left(s^{2}+34 R r+r^{2}\right) \\
\Leftrightarrow\left(s^{2}+4 R r+r^{2}\right)^{3}+16 R r s^{2}\left(s^{2}+4 R r+r^{2}\right)+8 R r s^{2}\left(s^{2}-6 R r-3 r^{2}\right) \\
\leq s^{2}\left(s^{2}+34 R r+r^{2}\right)\left(s^{2}+2 R r+r^{2}\right)
\end{array}
\end{align*}
$$

$\Leftrightarrow s^{4}-s^{2}\left(4 R^{2}+20 R r-2 r^{2}\right)+r(4 R+r)^{3} \stackrel{\text { m }}{\leq} 0$
Now, Rouche $\Rightarrow s^{2}-(m-n) \geq 0$ and $s^{2}-(m+n) \leq 0$, where $m$

$$
=2 R^{2}+10 R r-r^{2} \text { and } n=2(R-2 r) \sqrt{R^{2}-2 R r}
$$

$$
\therefore\left(s^{2}-(m+n)\right)\left(s^{2}-(m-n)\right) \leq \mathbf{0} \Rightarrow s^{4}-s^{2}(2 m)+m^{2}-n^{2} \leq \mathbf{0}
$$

$$
\Rightarrow s^{4}-s^{2}\left(4 R^{2}+20 R r-2 r^{2}\right)+r(4 R+r)^{3} \leq 0
$$

$\Rightarrow(b) \Rightarrow(a)$ is true (Proved)
1534. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} \frac{\frac{s-b}{s-c}+\frac{s-c}{s-a}+\left(\frac{s-a}{s-b}\right)^{3}}{\left(\frac{s-b}{s-c}\right)^{9}+\left(\frac{s-c}{s-a}\right)^{9}+\frac{s-b}{s-a}} \leq \sum_{c y c}\left(\frac{s-b}{s-a}\right)^{4}
$$

Proposed by Daniel Sitaru-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{equation*}
\sum_{c y c} \frac{\frac{s-b}{s-c}+\frac{s-c}{s-a}+\left(\frac{s-a}{s-b}\right)^{3}}{\left(\frac{s-b}{s-c}\right)^{9}+\left(\frac{s-c}{s-a}\right)^{9}+\frac{s-b}{s-a}} \leq \sum_{c y c}\left(\frac{s-b}{s-a}\right)^{4} \tag{*}
\end{equation*}
$$

Let: $x=\frac{s-a}{s-b} ; y=\frac{s-b}{s-c} ; z=\frac{s-c}{s-a} \Rightarrow x y z=1$ then


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$$
(*) \Leftrightarrow \sum_{c y c} \frac{x^{3}+y+z}{y^{9}+z^{9}+\frac{1}{x}} \leq \sum_{c y c} x^{4}
$$

We have:

$$
\begin{gathered}
y^{9}+z^{9} \geq(y z)^{4}(y+z) \stackrel{\substack{x y z=1 \\
=} \frac{y+z}{x^{4}} \Rightarrow}{y^{9}+z^{9}+\frac{1}{x} \geq \frac{y+z}{x^{4}}+\frac{1}{x}=\frac{y+z+x^{3}}{x^{4}} \Rightarrow} \\
\sum_{c y c} \frac{x^{3}+y+z}{y^{9}+z^{9}+\frac{1}{x}} \leq \sum_{c y c} \frac{y+z+x^{3}}{\frac{y+z+x^{3}}{x^{4}}}=\sum_{c y c} x^{4} \Rightarrow(1) \text { is true. }
\end{gathered}
$$

Lastly, we will to prove: $x^{9}+y^{9} \geq(x y)^{4}(x+y), \forall x, y>0$

$$
\begin{gathered}
\Leftrightarrow x^{9}+y^{9} \geq x^{5} y^{4}+x^{4} y^{5}, \forall x, y>0 \\
\Leftrightarrow x^{5}\left(x^{4}-y^{4}\right)+y^{5}\left(x^{4}-y^{4}\right) \geq 0, \forall x, y>0 \\
\Leftrightarrow\left(x^{4}-y^{4}\right)\left(x^{5}-y^{5}\right) \geq 0, \forall x, y>0 \\
\Leftrightarrow(x-y)^{2}(x+y)\left(x^{2}+y^{2}\right)\left(x^{4}+x^{3} y+x^{2} y^{2}+x y^{3}+y^{4}\right) \geq 0
\end{gathered}
$$

Which is true.

$$
\text { Equality } \Leftrightarrow x=y=z=1 \Leftrightarrow s-a=s-b=s-c \Leftrightarrow a=b=c
$$

1535. In $\triangle A B C$ the following relationship holds:

$$
\frac{1}{27}\left(\frac{7 R-2 r}{2 R}\right)^{4} \leq \frac{\mathbf{m}_{\mathrm{a}}^{4}}{\mathrm{~h}_{\mathrm{a}}^{4}}+\frac{\mathrm{m}_{\mathrm{b}}^{4}}{\mathrm{~h}_{\mathrm{b}}^{4}}+\frac{\mathrm{m}_{\mathrm{c}}^{4}}{\mathrm{~h}_{\mathrm{c}}^{4}} \leq \frac{3}{64}\left(\frac{27 \mathrm{R}^{2}}{4 \mathrm{r}^{2}}-19\right)^{2}
$$

Proposed by Marin Chirciu-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Panaitopol } \Rightarrow \frac{\mathbf{m}_{a}^{2}}{\mathbf{h}_{a}^{2}} \leq \frac{\mathbf{R}^{2}}{4 \mathbf{r}^{2}} \Rightarrow \frac{\mathbf{m}_{a}^{4}}{\mathbf{h}_{a}^{4}} \leq\left(\frac{\mathbf{R}^{2}}{4 r^{2}}\right) \frac{\mathbf{m}_{a}^{2}}{\mathbf{h}_{a}^{2}} \text { and analogs }
\end{aligned}
$$

$$
\begin{aligned}
& =\left(\frac{\mathbf{R}^{2}}{64 \mathbf{r}^{2} \mathbf{s}^{2}}\right)\left(2 \sum \mathbf{a}^{2} \mathbf{b}^{2}+2 \sum \mathrm{a}^{2} \mathbf{b}^{2}-\sum \mathrm{a}^{4}\right) \\
& =\left(\frac{\mathbf{R}^{2}}{64 \mathbf{r}^{2} \mathbf{s}^{2}}\right)\left[\mathbf{2}\left\{\left(\mathbf{s}^{2}+4 \mathbf{R r}+\mathbf{r}^{2}\right)^{2}-16 \operatorname{Rrs}^{2}\right\}+16 \mathbf{r}^{2} \mathbf{s}^{2}\right] \stackrel{?}{\dot{\sim}} \frac{\mathbf{3}}{\mathbf{6 4}}\left(\frac{\mathbf{2 7} \mathbf{R}^{2}}{4 \mathbf{r}^{2}}-\mathbf{1 9}\right)^{2}
\end{aligned}
$$



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$$
\Leftrightarrow 32 \mathbf{R}^{2}\left\{\left(\mathbf{s}^{2}+4 \mathbf{R r}+\mathbf{r}^{2}\right)^{2}-16 \mathbf{R r s}^{2}\right\}+256 \mathbf{R}^{2} \mathbf{r}^{2} \mathbf{s}^{2} \stackrel{\sim}{\stackrel{2}{\leftrightarrows}} 3 \mathbf{s}^{2}\left(27 \mathbf{R}^{2}-76 \mathbf{r}^{2}\right)^{2}
$$

Now, Rouche $\Rightarrow \mathbf{s}^{2}-(\mathbf{m}-\mathbf{n}) \geq 0$ and $\mathbf{s}^{2}-(\mathbf{m}+\mathbf{n}) \leq 0$, where $m$

$$
=2 R^{2}+10 R r-r^{2} \text { and } n=2(R-2 r) \sqrt{R^{2}-2 R r}
$$

$$
\therefore\left(\mathbf{s}^{2}-(\mathbf{m}+\mathbf{n})\right)\left(\mathbf{s}^{2}-(\mathbf{m}-\mathbf{n})\right) \leq \mathbf{0} \Rightarrow \mathbf{s}^{4}-\mathbf{s}^{2}(2 \mathbf{m})+\mathbf{m}^{2}-\mathbf{n}^{2} \leq \mathbf{0}
$$

$$
\Rightarrow s^{4}-s^{2}\left(4 R^{2}+20 R r-2 r^{2}\right)+r(4 R+r)^{3} \leq 0
$$

$$
\begin{aligned}
& (R-2 r)\left(16 R r-5 r^{2}\right)\left(2059 R^{3}+3734 R^{2} r-5100 R r^{2}-10200 r^{3}\right) \\
& +32 R^{2} r(4 R+r)^{3}-r^{2}\left(512 R^{4}+256 R^{3} r+32 R^{2} r^{2}\right)
\end{aligned}
$$


and RHS of (ii) $\qquad$ $3072 \mathrm{r}^{4}\left(4 \mathrm{R}^{2}+4 \mathrm{Rr}+3 \mathrm{r}^{2}\right) \therefore(\mathrm{a}),(\mathrm{b})$ (b)
$\Rightarrow$ in order to prove (ii), it suffices to prove :

$$
\begin{aligned}
& \Rightarrow 32 R^{2} s^{4}-s^{2}\left(128 R^{4}+640 R^{3} r-64 R^{2} r^{2}\right)+32 R^{2} r(4 R+r)^{3} \stackrel{(m)}{\leftrightarrows} 0 \\
& (m) \Rightarrow \text { in order to prove (i), it suffices to prove : } \\
& 32 R^{2} s^{4}-s^{2}\left(2187 R^{4}+256 R^{3} r-12632 R^{2} \mathbf{r}^{2}+17328 r^{4}\right) \\
& +r^{2}\left(512 R^{4}+256 R^{3} r+32 R^{2} r^{2}\right) \\
& \leq 32 R^{2} s^{4}-s^{2}\left(128 R^{4}+640 R^{3} r-64 R^{2} r^{2}\right)+32 R^{2} r(4 R+r)^{3} \\
& \Leftrightarrow s^{2}\left(2059 R^{4}-384 R^{3} r-12568 R^{2} r^{2}+17328 r^{4}\right)+32 R^{2} r(4 R+r)^{3} \\
& -r^{2}\left(512 R^{4}+256 R^{3} r+32 R^{2} r^{2}\right) \geq 0 \\
& \Leftrightarrow \mathbf{s}^{2}(\mathbf{R}-2 \mathbf{r})\left(2059 \mathbf{R}^{3}+3734 \mathbf{R}^{2} \mathbf{r}-5100 \mathrm{Rr}^{2}-10200 r^{3}\right)+32 \mathbf{R}^{2} \mathbf{r}(4 \mathrm{R}+\mathbf{r})^{3} \\
& \text { (ii) } \\
& -\mathbf{r}^{2}\left(512 R^{4}+256 R^{3} r+32 R^{2} r^{2}\right) \geq 3072 r^{4} \mathbf{s}^{2} \\
& \because 2059 R^{3}+3734 R^{2} \mathbf{r}-5100 \text { Rr }^{2}-10200 r^{3} \\
& =(\mathbf{R}-\mathbf{2 r})\left(\mathbf{2 0 5 9} \mathrm{R}^{2}+\mathbf{7 8 5} 2 \mathrm{Rr}+10604 \mathrm{r}^{2}\right)+11008 \mathrm{r}^{3} \stackrel{\text { Euler }}{\geq} 11008 \mathrm{r}^{3}>0 \\
& \therefore \text { LHS of (ii) }
\end{aligned}
$$



$$
\rightarrow \text { true } \because \mathrm{t} \stackrel{\text { Euler }}{\geqq} 2 \Rightarrow(\mathrm{ii}) \Rightarrow(\mathrm{i}) \text { is true }
$$

$$
\Rightarrow \frac{\mathbf{m}_{\mathrm{a}}^{4}}{\mathrm{~h}_{\mathrm{a}}^{4}}+\frac{\mathbf{m}_{\mathrm{b}}^{4}}{\mathrm{~h}_{\mathrm{b}}^{4}}+\frac{\mathbf{m}_{\mathrm{c}}^{4}}{\mathbf{h}_{\mathrm{c}}^{4}} \stackrel{(1)}{\leftrightarrows} \frac{3}{64}\left(\frac{27 \mathrm{R}^{2}}{4 \mathbf{r}^{2}}-19\right)^{2}
$$

$$
\text { Also, } \frac{\mathbf{m}_{\mathrm{a}}^{4}}{\mathbf{h}_{\mathrm{a}}^{4}}+\frac{\mathbf{m}_{\mathrm{b}}^{4}}{\mathbf{h}_{\mathrm{b}}^{4}}+\frac{\mathbf{m}_{\mathrm{c}}^{4}}{\mathbf{h}_{\mathrm{c}}^{4}} \geq \frac{1}{3}\left(\sum \frac{\mathbf{m}_{\mathrm{a}}^{2}}{\mathbf{h}_{\mathrm{a}}^{2}}\right)^{2} \stackrel{?}{\geq} \frac{1}{27}\left(\frac{7 \mathrm{R}-2 \mathrm{r}}{2 \mathrm{R}}\right)^{4} \Leftrightarrow \sum \frac{\mathbf{m}_{\mathrm{a}}^{2}}{\mathbf{h}_{\mathrm{a}}^{2}} \stackrel{?}{2} \frac{1}{3}\left(\frac{7 \mathrm{R}-2 \mathbf{r}}{2 \mathrm{R}}\right)^{2}
$$

$$
\Leftrightarrow \frac{\sum\left\{\mathbf{a}^{2}\left(2 \mathbf{b}^{2}+2 \mathbf{c}^{2}-\mathbf{a}^{2}\right)\right\}}{16 \mathbf{r}^{2} \mathbf{s}^{2}} \stackrel{?}{2} \frac{1}{3}\left(\frac{7 \mathbf{R}-2 r}{2 R}\right)^{2}
$$

$$
\Leftrightarrow \frac{\left(2 \sum a^{2} \mathbf{b}^{2}+2 \sum a^{2} \mathbf{b}^{2}-\sum \mathbf{a}^{4}\right)}{16 \mathbf{r}^{2} \mathbf{s}^{2}} \underset{\sim}{2} \frac{1}{3}\left(\frac{7 R-2 r}{2 R}\right)^{2}
$$

$$
\Leftrightarrow 3 R^{2}\left[2\left\{\left(s^{2}+4 R r+r^{2}\right)^{2}-16 R r s^{2}\right\}+16 r^{2} s^{2}\right] \stackrel{\text { ? }}{\stackrel{\sim}{y}} 4 r^{2} s^{2}(7 R-2 r)^{2}
$$

$$
\Leftrightarrow 3 R^{2} s^{4}-s^{2}\left(24 R^{3} r+68 R^{2} r^{2}-56 R r^{3}+8 r^{4}\right)+3 R^{2} r^{2}(4 R+r)^{2} \sum_{(i i i)}^{\text {? }} 0
$$

Gerretsen
Now, LHS of (iii) $\underset{\sim}{\text { C }}\left\{3 \mathrm{R}^{2}\left(16 \mathrm{Rr}-5 \mathrm{r}^{2}\right)-\left(24 \mathrm{R}^{3} \mathrm{r}+68 \mathrm{R}^{2} \mathrm{r}^{2}-56 \mathrm{Rr}^{3}+8 \mathrm{r}^{4}\right)\right\} \mathrm{s}^{2}$

$$
\begin{aligned}
& +3 R^{2} r^{2}(4 R+r)^{2} \xrightarrow{\stackrel{?}{2}} 0 \\
& \Leftrightarrow\left(24 R^{3}-83 R^{2} r+56 R r^{2}-8 r^{3}\right) s^{2}+3 R^{2} r(4 R+r)^{2} \stackrel{\text { ? }}{\geq} 0 \\
& \Leftrightarrow(R-2 r)\left(24 R^{2}-35 R r-14 r^{2}\right) s^{2}+3 R^{2} r(4 R+r)^{2} \sum_{\text {(iv) }}^{?} 36 r^{3} s^{2} \\
& \because 24 R^{2}-35 R r-14 r^{2}=(R-2 r)(24 R+13 r)+12 r^{2} \stackrel{\text { Euler }}{\geq} 12 r^{2}>0
\end{aligned}
$$

$$
\begin{aligned}
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& \text { www.ssmrmh.ro } \\
& (R-2 r)\left(16 R r-5 r^{2}\right)\left(2059 R^{3}+3734 R^{2} r-5100 R r^{2}-10200 r^{3}\right) \\
& +32 R^{2} r(4 R+r)^{3}-r^{2}\left(512 R^{4}+256 R^{3} r+32 R^{2} r^{2}\right) \\
& \geq 3072 \mathrm{r}^{4}\left(4 \mathrm{R}^{2}+4 \mathrm{Rr}+3 \mathrm{r}^{2}\right) \\
& \Leftrightarrow 34992 t^{5}-15415 t^{4}-199040 t^{3}+50552 t^{2}+314112 t-111216 \\
& \geq 0\left(\text { where } t=\frac{R}{r}\right) \\
& \Leftrightarrow(t-2)\left\{(t-2)\left(34992 t^{3}+124553 t^{2}+159204 t+189156\right)+433920\right\} \geq 0
\end{aligned}
$$



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$\therefore$ LHS of (iv) $\qquad$ $(R-2 r)\left(24 R^{2}-35 R r-14 r^{2}\right)(16 R r$

$$
\left.-5 r^{2}\right) \text { and RHS of }(\mathrm{iv}) \underbrace{\stackrel{\text { Gerretsen }}{\mathrm{m}^{\prime}}}_{(\mathrm{d})} 36 \mathrm{r}^{3}\left(4 \mathrm{R}^{2}+4 R r+3 \mathrm{r}^{2}\right)
$$

$\therefore(\mathrm{c}),(\mathrm{d}) \Rightarrow$ in order to prove (iv), it suffices to prove
: $(R-2 r)\left(24 R^{2}-35 R r-14 r^{2}\right)\left(16 R r-5 r^{2}\right)$
$\geq 36 r^{3}\left(4 R^{2}+4 R r+3 r^{2}\right)$
$\Leftrightarrow 216 t^{4}-712 t^{3}+585 t^{2}+12 t-124 \geq 0\left(\right.$ where $\left.t=\frac{R}{r}\right)$
$\Leftrightarrow(\mathbf{t}-\mathbf{2})\left\{(\mathbf{t}-\mathbf{2})\left(216 \mathbf{t}^{2}+\mathbf{1 5 2 t}+\mathbf{3 2 9}\right)+\mathbf{7 2 0}\right\} \geq 0$
$\rightarrow$ true $\because \mathbf{t} \stackrel{\text { Euler }}{\geq} 2 \Rightarrow(\mathrm{iv}) \Rightarrow(\mathrm{iii})$ is true $\Rightarrow \frac{\mathbf{m}_{\mathrm{a}}^{4}}{\mathbf{h}_{\mathrm{a}}^{4}}+\frac{\mathbf{m}_{\mathrm{b}}^{4}}{\mathbf{h}_{\mathrm{b}}^{4}}+\frac{\mathbf{m}_{\mathrm{c}}^{4}}{\mathbf{h}_{\mathrm{c}}^{4}} \stackrel{(2)}{\geq} \frac{1}{27}\left(\frac{7 \mathrm{R}-2 \mathrm{r}}{2 R}\right)^{4}$
$(1),(2) \Rightarrow \frac{1}{27}\left(\frac{7 R-2 r}{2 R}\right)^{4} \leq \frac{m_{a}^{4}}{h_{a}^{4}}+\frac{m_{b}^{4}}{h_{b}^{4}}+\frac{m_{c}^{4}}{h_{c}^{4}} \leq \frac{3}{64}\left(\frac{27 R^{2}}{4 r^{2}}-19\right)^{2}$ (Proved)
1536. In $\triangle A B C$ the following relationship holds:

$$
\frac{a^{2}+b^{2}+c^{2}}{4 S \sqrt{3}} \leq \frac{1}{9}+\frac{2}{9}\left(\frac{R}{r}\right)^{2}
$$

## Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution by Marian Ursărescu-Romania

$$
\begin{align*}
& \text { In any } \triangle A B C \text { we have: } \cot A+\cot B+\cot =\frac{a^{2}+b^{2}+c^{2}}{4 S} \\
& \text { We must show }: \frac{\cot A+\cot B+\cot }{\sqrt{3}} \leq \frac{1}{9}+\frac{2}{9}\left(\frac{R}{r}\right)^{2}  \tag{1}\\
& \text { But cot } A+\cot B+\cot =\frac{s^{2}-r(4 R+r)}{2 s r}  \tag{2}\\
& \text { From (1) and (2) we must show: } \frac{s^{2}-4 R r-r^{2}}{2 s r \sqrt{3}} \leq \frac{1}{9}+\frac{2}{9}\left(\frac{R}{r}\right)^{2} \tag{3}
\end{align*}
$$

From Gerretsen: $s^{2} \leq 4 R^{2}+4 R r+3 r^{2}$ and from Mitrinovici we have: $s \geq 3 \sqrt{3} r$ then

$$
\frac{s^{2}-4 R r-r^{2}}{2 s r \sqrt{3}} \leq \frac{4 R^{2}+4 R r+3 r^{2}-4 R r-r^{2}}{2 \cdot 9 r^{2}}=\frac{4 R^{2}+2 r^{2}}{2 \cdot 9 r^{2}}
$$



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=\frac{2 R^{2}+r^{2}}{9 r^{2}}=\frac{1}{9}+\frac{2}{9}\left(\frac{R}{r}\right)^{2} \Rightarrow(3) \text { is true.Proved. }
$$

1537. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} \frac{m_{a}^{2}}{b^{2}+c^{2}} \geq \frac{3 r}{2 R} \sum_{c y c} \sin ^{2} \frac{A}{2}
$$

Proposed by Marin Chirciu-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
m_{a}^{2} \geq \frac{b^{2}+c^{2}}{4 R} \text { and analogs } \\
\sum_{c y c} \frac{m_{a}^{2}}{b^{2}+c^{2}} \geq \sum_{c y c} \frac{\left(b^{2}+c^{2}\right)^{2}}{(4 R)^{2}\left(b^{2}+c^{2}\right)}=\frac{1}{(4 R)^{2}} \sum_{c y c}\left(b^{2}+c^{2}\right)=\frac{a^{2}+b^{2}+c^{2}}{8 R^{2}} \\
=\frac{a^{2}+b^{2}+c^{2}}{8 R^{2}}=\frac{2\left(s^{2}-4 R r-r^{2}\right)}{8 R^{2}}=\frac{s^{2}-4 R r-r^{2}}{R^{2}} \\
\sum_{c y c} \sin ^{2} \frac{A}{2}=\frac{2 R-r}{2 R} \Rightarrow \frac{3 r}{2 R} \cdot \sum_{c y c} \sin ^{2} \frac{A}{2}=\frac{3 r}{2 R} \cdot \frac{2 R-r}{2 R} \\
\text { We need to prove: } \\
\frac{s^{2}-4 R r-r^{2}}{R^{2}} \geq \frac{3 r}{2 R} \cdot \frac{2 R-r}{2 R} \Leftrightarrow \\
s^{2}-4 R r-r^{2} \geq 6 R r-3 r^{2} \Leftrightarrow s^{2} \geq 10 R r-2 r^{2} \\
\text { But: } s^{2} \geq 16 R r-5 r^{2} \\
\text { We must show that: } 16 R r-5 r^{2} \geq 10 R r-2 r^{2} \Leftrightarrow
\end{gathered}
$$

$6 R r \geq 3 r^{2} \Leftrightarrow 2 R \geq r \Leftrightarrow R \geq \frac{r}{2}$ which is true because $R \geq 2 r>\frac{r}{2}$. Proved.
1538. In $\triangle A B C$ the following relationship holds:

$$
\frac{n(b+c)-a}{\sqrt{a+2 b}}+\frac{n(c+a)-b}{\sqrt{b+2 c}}+\frac{n(a+b)-c}{\sqrt{c+2 a}} \geq(2 n-1) \sqrt{a+b+c}, n \geq 1
$$



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Solution 1 by Marian Ursărescu-Romania
We must show:

$$
\begin{gather*}
\frac{n(b+c)-a}{\sqrt{(a+2 b)(a+b+c)}}+\frac{n(c+a)-b}{\sqrt{(b+2 c)(a+b+c)}}+\frac{n(a+b)-c}{\sqrt{(c+2 a)(a+b+c)}} \geq(2 n-1) \\
\text { But: } \sqrt{(a+2 b)(a+b+c)} \leq \frac{a+2 b+a+b+c}{2}=\frac{2 a+3 b+c}{2} \Rightarrow \\
\frac{1}{\sqrt{(a+2 b)(a+b+c)}} \geq \frac{2}{2 a+3 b+c} \quad \text { (2) } \tag{2}
\end{gather*}
$$

From (1)+(2) we must show:

$$
\begin{equation*}
\sum_{c y c} \frac{n(b+c)-a}{2 a+3 b+c} \geq \frac{2 n-1}{2} \Leftrightarrow \sum_{c y c} \frac{(n(b+c)-a)^{2}}{(2 a+3 b+c)(n(b+c)-a)} \geq \frac{2 n-1}{2} \tag{3}
\end{equation*}
$$

From Bergstrom inequality we have:

$$
\begin{aligned}
& \sum_{c y c} \frac{(n(b+c)-a)^{2}}{(2 a+3 b+c)(n(b+c)-a)} \geq \frac{(n(b+c)-a+n(a+c)-b+n(a+b)-c)^{2}}{\sum_{c y c}(2 a+3 b+c)(n(b+c)-a)} \Leftrightarrow \\
& \sum_{c y c} \frac{(n(b+c)-a)^{2}}{(2 a+3 b+c)(n(b+c)-a)} \geq \frac{(2 n-1)^{2}(a+b+c)^{2}}{n \sum(2 a+3 b+c)(b+c)-\sum(2 a+3 b+c) a} \Leftrightarrow \\
& \sum_{c y c} \frac{(n(b+c)-a)^{2}}{(2 a+3 b+c)(n(b+c)-a)} \geq \frac{(2 n-1)^{2}\left(a^{2}+b^{2}+c^{2}+2 a b+2 b c+2 c a\right)}{2(2 n-1)\left(a^{2}+b^{2}+c^{2}+2 a b+2 b c+2 c a\right)} \Rightarrow
\end{aligned}
$$

(3) it's true.Proved.

## Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\text { Let } \varphi(t)=\frac{1}{\sqrt{t}}(t>0) \Rightarrow \varphi^{\prime}(t)=-\frac{1}{2} t^{-\frac{3}{2}} \Rightarrow \varphi^{\prime \prime}(t)=\frac{3}{4} t^{-\frac{5}{2}}>0
$$

In any $\triangle A B C, n \geq 2 \Rightarrow \boldsymbol{n}(b+c)-\boldsymbol{a} \geq \boldsymbol{b}+\boldsymbol{c}-\boldsymbol{a}>0 ; n(\boldsymbol{a}+\boldsymbol{c})-\boldsymbol{b} \geq \boldsymbol{a}+\boldsymbol{c}-\boldsymbol{b}>0$;

$$
\begin{aligned}
& n(a+b)-c>a+b-c>0 \\
& \text { Now, } L H S=\frac{n(b+c)-a}{\sqrt{a+2 b}}+\frac{n(c+a)-b}{\sqrt{b+2 c}}+\frac{n(a+b)-c}{\sqrt{c+2 a}} \\
& =(n(b+c)-a) \varphi(a+2 b)+(n(c+a)-b) \varphi(b+2 c)+(n(a+b)-c) \varphi(c+2 a) \\
& \stackrel{\text { Jensen }}{\geq} \frac{(2 n-1)(a+b+c)}{\sqrt{\frac{(n(b+c)-a)(a+2 b)+(n(c+a)-b)(b+2 c)+(n(a+b)-c)(c+2 a)}{(2 n-1)(a+b+c)}}}
\end{aligned}
$$



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$$
=\frac{(2 n-1) \sqrt{a+b+c} \cdot \sqrt{2 n-1}(a+b+c)}{\sqrt{(n(b+c)-a)(a+2 b)+(n(c+a)-b)(b+2 c)+(n(a+b)-c)(c+2 a)}}=\Phi
$$

We must show that:

$$
\begin{gathered}
\sqrt{2 n-1}(a+b+c)= \\
\sqrt{(n(b+c)-a)(a+2 b)+(n(c+a)-b)(b+2 c)+(n(a+b)-c)(c+2 a)} \\
\Leftrightarrow 2 n(a+b+c)^{2}=n\left(2 a^{2}+2 b^{2}+2 c^{2}+4 a b+4 a c+4 b c\right) \\
\Leftrightarrow 2 n(a+b+c)^{2}=2 n(a+b+c)^{2} \quad \text { (true) } \\
\Rightarrow L H S \geq \Phi=(2 n-1) \sqrt{a+b+c} . \text { Proved. } \\
\text { Equality } a=b=c .
\end{gathered}
$$

1539. In $\triangle A B C$ the following relationship holds:

$$
\frac{r(a+b+c)}{R}\left(x+\frac{R-2 r}{4 R+r}\right) \leq 2 x \sum_{c y c} \frac{(s-b)(s-c)}{a}, x \geq 1
$$

Proposed by Marin Chirciu-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\sum_{c y c} \frac{(s-b)(s-c)}{a}=\frac{a b(s-a)(s-b)+b c(s-b)(s-c)+c a(s-c)(s-a)}{a b c} \\
=\frac{(a b)^{2}+(b c)^{2}+(c a)^{2}+3 a b c s-(a b+b c+c a) s^{2}}{a b c} \\
=\frac{(a b+b c+c a)^{2}+2 a b c(a+b+c)+3 a b c s-(a b+b c+c a) s^{2}}{a b c} \\
=\frac{\left(s^{2}+4 R r+r^{2}\right)^{2}+28 R r s^{2}-\left(s^{2}+4 R r+r^{2}\right) s^{2}}{4 R r s} \\
=\frac{s^{4}+16 R^{2} r^{2}+r^{4}+8 R r s^{2}+2 s^{2} r^{2}+8 R r^{3}+28 R r s^{2}-s^{4}-4 R r s^{2}-s^{2} r^{2}}{4 R r s} \\
=\frac{16 r^{2} R^{2}+r^{4}+32 R r s^{2}+s^{2} r^{2}+8 R r^{3}}{4 R r s} \\
=\frac{16 r R^{2}+r^{3}+(32 R+r) s^{2}+8 R r^{2}}{4 R s}
\end{gathered}
$$



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$$
\begin{gathered}
R H S=2 x \sum_{c y c} \frac{(s-b)(s-c)}{a}=x \cdot \frac{16 r R^{2}+r^{3}+(32 R+r) s^{2}+8 R r^{2}}{4 R s} \\
L H S=\frac{r(a+b+c)}{R}\left(x+\frac{R-2 r}{4 R+r}\right)=\frac{2 s r}{R} \cdot \frac{[(4 x+1) R+(x-2) r]}{4 R+r}
\end{gathered}
$$

We need to prove:

$$
\begin{gathered}
x \cdot \frac{16 r R^{2}+8 R r^{2}+r^{3}+(32 R+r) s^{2}}{2 R s} \geq \frac{2 s r}{R} \cdot \frac{[(4 x+1) R+(x-2) r]}{4 R+r} \\
{\left[x(32 R+r)(4 R+r)-4 r((4 x+1) R+(x-2) r] s^{2}+\right.} \\
+x(4 R+r)\left(16 r R^{2}+8 R r^{2}+r^{3}\right) \geq 0
\end{gathered}
$$

$$
\left[128 x R^{2}+4(5 x-1) R r+(8-3 x) r^{2}\right] s^{2}+x(4 R+r)\left(16 r R^{2}+8 R r^{2}+r^{3}\right) \geq 0 ;(*)
$$ We have

$$
\begin{aligned}
& 128 x R^{2}+4(5 x-1) R r+(8-3 x) r^{2} \stackrel{R \geq 2 r ; x \geq 1}{\geq} 128 x \cdot 4 r^{2}+4(5 x-1) 2 r^{2}+ \\
&(8-3 x) r^{2}=549 x r^{2}>0 \xrightarrow{R, r, s>0}(*) \text { is true } \Rightarrow(1) \text { is true.Proved. }
\end{aligned}
$$

1540. In $\triangle A B C$ the following relationship holds:

$$
\sqrt{\frac{b c}{r_{b}+r_{c}}}+\sqrt{\frac{c a}{r_{c}+r_{a}}}+\sqrt{\frac{a b}{r_{a}+r_{b}}} \leq \frac{3 R}{2 r} \sqrt{R}
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Avishek Mitra-West Bengal-India

$$
\begin{aligned}
& \sum_{c y c} \sqrt{\frac{b c}{r_{b}+r_{c}}}=\sum_{c y c} \sqrt{\frac{b c}{F\left(\frac{1}{s-b}+\frac{1}{s-c}\right)}}=\frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{b c(s-b)(s-c)}{s-b+s-c}}{ }^{A M-G M} \\
& \leq \frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{b c}{a}} \cdot \frac{s-b+s-c}{2}=\frac{1}{\sqrt{F}} \sum_{c y c} \sqrt{\frac{b c}{a}} \cdot \frac{a}{2}=\frac{1}{2 \sqrt{F}} \sum_{c y c} \sqrt{a b c}= \\
& =\frac{3}{2} \sqrt{\frac{4 R F}{F}}=3 \sqrt{R}=3 \cdot \frac{1}{2} \cdot 2 \sqrt{R} \stackrel{\text { EULER }}{\stackrel{3 R}{\leq}} \frac{\sqrt{R}}{2 r} \sqrt{R}
\end{aligned}
$$



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Solution 2 by Marian Ursărescu-Romania

$$
\begin{equation*}
\text { We must show that: }\left(\sqrt{\frac{b c}{r_{b}+r_{c}}}+\sqrt{\frac{c a}{r_{c}+r_{a}}}+\sqrt{\frac{a b}{r_{a}+r_{b}}}\right)^{2} \leq \frac{9 R^{3}}{4 r^{2}} \tag{1}
\end{equation*}
$$

From Chauchy inequality we have:

$$
\begin{equation*}
\left(\sqrt{\frac{b c}{r_{b}+r_{c}}}+\sqrt{\frac{c a}{r_{c}+r_{a}}}+\sqrt{\frac{a b}{r_{a}+r_{b}}}\right)^{2} \leq 3\left(\frac{b c}{r_{b}+r_{c}}+\frac{c a}{r_{c}+r_{a}}+\frac{a b}{r_{a}+r_{b}}\right) \tag{2}
\end{equation*}
$$

From (1)+(2) we must show: $\frac{b c}{r_{b}+r_{c}}+\frac{c a}{r_{c}+r_{a}}+\frac{a b}{r_{a}+r_{b}} \leq \frac{3 R^{3}}{4 r^{2}}$

$$
\begin{gather*}
\text { But: } \frac{b c}{r_{b}+r_{c}}=\frac{b c}{\frac{s}{s-b}+\frac{s}{s-c}}=\frac{b c(s-b)(s-c)}{S(s-b+s-c)}=\frac{b c(s-b)(s-c)}{a S}  \tag{3}\\
\sqrt{(s-b)(s-c)} \leq \frac{s-b+s-c}{2}=\frac{a}{2} \Rightarrow(s-b)(s-c) \leq \frac{a^{2}}{4} \text { then } \\
\frac{b c}{r_{b}+r_{c}} \leq \frac{b c a^{2}}{4 a S}=\frac{a b c}{4 S}=R \text { and similary, then } \frac{b c}{r_{b}+r_{c}}+\frac{c a}{r_{c}+r_{a}}+\frac{a b}{r_{a}+r_{b}} \leq 3 R \tag{4}
\end{gather*}
$$

From (3)+(4) we must show: $3 R \leq \frac{3 R^{3}}{4 r^{2}} \Leftrightarrow 2 r \leq R$ true from Euler.

## Solution 3 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\text { Suppose: } a \geq b \geq c \Rightarrow b c \leq c a \leq a b ; \cos \frac{A}{2} \leq \cos \frac{B}{2} \leq \cos \frac{C}{2} \Rightarrow \\
\cos ^{2} \frac{A}{2} \leq \cos ^{2} \frac{B}{2} \leq \cos ^{2} \frac{C}{2} \Rightarrow \frac{1}{\cos ^{2} \frac{A}{2}} \geq \frac{1}{\cos ^{2} \frac{B}{2}} \geq \frac{1}{\cos ^{2} \frac{C}{2}} \\
r_{b}+r_{c}=4 R \cos ^{2} \frac{A}{2} \text { and analogs } \\
\Rightarrow \sum_{c y c} \sqrt{\frac{b c}{r_{b}+r_{c}} \stackrel{B C S}{\leq}} \sqrt{3 \cdot \sum_{c y c} \frac{b c}{4 R \cos ^{2} \frac{A}{2}}} \\
=\sqrt{\frac{s^{2}+4 R r+r^{2}}{4 R} \cdot \frac{s^{2}+(4 R+r)^{2}}{s^{2}}}=\sqrt{\frac{\left(s^{2}+4 R r+r^{2}\right)\left(s^{2}+(4 R+r)^{2}\right)}{4 R s^{2}}} \\
\begin{array}{l}
\text { Cebyshev } \\
\underline{3} \cdot \frac{1}{3}(b c+c a+a b) \cdot \frac{1}{4 R} \cdot\left(\frac{1}{\cos ^{2} \frac{A}{2}}+\frac{1}{\cos ^{2} \frac{B}{2}}+\frac{1}{\cos ^{2} \frac{C}{2}}\right) \\
\text { We must show that: } \sqrt{\frac{\left(s^{2}+4 R r+r^{2}\right)\left(s^{2}+(4 R+r)^{2}\right)}{4 R s^{2}}} \leq \frac{3 R}{2 r} \sqrt{R}
\end{array}
\end{gathered}
$$



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$$
\begin{gathered}
r^{2}\left(s^{2}+4 R r+r^{2}\right)\left(s^{2}+(4 R+r)^{2}\right) \leq 9 R^{4} s^{2} \\
s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \Rightarrow \\
r^{2}\left(s^{2}+4 R r+r^{2}\right)\left(s^{2}+(4 R+r)^{2}\right) \leq r^{2}\left(4 R^{2}+8 R r+4 r^{2}\right)\left(20 R^{2}+12 R r+4 r^{2}\right) \\
s^{2} \geq 16 R r-5 r^{2} \Rightarrow \\
9 R^{4} s^{2} \geq 9 R^{4}\left(16 R r-5 r^{2}\right)=144 R^{5} r-45 R^{4} r^{2}
\end{gathered}
$$

We must show that:

$$
\begin{aligned}
& r\left(4 R^{2}+8 R r+4 r^{2}\right)\left(20 R^{2}+12 R r+4 r^{2}\right) \leq 144 R^{5}-45 R^{4} r \\
& \Leftrightarrow 16 r\left(R^{2}+2 R r+r^{2}\right)\left(5 R^{2}+3 R r+r^{2}\right) \leq 144 R^{5}-45 R^{4} r \stackrel{t=\frac{R}{r} \geq 2}{\Longleftrightarrow} \\
& 144 t^{5}-125 t^{4}-208 t^{3}-192 t^{2}-80 t-16 \geq 0 \\
&(t-2)\left(144 t^{4}+163 t^{2}+118 t^{2}+44 t+8\right) \geq 0 \text { true from } t \geq 2
\end{aligned}
$$

Solution 4 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\operatorname{s\operatorname {sin}(\frac {B+C}{2})\operatorname {cos}\frac {A}{2}}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{sos}^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
& \text { (i) } \\
& \therefore r_{b}+r_{c} \stackrel{(i)}{\cong} 4 R \cos ^{2} \frac{A}{2} N o w, \frac{b+c}{2}=\frac{4 R \cos \frac{A}{2} \cos \left(\frac{B-C}{2}\right)}{2} \\
& \leq 2 R \cos \frac{A}{2}\left(\because 0<\cos \frac{B-C}{2} \leq 1\right)=\sqrt{R} \sqrt{4 R \cos ^{2} \frac{A}{2}} \stackrel{b y(i)}{=} \sqrt{R\left(r_{b}+r_{c}\right)} \Rightarrow \sqrt{\frac{b c}{r_{b}+r_{c}}} \leq \sqrt{R}\left(\frac{2 \sqrt{b c}}{b+c}\right) \\
& \stackrel{G M \leq A M}{\sim} \sqrt{幺} \sqrt{R} \therefore \sqrt{\frac{b c}{r_{b}+r_{c}}} \leq \sqrt{R} \text { and analogs and } s o, \sum \sqrt{\frac{b c}{r_{b}+r_{c}}} \leq 3 \sqrt{R} \\
& \leq 3 \sqrt{R}\left(\frac{R}{2 r}\right)\left(\because 1 \stackrel{\text { Euler }}{\underset{\sim}{\leq}} \frac{R}{2 r}\right)=\frac{3 R}{2 r} \sqrt{R} \text { (Proved) }
\end{aligned}
$$

## Solution 5 by Bogdan Fuştei-Romania

We know that: $\cos \frac{A}{2}=\sqrt{\frac{r_{b}+r_{c}}{4 R}} \Rightarrow 2 \sqrt{R} \cos \frac{A}{2}=\sqrt{r_{b}+r_{c}}$ and analogs.

$$
\begin{gathered}
\frac{b+c}{2}=2 R \cos \frac{A}{2} \cdot \frac{h_{a}}{l_{a}} \text { and analogs } \\
h_{a} \leq l_{a} \text { and analogs } \\
\frac{\sqrt{b c}}{\sqrt{r_{b}++r_{c}}} \stackrel{\text { an-Gm }}{\sim} \frac{b+c}{2 \sqrt{r_{b}++r_{c}}}
\end{gathered}
$$



$$
\begin{align*}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \frac{\boldsymbol{b}+\boldsymbol{c}}{2 \sqrt{\boldsymbol{r}_{b}++\boldsymbol{r}_{c}}}=2 \boldsymbol{R} \boldsymbol{\operatorname { c o s }} \frac{\boldsymbol{A}}{\mathbf{2}} \cdot \frac{\boldsymbol{h}_{a}}{\boldsymbol{l}_{a}} \cdot \frac{\mathbf{1}}{2 \sqrt{\boldsymbol{R}} \boldsymbol{\operatorname { c o s }} \frac{\boldsymbol{A}}{2}}=\sqrt{\boldsymbol{R}} \cdot \frac{\boldsymbol{h}_{a}}{\boldsymbol{l}_{a}} \leq \sqrt{\boldsymbol{R}} \cdot \mathbf{1} \leq \sqrt{\boldsymbol{R}} \Rightarrow \\
& \sum_{c y c} \frac{\boldsymbol{b + c}}{2 \sqrt{r_{b}+r_{c}}} \leq \mathbf{3} \sqrt{\boldsymbol{R}} \Rightarrow \sum_{c y c} \sqrt{\frac{b c}{r_{b}+r_{c}}} \leq \mathbf{3} \sqrt{\boldsymbol{R}} \text { (1) }
\end{align*}
$$

We must show: $3 \sqrt{R} \leq \frac{3 R}{2 r} \sqrt{R} \Rightarrow 1 \leq \frac{R}{2 r} \Rightarrow 2 r \leq R$ true from Euler (2)
From (1)+(2) we get inequality.
1541. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{3} \geq 27 \cdot \sqrt{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}}
$$

## Proposed by Adil Abdullayev-Baku-Azerbaijan

Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{3} \stackrel{A m-G m}{\geq} 27 \cdot \frac{m_{a} m_{b} m_{c}}{w_{a} w_{b} w_{c}} \stackrel{(1)}{\geq} 27 \cdot \sqrt{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}} \\
(1) \Leftrightarrow\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{2} \geq \frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}} \Leftrightarrow m_{a} m_{b} m_{c} \geq \frac{\left(m_{a} m_{b} m_{c}\right)^{2}}{s_{a} s_{b} s_{c}} \\
m_{a} \geq \frac{w_{a}^{2}}{s_{a}}=\frac{w_{a}^{2}}{\frac{2 b c}{b^{2}+c^{2}} \cdot m_{a}} \Leftrightarrow m_{a}^{2} \geq \frac{b^{2}+c^{2}}{2 b c} \cdot w_{a}^{2} \Leftrightarrow \\
\frac{2 b^{2}+2 c^{2}-a^{2}}{4} \geq \frac{b^{2}+c^{2}}{2 b c} \cdot \frac{4 b c}{(b+c)^{2}} \cdot s(s-a) \\
\frac{2 b^{2}+2 c^{2}-a^{2}}{4} \geq \frac{b^{2}+c^{2}}{2 b c} \cdot \frac{4 b c}{(b+c)^{2}} \cdot \frac{a+b+c}{2} \cdot \frac{b+c-a}{2} \\
\left(2\left(b^{2}+c^{2}\right)-a^{2}\right)(b+c)^{2} \geq 2\left(b^{2}+c^{2}\right)(a+b+c)(b+c-a) \\
(b+c)^{2}-a^{2}(b+c)^{2} \geq 2\left(b^{2}+c^{2}\right)(b+c)^{2}-2\left(b^{2}+c^{2}\right) a^{2} \\
\left(2 b^{2}+2 c^{2}-(b+c)^{2}\right) a^{2} \geq 0
\end{gathered}
$$

$(b-c)^{2} a^{2} \geq 0$ true then $m_{a} \geq \frac{w_{a}^{2}}{s_{a}}$ and similary $m_{b} \geq \frac{w_{b}^{2}}{s_{b}} ; m_{c} \geq \frac{w_{c}^{2}}{s_{c}}$ then (*) is true.


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Solution 2 by Soumava Chakraborty－Kolkata－India

$$
\begin{aligned}
& \left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{3} \stackrel{\operatorname{A-G}}{\sum} 27 \frac{m_{a} m_{b} m_{c}}{w_{a} w_{b} w_{c}} \stackrel{?}{\geq} 27 \sqrt{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}} \Leftrightarrow\left(\frac{m_{a} m_{b} m_{c}}{w_{a} w_{b} w_{c}}\right)^{2} \stackrel{?}{\geq} \frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}} \\
& \Leftrightarrow \prod\left(m_{a} s_{a}\right){\underset{(i)}{?}}_{\substack{i \\
\prod}} \boldsymbol{w}_{\boldsymbol{a}}^{\mathbf{a}} \\
& \text { Now, } m_{a} s_{a} \stackrel{?}{\sim} w_{a}^{2} \Leftrightarrow \frac{2 b c}{b^{2}+c^{2}} m_{a}^{2} \stackrel{?}{\sim} \frac{4 b^{2} c^{2}}{(b+c)^{2}}\left\{\frac{s(s-a)}{b c}\right\} \\
& \Leftrightarrow \frac{b c\left(2 b^{2}+2 c^{2}-a^{2}\right)}{2\left(b^{2}+c^{2}\right)} \underset{\sim}{\sum} \frac{b c(b+c+a)(b+c-a)}{(b+c)^{2}}=b c\left\{\frac{(b+c)^{2}-a^{2}}{(b+c)^{2}}\right\} \\
& \Leftrightarrow b c-\frac{a^{2} b c}{2\left(b^{2}+c^{2}\right)} \stackrel{?}{乌} \boldsymbol{b} c-\frac{a^{2} b c}{(b+c)^{2}} \Leftrightarrow \frac{a^{2} b c}{(b+c)^{2}} \stackrel{?}{\sim} \frac{a^{2} b c}{2\left(b^{2}+c^{2}\right)} \Leftrightarrow 2\left(b^{2}+c^{2}\right) \stackrel{?}{\sim}(b+c)^{2} \\
& \Leftrightarrow(b-c)^{2} \stackrel{?}{2} \mathbf{0} \rightarrow \text { true } \\
& \therefore \boldsymbol{m}_{a} \boldsymbol{s}_{a} \stackrel{?}{\stackrel{?}{n}} \boldsymbol{w}_{a}^{2} \text { and analog } s \Rightarrow \prod\left(\boldsymbol{m}_{a} s_{a}\right) \stackrel{?}{乌} \prod w_{a}^{2} \Rightarrow \\
& \text { (1) is true } \therefore\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{3} \geq 27 \sqrt{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}} \text { (Proved) }
\end{aligned}
$$

Solution 3 by proposer

$$
\begin{gathered}
\text { Lemma 1: } \frac{m_{a}}{w_{a}} \geq \frac{(b+c)^{2}}{4 b c} \\
\text { Lemma } 2: \frac{m_{a}}{s_{a}}=\frac{b^{2}+c^{2}}{2 b c} \\
\frac{m_{a}}{w_{a}} \geq \frac{b^{2}+c^{2}}{4 b c}+\frac{1}{2}=\frac{m_{a}}{2 s_{a}}+\frac{1}{2} \Rightarrow \frac{2 m_{a}}{w_{a}} \geq \frac{m_{a}}{s_{a}}+1 \\
2\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right) \geq \frac{m_{a}}{s_{a}}+\frac{m_{b}}{s_{b}}+\frac{m_{c}}{s_{c}}+1+1+1 \stackrel{A m-G m}{\geq} 6 \cdot \sqrt[6]{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}} \\
\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}} \geq 3 \cdot \sqrt[6]{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}} \\
\left(\frac{m_{a}}{w_{a}}+\frac{m_{b}}{w_{b}}+\frac{m_{c}}{w_{c}}\right)^{3} \geq 27 \cdot \sqrt{\frac{m_{a} m_{b} m_{c}}{s_{a} s_{b} s_{c}}}
\end{gathered}
$$



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1542. In $\triangle A B C$ the following relationship holds:

$$
\left(\frac{h_{a}}{r_{a}}\right)^{2}+\left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{10 r}{R} \geq 8
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\frac{h_{a}}{r_{a}}=\frac{\frac{2 S}{a}}{s \cdot \tan \frac{A}{2}}=\frac{2 s r}{\operatorname{sa\cdot \operatorname {tan}\frac {A}{2}}=\frac{2 r}{\left(4 R \sin \frac{A}{2} \cos \frac{A}{2}\right)\left(\frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}\right)}=\frac{r}{2 R} \cdot \frac{1}{\sin ^{2} \frac{A}{2}}} \begin{aligned}
& \sum_{c y c}\left(\frac{h_{a}}{r_{a}}\right)^{2}=\frac{r^{2}}{4 R^{2}} \sum_{c y c}\left(\frac{1}{\sin ^{2} \frac{A}{2}}\right)^{2}=\frac{r^{2}}{4 R^{2}}\left[-2 \sum_{c y c} \frac{1}{\left(\sin \frac{A}{2} \cos \frac{A}{2}\right)^{2}}\right] \\
&=\frac{r^{2}}{4 R^{2}}\left[\left(\frac{s^{2}+r^{2}-8 R r}{r^{2}}\right)^{2}-2 \cdot \frac{8 R(2 R-r)}{r^{2}}\right] \\
&=\frac{\left(s^{2}+r^{2}-8 R r\right)^{2}-32 R^{2} r^{2}+16 R r^{3}}{4 R^{2} r^{2} \geq 16 R r-5 r^{2}} \underset{\geq}{\geq} \frac{\left(16 R r-4 r^{2}-8 R r\right)^{2}}{4 R^{2} r^{2}}-8+\frac{4 r}{R} \\
&=\left(4-\frac{2 r}{R}\right)^{2}+\frac{4 r}{R}-8=8-\frac{12 r}{R}+\frac{4 r^{2}}{R^{2}} \\
& L H S \geq 8-\frac{12 r}{R}+\frac{4 r^{2}}{R^{2}}+\frac{10 r}{R}=8-\frac{2 r}{R}+\frac{4 r^{2}}{R^{2}}=8-\frac{2 r}{R}\left(1-\frac{2 r}{R}\right)^{\frac{2 r}{R} \leq 1} \geq 8 \\
& P r o v e d .
\end{aligned}
\end{gathered}
$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\frac{h_{a}}{r_{a}}=\frac{2 r s}{4 R \operatorname{stan} \frac{A}{2} \cos \frac{A}{2} \sin \frac{A}{2}}=\frac{r}{2 R} \operatorname{cosec}^{2} \frac{A}{2} \text { and analogs } \\
\Rightarrow \sum\left(\frac{h_{a}}{r_{a}}\right)^{2}=\frac{r^{2}}{4 R^{2}} \sum \operatorname{cosec}^{4} \frac{A}{2}
\end{gathered}
$$



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$=\frac{r^{2}}{4 R^{2}}\left[\left(\sum \operatorname{cosec}^{2} \frac{A}{2}\right)^{2}-2 \sum \operatorname{cosec}^{2} \frac{B}{2} \operatorname{cosec}^{2} \frac{C}{2}\right]$
$=\frac{r^{2}}{4 R^{2}}\left[\left(\sum \frac{b c(s-a)}{(s-b)(s-c)(s-a)}\right)^{2}-2 \sum \frac{c a \cdot a b}{(s-c)(s-a) \cdot(s-a)(s-b)}\right]$

$$
=\frac{r^{2}}{4 R^{2}}\left[\left(\frac{s\left(s^{2}+4 R r+r^{2}\right)-12 R r s}{r^{2} s}\right)^{2}-\frac{8 R r s}{r^{2} s} \sum \frac{a-s+s}{s-a}\right]
$$

$$
=\frac{r^{2}}{4 R^{2}}\left[\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{r^{4}}-\frac{8 R}{r}\left(-3+\frac{s \sum(s-b)(s-c)}{r^{2} s}\right)\right]
$$

$$
=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-\frac{2 r}{R}\left(\frac{4 R+r}{r}-3\right)=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-8+\frac{4 r}{R}
$$

$$
\therefore \sum\left(\frac{h_{a}}{r_{a}}\right)^{2}+\frac{10 r}{R}=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-8+\frac{14 r}{R} \geq 8
$$

$$
\Leftrightarrow \frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}+\frac{14 r}{R} \geq 16 \Leftrightarrow \frac{\left(s^{2}-8 R r+r^{2}\right)^{2}+56 R r^{3}}{4 R^{2} r^{2}} \geq 16
$$

$$
\Leftrightarrow s^{4}-2 s^{2}\left(8 R r-r^{2}\right)+\left(8 R r-r^{2}\right)^{2}+56 R r^{3} \geq 64 R^{2} r^{2}
$$

$$
\Leftrightarrow s^{4}-2 s^{2}\left(8 R r-r^{2}\right)+40 R r^{3}+r^{4} \underset{\sim}{(1)} 0
$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c$

$$
=x+y
$$

Then, $8 R r-r^{2}=\frac{8 a b c \Delta}{4 \Delta s}-\frac{(s-b)(s-c)(s-a)}{s} \stackrel{(i)}{=} \frac{2 \prod(y+z)-x y z}{\sum x}$ and, $40 R r^{3}+r^{4}$

$$
=\frac{40 a b c \Delta}{4 \Delta s} \cdot \frac{(s-b)(s-c)(s-a)}{s}
$$

$$
+\left(\frac{(s-b)(s-c)(s-a)}{s}\right)^{2} \stackrel{(i i)}{\bar{j}} \frac{10 x y z \Pi(y+z)+x^{2} y^{2} z^{2}}{\left(\sum x\right)^{2}}
$$

(i), (ii) $\Rightarrow(1) \Leftrightarrow\left(\sum x\right)^{4}-2 \sum x\{2 \Pi(y+z)-x y z\}+\frac{10 x y z \prod(y+z)+x^{2} y^{2} z^{2}}{\left(\sum x\right)^{2}} \geq 0$

$$
\Leftrightarrow\left(\sum x\right)^{6}-2\left(\sum x\right)^{3}\{2 \Pi(y+z)-x y z\}+10 x y z \Pi(y+z)+x^{2} y^{2} z^{2} \geq 0
$$



$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \text { www.ssmrmh.ro } \\
& \Leftrightarrow \sum x^{6}+2 \sum x^{5} y+2 \sum x y^{5}+3 x^{2} y^{2} z^{2} \stackrel{(2)}{\sum} \sum x^{4} y^{2}+\sum x^{2} y^{4}+4 \sum x^{3} y^{3} \\
& \text { Now, } \sum x^{6}+3 x^{2} y^{2} z^{2} \stackrel{\text { Schur }}{>} \sum x^{4} y^{2}+\sum x^{2} y^{4} \text { and } 2 \sum x^{5} y+2 \sum x y^{5} \\
& =2 \sum\left(x^{5} y+x y^{5}\right)^{A-G} 4 \sum x^{3} y^{3} \text { and adding these two, } \\
& (2) \Rightarrow(1) \text { is true } \therefore\left(\frac{h_{a}}{r_{a}}\right)^{2}+\left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{10 r}{R} \geq 8 \text { (Proved) }
\end{aligned}
$$

1543. In $\triangle \mathrm{ABC}$ the following relationship holds:

$$
\left(\frac{h_{a}}{r_{a}}\right)^{2}+\left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{2 \mu r}{R} \geq \mu+3, \mu \leq 5
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\frac{h_{a}}{r_{a}}=\frac{2 r s}{4 R \operatorname{stan} \frac{A}{2} \cos \frac{A}{2} \sin \frac{A}{2}}=\frac{r}{2 R} \operatorname{cosec}^{2} \frac{A}{2} \text { and analogs } \Rightarrow \\
\sum\left(\frac{h_{a}}{r_{a}}\right)^{2}=\frac{r^{2}}{4 R^{2}} \sum \operatorname{cosec}^{4} \frac{A}{2} \\
=\frac{r^{2}}{4 R^{2}}\left[\left(\sum \frac{r^{2}}{4 R^{2}}\left[\left(\sum \operatorname{cosec}^{2} \frac{A}{2}\right)^{2}-2 \sum \operatorname{cosec}^{2} \frac{B}{2} \operatorname{cosec}^{2} \frac{C}{2}\right]\right.\right. \\
=\frac{r^{2}}{4 R^{2}}\left[\left(\frac{s\left(s^{2}+4 R(s-a)(s-c)(s-a)\right.}{}\right)^{2}-2 \sum \frac{c a \cdot a b}{r^{2} s}(s-c)(s-a) .(s-a)(s-b)\right] \\
=\frac{r^{2}}{4 R^{2}}\left[\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{r^{4}}-\frac{8 R}{r}\left(-3+\frac{s \sum(s-b)(s-c)}{r^{2} s}\right)\right] \\
=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-\frac{2 r}{R}\left(\frac{4 R+r}{r}-3\right)=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-8+\frac{4 r}{R} \\
\therefore \sum\left(\frac{h_{a}}{r_{a}}\right)^{2}+\frac{10 r}{R}=\frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}-8+\frac{14 r}{R} \geq 8
\end{gathered}
$$



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$$
\begin{gathered}
\Leftrightarrow \frac{\left(s^{2}-8 R r+r^{2}\right)^{2}}{4 R^{2} r^{2}}+\frac{14 r}{R} \geq 16 \Leftrightarrow \frac{\left(s^{2}-8 R r+r^{2}\right)^{2}+56 R r^{3}}{4 R^{2} r^{2}} \geq 16 \\
\Leftrightarrow s^{4}-2 s^{2}\left(8 R r-r^{2}\right)+\left(8 R r-r^{2}\right)^{2}+56 R r^{3} \geq 64 R^{2} r^{2} \\
\Leftrightarrow s^{4}-2 s^{2}\left(8 R r-r^{2}\right)+40 R r^{3}+r^{4} \stackrel{(1)}{\geq} 0
\end{gathered}
$$

Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c$ $=x+y$
Then, $8 R r-r^{2}=\frac{8 a b c \Delta}{4 \Delta s}-\frac{(s-b)(s-c)(s-a)}{s} \stackrel{(i)}{=} \frac{2 \prod(y+z)-x y z}{\sum x}$

$$
a n d, 40 R r^{3}+r^{4}
$$

$$
=\frac{40 a b c \Delta}{4 \Delta s} \cdot \frac{(s-b)(s-c)(s-a)}{s}
$$

$$
+\left(\frac{(s-b)(s-c)(s-a)}{s}\right)^{2} \stackrel{(i i)}{=} \frac{10 x y z \prod(y+z)+x^{2} y^{2} z^{2}}{\left(\sum x\right)^{2}}
$$

(i), $(i i) \Rightarrow(1) \Leftrightarrow\left(\sum x\right)^{4}-2 \sum x\left\{2 \prod(y+z)-x y z\right\}+\frac{10 x y z \prod(y+z)+x^{2} y^{2} z^{2}}{\left(\sum x\right)^{2}} \geq 0$

$$
\Leftrightarrow\left(\sum x\right)^{6}-2\left(\sum x\right)^{3}\{2 \Pi(y+z)-x y z\}+10 x y z \Pi(y+z)+x^{2} y^{2} z^{2} \geq 0
$$

(2)
$\Leftrightarrow \sum x^{6}+2 \sum x^{5} y+2 \sum x y^{5}+3 x^{2} y^{2} z^{2} \geq \sum x^{4} y^{2}+\sum x^{2} y^{4}+4 \sum x^{3} y^{3}$
Now, $\sum x^{6}+3 x^{2} y^{2} z^{2} \stackrel{\text { Schur }}{\geq} \sum x^{4} y^{2}+\sum x^{2} y^{4}$ and $2 \sum x^{5} y+2 \sum x y^{5}$

$$
=2 \sum\left(x^{5} y+x y^{5}\right) \stackrel{A-G}{\geq} 4 \sum x^{3} y^{3} \text { and adding these two }
$$

(2) $\Rightarrow(1)$ is true $\therefore{\left(\frac{h_{a}}{r_{a}}\right)^{2}+\left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{10 r}{R}-8 \geq 0}_{\Rightarrow \text { it suffices to prove : }}$

$$
\begin{aligned}
\left(\frac{h_{a}}{r_{a}}\right)^{2}+ & \left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{2 \mu r}{R}-\mu-3 \geq\left(\frac{h_{a}}{r_{a}}\right)^{2}+\left(\frac{h_{b}}{r_{b}}\right)^{2}+\left(\frac{h_{c}}{r_{c}}\right)^{2}+\frac{10 r}{R}-8 \\
& \Leftrightarrow \frac{2 r}{R}(\mu-5) \geq \mu-5 \Leftrightarrow(\mu-5)\left(\frac{2 r}{R}-1\right) \geq 0 \\
& \rightarrow \text { true } \because \mu-5 \leq 0 \text { and } \frac{2 r}{R}-1 \stackrel{\text { Euler }}{\sim} 0 \text { (Hence proved) }
\end{aligned}
$$



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Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\frac{s-a}{a}+\frac{s-b}{b}+\frac{s-c}{c}=\frac{b c(s-a)+c a(s-b)+a b(s-c)}{a b c} \\
=\frac{s(a b+b c+c a)-3 a b c}{a b c}=\frac{s\left(s^{2}+4 R r+r^{2}\right)-12 R r s}{4 R r s} \\
=\frac{s^{2}-8 R r+r^{2}}{4 R r} \stackrel{s^{2} \geq 16 R r-5 r^{2}}{\sum^{2}} \frac{8 R r-4 r^{2}}{4 R r}=2-\frac{r}{R} \\
\quad \frac{s-a}{a} \cdot \frac{s-b}{b}+\frac{s-b}{b} \cdot \frac{s-c}{c}+\frac{s-c}{c} \cdot \frac{s-a}{a}= \\
=\frac{c(s-a)(s-b)+a(s-b)(s-c)+b(s-a)(s-c)}{a b c} \\
=\frac{2 s \cdot s^{2}-2(a b+b c+c a)+3 a b c}{a b c}=\frac{2 s^{3}-2\left(s^{2}+4 R r+r^{2}\right) s+12 R r s}{4 R r s} \\
=\frac{s^{2}-\left(s^{2}+4 R r+r^{2}\right)+6 R r}{2 R r}=1-\frac{r}{2 R} \\
\Rightarrow \sum_{c y c}\left(\frac{h_{a}}{r_{a}}\right)^{2}=4 \sum_{c y c}^{a}\left(\frac{s-a}{a}\right)^{2}=4\left[\left(\sum_{c y c}^{\frac{s}{2}} \frac{s-a}{a}\right)^{2}-2 \sum_{c y c} \frac{(s-a)(s-b)}{a b}\right] \\
\geq 4\left[\left(2-\frac{r}{R}\right)^{2}-2+\frac{r}{R}\right]^{t=\frac{r}{R} \leq \frac{1}{2}} \frac{\stackrel{m}{2}}{=} 4\left(t^{2}-3 t+2\right)
\end{gathered}
$$

We must show that: $4\left(t^{2}-3 t+2\right)+2 \mu t \geq \mu+1$

$$
\begin{equation*}
\Leftrightarrow 4 t^{2}+2(\mu-6) t+7-\mu \geq 0 \tag{*}
\end{equation*}
$$

$0<t \leq \frac{1}{2} ; \mu \leq 5<6 \Rightarrow 4 t^{2}+2(\mu-6) t+7-\mu \geq 0+2(\mu-6) t \cdot \frac{1}{2}+7-\mu=1>0$
$\Rightarrow$ (*) true.Proved.
1544. In $\triangle A B C$ the following relationship holds:

$$
\sqrt[n]{\frac{2 a}{b+c}}+\sqrt[n]{\frac{2 b}{c+a}}+\sqrt[n]{\frac{2 c}{a+b}} \leq \frac{3 R}{2 r}, \quad \forall n \geq 2
$$



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Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\operatorname{ssin}\left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{scos} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
& \therefore \mathrm{r}_{\mathrm{b}}+\mathrm{r}_{\mathrm{c}} \stackrel{(\mathrm{i})}{=} 4 \mathrm{Rcos}^{2} \frac{\mathrm{~A}}{2}
\end{aligned}
$$

$$
\begin{aligned}
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a) \\
& \Leftrightarrow(b+c)^{2}+4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \text { true } \\
& \therefore b+c \geq 4 \sqrt{2 R r} \cos \frac{A}{2} \Rightarrow \frac{2 a}{b+c} \leq \frac{2.4 R \sin \frac{A}{2} \cos \frac{A}{2}}{4 \sqrt{2 R r} \cos \frac{A}{2}}=\sqrt{\frac{2 R}{r}} \sin \frac{A}{2} \\
& \Rightarrow \ln \left(\frac{2 \mathrm{a}}{\mathrm{~b}+\mathrm{c}}\right) \leq \ln \left(\sqrt{\frac{2 R}{r}} \sin \frac{A}{2}\right) \Rightarrow \frac{1}{\mathrm{n}}\left\{\ln \left(\frac{2 \mathrm{a}}{\mathrm{~b}+\mathrm{c}}\right)-\ln \left(\sqrt{\frac{2 R}{r}} \sin \frac{A}{2}\right)\right\} \\
& \leq \mathbf{0}\left(\because \frac{1}{n}>0 \text { as } \mathbf{n} \geq 2\right) \Rightarrow \frac{1}{n} \cdot \ln \left(\frac{2 a}{b+c}\right) \leq \frac{1}{n} \cdot \ln \left(\sqrt{\frac{2 R}{r}} \sin \frac{A}{2}\right) \\
& \Rightarrow \ln \left(\sqrt[n]{\frac{2 \mathrm{a}}{\mathrm{~b}+\mathrm{c}}}\right) \leq \ln \left(\sqrt[n]{\left(\sqrt{\frac{2 \mathrm{R}}{\mathrm{r}}} \sin \frac{\mathrm{~A}}{2}\right)}\right) \Rightarrow \sqrt[n]{\frac{2 \mathrm{a}}{\mathrm{~b}+\mathrm{c}}} \leq \sqrt[n]{\left(\sqrt{\frac{2 \mathrm{R}}{\mathrm{r}}} \sin \frac{\mathrm{~A}}{2}\right)} \text { and analogs } \\
& \Rightarrow \sum \sqrt[n]{\frac{2 a}{b+c}} \stackrel{(i i)}{n}_{s}^{\left(\sqrt{\frac{2 R}{r}}\right.}\left(\sum \sqrt[n]{\sin \frac{A}{2}}\right) \\
& \text { Let } f(x)=\sqrt[n]{\sin \frac{x}{2}} \forall x \in(0, \pi) \text { and } \forall n \geq 2 \therefore f \quad "(x) \\
& =-\frac{\left(\sin \frac{x}{2}\right)^{\frac{1}{n}-2}\left\{n \sin ^{2} \frac{x}{2}+(n-1) \cos ^{2} \frac{x}{2}\right\}}{4 n^{2}}<0 \Rightarrow f(x) \text { is concave }
\end{aligned}
$$



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$$
\begin{aligned}
& \therefore \text { by Jensen, } \sqrt[n]{\frac{2 R}{r}}\left(\sum^{n} \sqrt[n]{\sin \frac{A}{2}}\right) \leq 3 \sqrt[n]{\sqrt{\frac{2 R}{r}}}\left(\sqrt[n]{\sin \frac{A+B+C}{6}}\right)=3 \sqrt[n]{\frac{1}{2} \sqrt{\frac{2 R}{r}}} \\
& \Rightarrow \sqrt[n]{\frac{2 R}{r}}\left(\sum \sqrt[n]{\sin \frac{A}{2}}\right) \stackrel{\text { (iii) }}{\sim} 3^{\frac{n}{\leq}} \sqrt{\frac{1}{2} \sqrt{\frac{2 R}{r}}} \\
& \text { Now, } \frac{1}{n}-\frac{1}{2} \stackrel{\substack{n \geq 2}}{\sim} 0 \text { and } \ln \left(\frac{1}{2} \sqrt{\frac{2 \mathrm{R}}{\mathrm{r}}}\right) \geq 0 \text { as } \frac{1}{2} \sqrt{\frac{2 \mathrm{R}}{\mathrm{r}}} \stackrel{\text { Euler }}{\geq} 1 \text { and } \\
& \therefore\left(\frac{1}{n}-\frac{1}{2}\right) \cdot \ln \left(\frac{1}{2} \sqrt{\frac{2 R}{r}}\right) \leq 0 \Rightarrow \frac{1}{n} \cdot \ln \left(\frac{1}{2} \sqrt{\frac{2 R}{r}}\right) \leq \frac{1}{2} \cdot \ln \left(\frac{1}{2} \sqrt{\frac{2 R}{r}}\right) \\
& \Rightarrow \ln \sqrt[n]{\frac{1}{2} \sqrt{\frac{2 R}{r}}} \leq \ln \sqrt{\frac{1}{2} \sqrt{\frac{2 R}{r}}} \Rightarrow 3 \sqrt[n]{\frac{1}{2} \sqrt{\frac{2 R}{r}}} \stackrel{(\text { (iv) }}{\leq} 3 \sqrt{\frac{1}{2} \sqrt{\frac{2 R}{r}}} . \text { (ii), (iii), (iv) } \Rightarrow \\
& \sum \sqrt[n]{\frac{2 a}{b+c}} \leq 3 \sqrt{\frac{1}{2} \sqrt{\frac{2 R}{r}} \stackrel{?}{\dot{s}}} \frac{3 R}{2 r} \Leftrightarrow \frac{1}{2} \sqrt{\frac{2 R}{r}} \stackrel{?}{\sim} \frac{R^{2}}{4 r^{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow \sqrt[n]{\frac{2 a}{b+c}}+\sqrt[n]{\frac{2 b}{c+a}}+\sqrt[n]{\frac{2 c}{a+b}} \leq \frac{3 R}{2 r} \forall n \geq 2 \text { (Proved) }
\end{aligned}
$$

1545. In $\triangle A B C$ the following relationship holds:

$$
2 \leq \frac{r_{a}-r}{h_{a}}+\frac{r_{b}-r}{h_{b}}+\frac{r_{c}-r}{h_{c}} \leq \frac{R^{2}-2 r^{2}}{r^{2}}
$$

Proposed by K.Geronikolas,George Apostolopoulos-Greece
Solution by Daniel Sitaru-Romania

$$
\sum_{c y c} \frac{r_{a}-r}{h_{a}}=\sum_{c y c} \frac{\frac{S}{s-a}-\frac{S}{s}}{\frac{2 S}{a}}=\sum_{c y c} \frac{(s-s+a) a}{2 s(s-a)}=
$$



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$$
\begin{gathered}
=\frac{1}{2 s} \sum_{c y c} \frac{a^{2}}{s-a}=\frac{1}{2 s} \cdot \frac{4 s(R-r)}{r}=\frac{2(R-r)}{r} \\
\sum_{c y c} \frac{r_{a}-r}{h_{a}} \geq 2 \leftrightarrow \frac{2(R-r)}{r} \geq 2 \leftrightarrow R \geq 2 r \\
\sum_{c y c} \frac{r_{a}-r}{h_{a}} \leq \frac{R^{2}-2 r^{2}}{r^{2}} \leftrightarrow \frac{2(R-r)}{r} \leq \frac{R^{2}-2 r^{2}}{r^{2}} \leftrightarrow R \geq 2 r
\end{gathered}
$$

1546. In $\triangle A B C, n_{a}$-Nagel's cevian, the following relationship holds:

$$
2 \sum \frac{r_{a}}{s-n_{a}} \geq 3 \sqrt{3}+\sum \frac{n_{a}}{h_{a}}
$$

## Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathbf{s}^{2}-\mathrm{s}(2 \mathrm{~s}-\mathrm{a})+\mathrm{bc}\right) \Rightarrow \mathrm{s}\left(\mathrm{~b}^{2}+\mathrm{c}^{2}\right)-2 \mathrm{sbc} \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+c^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{\mathrm{a}}{ }^{2}-\mathbf{a s}^{2} \Rightarrow \mathbf{a n}_{\mathrm{a}}{ }^{2}=\mathbf{a s}{ }^{2}+\mathbf{s}(2 b c \cos A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \mathbf{s b c}(s-b)(s-c)(s-a)}{b c(s-a)} \\
& =a s^{2}-\frac{4 \Delta^{2}}{s-a}=a s^{2}-2 a\left(\frac{2 \Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=a s^{2}-2 a h_{a} r_{a} \therefore n_{a} \stackrel{(1)}{\mathscr{m}} s^{2}-2 h_{a} r_{a} \\
& \text { Now, } 2 \sum \frac{\mathbf{r}_{\mathrm{a}}\left(\mathbf{s}+\mathbf{n}_{\mathrm{a}}\right)}{\left(\mathbf{s}-\mathbf{n}_{\mathrm{a}}\right)\left(\mathbf{s}+\mathbf{n}_{\mathrm{a}}\right)}=2 \sum \frac{\mathbf{r}_{\mathrm{a}}\left(\mathbf{s}+\mathbf{n}_{\mathrm{a}}\right)}{\mathbf{s}^{2}-\mathbf{n}_{\mathrm{a}}{ }^{2}} \stackrel{\text { by }(1) \text { and analogs }}{\cong} 2 \sum \frac{\mathbf{r}_{\mathrm{a}}\left(\mathbf{s}+\mathbf{n}_{\mathrm{a}}\right)}{2 \mathbf{h}_{\mathrm{a}} \mathbf{r}_{\mathrm{a}}} \\
& =\sum \frac{\mathbf{s}+\mathbf{n}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}}=\frac{\mathbf{s} \sum \mathbf{a}}{2 \mathbf{r} \mathbf{s}}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}}=\frac{\mathbf{s}}{\mathbf{r}}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}} \\
& \stackrel{\text { Mitrinovic }}{\underset{\sim}{2}} 3 \sqrt{3}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{h}_{\mathrm{a}}} \text { (Proved) }
\end{aligned}
$$



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1547. In $\triangle A B C, n_{a}$-Nagel's cevian, $\boldsymbol{g}_{a}$-Gergonne's cevian, the following relationship holds:

$$
n_{a}+g_{a} \geq \max \left(2 m_{a}, \frac{b^{2}+c^{2}}{2 R}\right)
$$

## Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow b^{2}(s-c)+c^{2}(s-b) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(s-b)+\mathbf{c}^{2}(s-c) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \text { Adding the above two, we get : }\left(b^{2}+c^{2}\right)(2 s-b-c) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a g}_{\mathrm{a}}{ }^{2}+2 \mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(n_{a}{ }^{2}+\mathrm{ga}^{2}\right)+\mathbf{a}(\mathbf{a}+\mathrm{b}-\mathbf{c})(\mathbf{c}+\mathrm{a}-\mathrm{b}) \Rightarrow \mathbf{2}\left(\mathrm{b}^{2}+\mathbf{c}^{2}\right) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \Rightarrow 4 m_{a}{ }^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \\
& \Rightarrow 4 m_{a}^{2}+(b-c)^{2}+4 r_{b} r_{c}=2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+4 r_{b} r_{c} \Rightarrow 4 m_{a}{ }^{2}+(b-c)^{2}+4 s(s-a) \\
& =2\left(n_{a}^{2}+g_{a}^{2}\right)+4 s(s-a) \\
& \text { (1) }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (a) } \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\}{ }^{\text {² }} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$

Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c $=x+y$

Using these substitutions, (a)
$\Leftrightarrow\left\{z(z+x)^{2}+y(x+y)^{2}-y z(y+z)\right\}\left\{y(z+x)^{2}+z(x+y)^{2}-y z(y\right.$
$+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$


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$\Leftrightarrow x^{2}+x z^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow$ true

$$
\Rightarrow(\mathbf{a}) \text { is true } \Rightarrow \mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}} \stackrel{(2)}{\geq} \mathrm{s}(\mathrm{~s}-\mathbf{a})
$$

by (1) and (2)

$$
\begin{gathered}
\text { Now, } \mathbf{n}_{\mathrm{a}}{ }^{2}+\mathbf{g}_{\mathrm{a}}{ }^{2}+2 \mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}} \stackrel{\mathrm{n}}{\geq} 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}-2 \mathrm{~s}(\mathrm{~s}-\mathrm{a})+2 \mathrm{~s}(\mathrm{~s}-\mathrm{a})=4 \mathrm{~m}_{\mathrm{a}}{ }^{2} \\
\Rightarrow\left(\mathbf{n}_{\mathrm{a}}+\mathbf{g}_{\mathrm{a}}\right)^{2} \geq 4 \mathbf{m}_{\mathrm{a}}^{2} \Rightarrow n_{a}+g_{a} \geq 2 m_{a} \stackrel{\text { Tereshin }}{\geq} \frac{b^{2}+c^{2}}{2 R} \\
\therefore \mathbf{n}_{\mathrm{a}}+\mathbf{g}_{\mathrm{a}} \geq \max \left(2 \mathrm{~m}_{\mathrm{a}} \frac{\mathbf{b}^{2}+\mathbf{c}^{2}}{2 \mathrm{R}}\right) \text { (Proved) }
\end{gathered}
$$

1548. In $\triangle A B C, n_{a}$-Nagel's cevian, $\boldsymbol{g}_{a}$-Gergonne's cevian, the following relationship holds:

$$
\sqrt{n_{a} g_{a} r_{a}}+\sqrt{n_{b} g_{b} r_{b}}+\sqrt{n_{c} g_{c} r_{c}} \geq 3 s \sqrt{r}
$$

## Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{align*}
& \quad \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \therefore \mathbf{a n}_{\mathrm{a}}{ }^{2} \cdot \mathbf{a g}_{\mathbf{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{(\mathbf{a})}{\geq} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \tag{a}
\end{align*}
$$

Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, (a)

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right. \\
& +z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2} \\
& \Leftrightarrow \mathrm{xy}^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq 2 \mathrm{xyz}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq 0 \rightarrow \text { true } \\
& \Rightarrow(a) \text { is true } \Rightarrow \mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}} \geq \mathbf{s}(\mathbf{s}-\mathbf{a})
\end{aligned}
$$



$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \begin{aligned}
\Rightarrow \mathbf{n}_{\mathbf{a}} \mathbf{g}_{\mathrm{a}} \mathbf{r}_{\mathbf{a}} \geq \mathbf{s}(\mathbf{s}-\mathbf{a})\left(\frac{\mathbf{r s}}{\mathbf{s}-\mathbf{a}}\right)=\mathbf{r s}^{2} \Rightarrow \sqrt{\mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}} \mathbf{r}_{\mathrm{a}}} \geq \mathbf{s} \sqrt{\mathbf{r}} \text { and analogs } \\
\Rightarrow \sqrt{\mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathbf{a}} \mathbf{r}_{\mathbf{a}}}+\sqrt{\mathbf{n}_{\mathrm{b}} \mathbf{g}_{\mathrm{b}} \mathbf{r}_{\mathbf{b}}}+\sqrt{\mathbf{n}_{\mathbf{c}} \mathbf{g}_{\mathbf{c}} \mathbf{r}_{\mathbf{c}}} \geq \mathbf{3 s} \sqrt{\mathbf{r}} \text { (Proved) }
\end{aligned}
\end{aligned}
$$

1549. In $\triangle A B C, n_{a}$-Nagel's cevian, $\boldsymbol{g}_{a}$-Gergonne's cevian, the following relationship holds:

$$
b^{2}+c^{2} \geq 2 n_{a} g_{a}+2 r r_{a} \geq 2 b c
$$

## Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathrm{c})+\mathbf{c}^{2}(\mathrm{~s}-\mathrm{b}) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c}) \text { and } \mathbf{b}^{2}(\mathrm{~s}-\mathrm{b})+\mathbf{c}^{2}(\mathrm{~s}-\mathrm{c}) \\
& =\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \text { Adding the above two, we get: }\left(\mathrm{b}^{2}+\mathrm{c}^{2}\right)(2 \mathrm{~s}-\mathrm{b}-\mathrm{c}) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{ag}_{\mathrm{a}}{ }^{2}+2 \mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathrm{c}^{2}\right)=\mathbf{2 a}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}_{\mathrm{a}}{ }^{2}\right)+\mathbf{a}(\mathrm{a}+\mathrm{b}-\mathbf{c})(\mathrm{c}+\mathrm{a}-\mathrm{b}) \Rightarrow \mathbf{2}\left(\mathrm{b}^{2}+\mathrm{c}^{2}\right) \\
& =2\left(n_{a}^{2}+g_{a}{ }^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+(\mathrm{b}-\mathrm{c})^{2}+4 \mathbf{r}_{\mathrm{b}} \mathrm{r}_{\mathrm{c}}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2}\right)+\mathbf{4} \mathrm{r}_{\mathrm{b}} \mathrm{r}_{\mathrm{c}} \Rightarrow \mathbf{4 m} \mathrm{a}_{\mathrm{a}}{ }^{2}+(\mathrm{b}-\mathrm{c})^{2}+\mathbf{4 s}(\mathrm{s}-\mathrm{a}) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+4 s(s-a) \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+4 \mathrm{~m}_{\mathrm{a}}{ }^{2}=2\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2}\right)+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a}) \Rightarrow \mathrm{na}^{2}+\mathrm{ga}^{2} \stackrel{(1)}{\cong} 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}-\mathbf{2 s}(\mathrm{s}-\mathrm{a})
\end{aligned}
$$

$$
\begin{aligned}
& +\frac{4(s-a)(s-b)(s-c)}{s-a} \\
& =4 b^{2}+4 c^{2}-2 a^{2}-(a+b+c)(b+c-a)+(c+a-b)(a+b-c) \\
& =4 b^{2}+4 c^{2}-2 a^{2}-(b+c)^{2}+a^{2}+a^{2}-(b-c)^{2} \\
& \text { (i) } \\
& =4 b^{2}+4 c^{2}-\left(2 b^{2}+2 c^{2}\right)=2 b^{2}+2 c^{2} \Rightarrow b^{2}+c^{2} \geq 2 n_{a} g_{a}+2 r r_{a}
\end{aligned}
$$



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$$
\text { Again, } 4 n_{a} g_{a}+4 r r_{a} \geq 4 b c \Leftrightarrow 4 n_{a} g_{a}+4\left(\frac{r^{2} s}{s-a}\right) \geq 4 b c
$$

$$
\Leftrightarrow 4 n_{\mathrm{a}} g_{\mathrm{a}}+\frac{4(s-a)(s-b)(s-c)}{s-a} \geq 4 b c
$$

$$
\Leftrightarrow 4 n_{a} g_{a}+(c+a-b)(a+b-c) \geq 4 b c \Leftrightarrow 4 n_{a} g_{a}+a^{2}-(b-c)^{2} \geq 4 b c \Leftrightarrow 4 n_{a} g_{a}
$$

$$
\geq(b+c)^{2}-a^{2}=4 s(s-a)
$$

$$
\Leftrightarrow \mathbf{a n}_{\mathbf{a}}^{2} \cdot \mathbf{a g}_{\mathbf{a}}^{2} \geq \boldsymbol{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
$$

$$
\Leftrightarrow\left\{\mathbf{b}^{2}(s-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right.
$$

(a)
$-b)(s-c)\} \stackrel{\text { m }}{\geq} a^{2} s^{2}(s-a)^{2}$
Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, (a)

$$
\Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathbf{x})^{2}+\mathbf{y}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathbf{x})^{2}+\mathrm{z}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}\right.
$$

$$
+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}
$$

$$
\begin{equation*}
\Leftrightarrow x^{2}+x^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow \operatorname{tr} u e \tag{ii}
\end{equation*}
$$

$\Rightarrow(a)$ is true $\Rightarrow 2 n_{a} g_{a}+2 r r_{a} \geq 2 b c$

$$
\text { (i), (ii) } \Rightarrow b^{2}+c^{2} \geq 2 n_{a} g_{a}+2 r r_{a} \geq 2 b c(\text { Proved })
$$

1550. In $\triangle A B C, n_{a}$-Nagel's cevian, the following relationship holds:

$$
2 \sum \frac{\mathbf{h}_{\mathrm{a}}}{s-\mathbf{n}_{\mathrm{a}}} \geq 3 \sqrt{3}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathrm{r}_{\mathrm{a}}}
$$

## Proposed by Bogdan Fuștei-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(s-c)+\mathbf{c}^{2}(s-b)=\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(s-b)(s-c) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}\left(s^{2}-s(2 s-a)+b c\right) \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c \\
& =\mathrm{an}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+\mathbf{c}^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{\mathrm{a}}{ }^{2}-\mathbf{a s}^{2} \Rightarrow \mathrm{an}_{\mathrm{a}}{ }^{2}=\mathbf{a s}{ }^{2}+\mathbf{s}(2 \mathrm{bccos} A-2 b c) \\
& =\mathbf{a s}^{2}-4 \mathbf{s b c s i n}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \operatorname{sbc}(s-b)(s-c)(s-a)}{b c(s-a)}
\end{aligned}
$$



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$$
\begin{aligned}
& =\mathbf{a s}^{2}-\frac{4 \Delta^{2}}{s-a}=\mathbf{a s}^{2}-2 \mathrm{a}\left(\frac{2 \Delta}{\mathrm{a}}\right)\left(\frac{\Delta}{s-a}\right)=\mathrm{as}^{2}-2 \mathrm{ah}_{\mathrm{a}} \mathrm{r}_{\mathrm{a}} \therefore \mathbf{n}_{\mathrm{a}}{ }^{2} \stackrel{(1)}{m} \mathbf{s}^{2}-2 h_{\mathrm{a}} \mathbf{r}_{\mathrm{a}} \\
& \text { Now, } 2 \sum \frac{h_{a}\left(s+n_{a}\right)}{\left(s-n_{a}\right)\left(s+n_{a}\right)}=2 \sum \frac{h_{a}\left(s+n_{a}\right)}{s^{2}-n_{a}^{2}} \stackrel{\text { by (1) and analogs }}{\left.\stackrel{m}{=} 2 \sum \frac{h_{a}\left(s+n_{a}\right)}{2 h_{a} r_{a}}\right) ~} \\
& =\sum \frac{s+\mathbf{n}_{\mathbf{a}}}{\mathbf{r}_{\mathrm{a}}}=\mathbf{s} \sum \frac{1}{\mathbf{r}_{\mathrm{a}}}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{r}_{\mathrm{a}}}=\frac{\mathbf{s}}{\mathbf{r}}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{r}_{\mathbf{a}}}
\end{aligned}
$$

$\stackrel{\text { Mitrinovic }}{\sum^{n}} 3 \sqrt{3}+\sum \frac{\mathbf{n}_{\mathrm{a}}}{\mathbf{r}_{\mathrm{a}}}$ (Proved)
1551. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} m_{a} \cdot \sqrt{h_{a}^{2}+h_{b} h_{c}+h_{c}^{2}} \geq\left(3 r^{2}-\frac{R^{2}}{4}\right) \cdot \frac{27 \sqrt{3}}{2}
$$

## Proposed by Marin Chirciu-Romania

## Solution by George Florin Şerban-Romania

We show that: $\sqrt{a^{2}+a b+b^{2}} \geq \frac{(a+b) \sqrt{3}}{2}, \forall a, b \geq 0 \Leftrightarrow$

$$
\begin{aligned}
& 4\left(a^{2}+a b+b^{2}\right) \geq 3(a+b)^{2} \Leftrightarrow(a-b)^{2} \geq 0 \text { true for any } a, b \geq 0 \\
& \sum_{c y c} m_{a} \cdot \sqrt{h_{a}^{2}+h_{b} h_{c}+h_{c}^{2}} \stackrel{A m-H m}{\geq} \sum_{c y c} h_{a} \frac{\sqrt{3}}{2}\left(h_{b}+h_{c}\right)=\frac{\sqrt{3}}{2} \sum_{c y c} h_{a}\left(h_{b}+h_{c}\right) \\
& =\frac{2 \sqrt{3}}{2} \sum_{c y c} h_{a} h_{b}=\frac{\sqrt{3}}{2} \cdot \frac{4 r s^{2}}{R} \stackrel{?}{\gtrless}\left(3 r^{2}-\frac{R^{2}}{4}\right) \cdot \frac{27 \sqrt{3}}{2} \\
& \frac{4 r s^{2}}{R} \stackrel{?}{2}_{\geq}^{\geq} 81 r^{2}-\frac{27 R^{2}}{4} \\
& \frac{4 r s^{2}}{R} \stackrel{\text { Gerretsen }}{\geq} \frac{4 r\left(16 R r-5 r^{2}\right)}{R} \stackrel{\sim}{2}_{\geq}^{?} 81 r^{2}-\frac{27 R^{2}}{4} \\
& \left.\frac{4 r\left(16 R r-5 r^{2}\right)}{R} \geq 81 r^{2}-\frac{27 R^{2}}{4} \right\rvert\,: r^{2} \\
& \frac{4\left(16 R r-5 r^{2}\right)}{R} \geq 81 r-\frac{27}{4}\left(\frac{R}{r}\right)^{2} \\
& 4\left(16-\frac{5 r}{R}\right) \geq 81-\frac{27}{4}\left(\frac{R}{r}\right)^{\frac{2}{r}=t \geq 2(\text { Euler })} \stackrel{27}{4} t^{2}-\frac{20}{t}-17 \geq 0
\end{aligned}
$$



## ROMANIAN MATHEMATICAL MAGAZINE <br> www.ssmrmh.ro <br> $27 t^{3}-68 t-80 \geq 0 \Leftrightarrow(t-2)\left(27 t^{2}+54 t+40\right) \geq 0$ true for $t \geq 2$ (Euler).

## Proved.

1552. If $x, y>0,4 x=3 y+1$ then in acute $\triangle A B C$ the following relationship holds:

$$
\frac{x-y \cos A}{x+\cos A}+\frac{x-y \cos B}{x+\cos B}+\frac{x-y \cos C}{x+\cos C} \geq 1
$$

## Proposed by Marin Chirciu-Romania

Solution 1 by Khaled Abd Imouti-Damascus-Syria

$$
\begin{gathered}
f(\theta)=\frac{x-y \cos \theta}{x+\cos \theta}, 0<\theta \leq \frac{\pi}{2} \\
f^{\prime}(\theta)=\frac{(x y+x) \sin \theta}{(x+\cos \theta)^{2}} \Rightarrow \\
f^{\prime \prime}(\theta)=\frac{(x y+x)\left(\cos \theta(x+\cos \theta)^{2}+2(x+\cos \theta) \sin ^{2} \theta\right)}{(x+\cos \theta)^{2}}>0
\end{gathered}
$$

$$
f(A)+f(B)+f(C) \stackrel{\text { Jensen }}{\approx} 3 f\left(\frac{A+B+C}{3}\right)=3 f\left(\frac{\pi}{3}\right)
$$

$$
f\left(\frac{\pi}{3}\right)=\frac{x-\frac{y}{2}}{x+\frac{1}{2}}=\frac{4 x-2 y}{4 x+2} \text { but } 4 x=3 y+1 \text { then }
$$

$$
f\left(\frac{\pi}{3}\right)=\frac{3 y+1-2 y}{3 y+1+2}=\frac{y+1}{3(y+1)}=\frac{1}{3}
$$

$$
\text { So, } \frac{x-y \cos A}{x+\cos A}+\frac{x-y \cos B}{x+\cos B}+\frac{x-y \cos C}{x+\cos C} \geq 1 . \text { Proved. }
$$

## Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y>0,4 x=3 y+1$ and acute $\triangle A B C$ we have $y=\frac{4 x-1}{3}$.

$$
\begin{gathered}
\text { Hence } \frac{x-y \cos A}{x+\cos A}+\frac{x-y \cos B}{x+\cos B}+\frac{x-y \cos C}{x+\cos C}= \\
=\frac{x-\frac{4 x-1}{3} \cos A}{x+\cos A}+\frac{x-\frac{4 x-1}{3} \cos B}{x+\cos B}+\frac{x-\frac{4 x-1}{3} \cos C}{x+\cos C} \\
=\frac{3 x+\cos A-4 x \cos A}{3(x+\cos A)}+\frac{3 x+\cos B-4 x \cos B}{3(x+\cos B)}+\frac{3 x+\cos C-4 x \cos C}{3(x+\cos C)}
\end{gathered}
$$



$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \text { www.ssmrmh.ro } \\
& =\frac{1}{3}+\frac{2 x-4 x \cos A}{3(x+\cos A)}+\frac{1}{3}+\frac{2 x-4 x \cos B}{3(x+\cos B)}+\frac{1}{3}+\frac{2 x-4 x \cos C}{3(x+\cos C)} \geq 1 \\
& \text { Iff } \\
& \frac{2 x}{3(x+\cos A)}+\frac{2 x}{3(x+\cos B)}+\frac{2 x}{3(x+\cos C)} \geq \frac{4 x \cos A}{3(x+\cos A)}+\frac{4 x \cos B}{3(x+\cos B)}+\frac{4 x \cos C}{3(x+\cos C)} \\
& \text { Iff } \frac{1}{x+\cos A}+\frac{1}{x+\cos B}+\frac{1}{x+\cos C} \geq \frac{2 \cos A}{x+\cos A}+\frac{2 \cos B}{x+\cos B}+\frac{2 \cos C}{x+\cos C} \\
& (x+\cos A)(x+\cos B)+(x+\cos B)(x+\cos C)+(x+\cos C)(x+\cos A) \geq \\
& \geq 2 \cos C(x+\cos A)(x+\cos B)+2 \cos A(x+\cos B)(x+\cos C) \\
& +2 \cos B(x+\cos C)(x+\cos A) \\
& \left(3 x^{2}+2 x \cos A \cos B \cos C\right)+\cos A \cos B+\cos B \cos C+\cos C \cos A \geq \\
& \geq 2 x^{2}(\cos A+\cos B+\cos C)+4 x(\cos A \cos B+\cos B \cos C+\cos C \cos A) \\
& +6 \cos A \cos B \cos C \text { true, because } \\
& \cos A+\cos B+\cos C \leq \frac{3}{2} \text { then } \\
& 3 x^{2} \geq 2 x^{2}(\cos A+\cos B+\cos C) \\
& 2 x \cos A \cos B \cos C \geq 4 x(\cos A \cos B+\cos B \cos C+\cos C \cos A) \text { and } \\
& (\cos A \cos B+\cos B \cos C+\cos C \cos A) \geq 6 \cos A \cos B \cos C \text {. Proved. }
\end{aligned}
$$

## Solution 3 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\text { Let: } \varphi(t)=\frac{x-y t}{x+t}(\forall t>0 ; x, y>0 ; x, y-\text { fixed. } \\
\varphi^{\prime}(t)=-\frac{x y+x}{(t+x)^{2}}<0(\forall t>0 ; x, y>0) \\
\varphi^{\prime \prime}(t)=\frac{2(x y+x)}{(t+x)^{3}}>0 \quad(\forall t>0 ; x, y>0) \\
\Delta A B C \text { acute } \Rightarrow \cos A, \cos B, \cos C>0 \Rightarrow \cos A+\cos B+\cos C \leq \frac{3}{2} \\
\sum_{c y c} \frac{x-y \cos A}{x+\cos A}=\varphi(\cos A)+\varphi(\cos B)+\varphi(\cos C) \stackrel{\sim}{\geq} 3 \cdot \frac{x-y \cdot\left(\frac{\sum_{c y c} \cos A}{3}\right)}{x+\frac{\sum_{c y c} \cos A}{3}} \\
\underset{\varphi \downarrow(0, \infty)}{\geq} 3 \cdot \frac{x-y \cdot \frac{1}{2}}{x+\frac{1}{2}}=3 \cdot \frac{2 x-y}{2 x+1}=\frac{6 x-3 y}{2 x+1} \stackrel{3 y=4 x-1}{=} \frac{2 x+1}{2 x+1}=1 . \text { Proved. }
\end{gathered}
$$



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1553. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} \frac{m_{a}^{2}}{b^{2}+c^{2}} \geq \frac{3 r}{2 R} \sum_{c y c} \sin ^{2} \frac{A}{2}
$$

Proposed by Marin Chirciu-Romania
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \sum \frac{4 m_{a}^{2}}{b^{2}+c^{2}}=\sum \frac{2\left(b^{2}+c^{2}\right)-a^{2}}{b^{2}+c^{2}}=6-\sum \frac{a^{2}}{b^{2}+c^{2}} \geq 6-2 \sum \frac{a^{2}}{(b+c)^{2}} \\
& \Rightarrow \sum \frac{4 m_{a}^{2}}{b^{2}+c^{2}} \stackrel{(1)}{\gtrless} 6-2 \sum \frac{a^{2}}{(b+c)^{2}} \\
& \text { Now, } \sum \frac{a^{2}}{(b+c)^{2}}=\sum \frac{(2 s-(b+c))^{2}}{(b+c)^{2}} \\
& =\sum \frac{4 s^{2}-4 s(b+c)+(b+c)^{2}}{(b+c)^{2}} \stackrel{(i)}{=} 4 s^{2}\left[\frac{\sum\left\{(c+a)^{2}(a+b)^{2}\right\}}{\left\{\prod(b+c)\right\}^{2}}\right] \\
& -4 s\left[\frac{\sum(c+a)(a+b)}{\prod(b+c)}\right]+3 \\
& \sum\left\{(c+a)^{2}(a+b)^{2}\right\}=\sum\left(\sum a b+a^{2}\right)^{2}=\sum\left\{\left(\sum a b\right)^{2}+2 a^{2} \sum a b+a^{4}\right\} \\
& =3\left(\sum a b\right)^{2}+2\left(\sum a b\right)\left(\sum a^{2}\right)+\left(\sum a^{2}\right)^{2}-2 \sum a^{2} b^{2} \\
& =\left(\sum a b\right)^{2}+2\left(\sum a b\right)\left(\sum a^{2}\right)+\left(\sum a^{2}\right)^{2}+2 \sum a^{2} b^{2}+4 a b c(2 s)-2 \sum a^{2} b^{2} \\
& =\left(\sum a b+\sum a^{2}\right)^{2}+32 R r s^{2} \\
& =\left(3 s^{2}-4 R r-r^{2}\right)^{2}+32 R r s^{2} \\
& \therefore \sum\left\{(c+a)^{2}(a+b)^{2}\right\} \stackrel{(i i)}{=}\left(3 s^{2}-4 R r-r^{2}\right)^{2}+32 R r s^{2} \\
& \text { Again, } \sum(c+a)(a+b)=\sum\left(\sum a b+a^{2}\right)=3 \sum a b+\sum a^{2} \\
& =\sum a^{2}+2 \sum a b+\sum a b=4 s^{2}+s^{2}+4 R r+r^{2}
\end{aligned}
$$



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$$
\begin{gathered}
\therefore \begin{array}{c}
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\therefore \sum(c+a)(a+b) \stackrel{(i i i)}{=} 5 s^{2}+4 R r+r^{2} \\
\because \prod(b+c)=s^{2}+2 R r+r^{2} \therefore(i),(i i),(i i i) \Rightarrow \sum \frac{a^{2}}{(b+c)^{2}} \\
=\frac{4 s^{2}\left\{\left(3 s^{2}-4 R r-r^{2}\right)^{2}+32 R r s^{2}\right\}}{4 s^{2}\left(s^{2}+2 R r+r^{2}\right)^{2}}-\frac{4 s\left(5 s^{2}+4 R r+r^{2}\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)}+3 \\
=\frac{\left(3 s^{2}-4 R r-r^{2}\right)^{2}+32 R r s^{2}-2\left(5 s^{2}+4 R r+r^{2}\right)\left(s^{2}+2 R r+r^{2}\right)+3\left(s^{2}+2 R r+r^{2}\right)^{2}}{\left(s^{2}+2 R r+r^{2}\right)^{2}}
\end{array}
\end{gathered}
$$

$$
\stackrel{(i v)}{\stackrel{\left(2 s^{4}-s^{2}\left(8 R r+12 r^{2}\right)+12 R^{2} r^{2}+8 R r^{3}+2 r^{4}\right.}{\left(s^{2}+2 R r+r^{2}\right)^{2}} \therefore(1),(i v) \Rightarrow \sum \frac{4 m_{a}^{2}}{b^{2}+c^{2}} \text {. }}
$$

$$
\geq 6-\frac{4 s^{4}-s^{2}\left(16 R r+24 r^{2}\right)+24 R^{2} r^{2}+16 R r^{3}+4 r^{4}}{\left(s^{2}+2 R r+r^{2}\right)^{2}}
$$

$$
\stackrel{?}{\underset{\sim}{\sim}} \frac{12 r}{R} \sum \sin ^{2} \frac{A}{2}=\frac{6 r}{R} \sum(1-\cos A)=\frac{6 r}{R}\left(3-1-\frac{r}{R}\right)=\frac{6 r(2 R-r)}{R^{2}}
$$

$$
\Leftrightarrow R^{2}\left[6\left(s^{2}+2 R r+r^{2}\right)^{2}\right.
$$

$$
\left.-\left\{4 s^{4}-s^{2}\left(16 R r+24 r^{2}\right)+24 R^{2} r^{2}+16 R r^{3}+4 r^{4}\right\}\right] \stackrel{?}{\sum} 6 r(2 R
$$

$$
-r)\left(s^{2}+2 R r+r^{2}\right)^{2}
$$

$$
\Leftrightarrow\left(R^{2}-6 R r+3 r^{2}\right) s^{4}+\left(20 R^{3} r-6 R^{2} r^{2}+6 r^{4}\right) s^{2} \stackrel{\text { ? }}{\geq} 20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}
$$

$$
-3 r^{6}
$$

$$
\Leftrightarrow(R-2 r)^{2} s^{4}+\left(20 R^{3} r-6 R^{2} r^{2}+6 r^{4}\right) s^{2} \underset{(2)}{{\underset{己}{2}}^{?}} 20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}-3 r^{6}
$$

$$
+\left(2 R r+r^{2}\right) s^{4}
$$

Now, LHS of (2)
$\qquad$
$\qquad$ $(R-2 r)^{2}\left(16 R r-5 r^{2}\right) s^{2}+\left(20 R^{3} r-6 R^{2} r^{2}\right.$ $\left.+6 r^{4}\right) s^{2}$ and and RHS of $(3) \underbrace{\substack{\text { Gerretsen }}}_{(b)}$

$$
20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}-3 r^{6}+\left(2 R r+r^{2}\right)\left(4 R^{2}+4 R r+3 r^{2}\right) s^{2} \therefore(a),(b)
$$

$\Rightarrow$ in order to prove (2), it suffices to prove :


## Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
m_{a} \geq \frac{b^{2}+c^{2}}{4 R} \text { and analogs. } \\
\sum_{c y c} \frac{m_{a}^{2}}{b^{2}+c^{2}} \geq \sum_{c y c} \frac{\left(b^{2}+c^{2}\right)^{2}}{(4 R)^{2}\left(b^{2}+c^{2}\right)}=\frac{1}{(4 R)^{2}} \sum_{c y c}\left(b^{2}+c^{2}\right)=\frac{1}{16 R^{2}}\left(2 a^{2}+2 b^{2}+2 c^{2}\right) \\
=\frac{a^{2}+b^{2}+c^{2}}{8 R^{2}}=\frac{2\left(s^{2}-4 R r-r^{2}\right)}{8 R^{2}}=\frac{s^{2}-4 R r-r^{2}}{4 R^{2}} \\
\sum_{c y c} \sin ^{2} \frac{A}{2}=\frac{2 R-r}{2 R} \Rightarrow \frac{3 r}{2 R} \sum_{c y c} \sin ^{2} \frac{A}{2}=\frac{3 r}{2 R} \cdot \frac{2 R-r}{2 R}
\end{gathered}
$$

$$
\text { We need to prove: } \frac{s^{2}-4 R r-r^{2}}{4 R^{2}} \geq \frac{3 r}{2 R} \cdot \frac{2 R-r}{2 R} \Leftrightarrow
$$

$$
s^{2}-4 R r-r^{2} \geq 6 R r-3 r^{2} \Leftrightarrow s^{2} \geq 10 R r-2 r^{2}
$$

But: $s^{2} \geq 16 R r-5 r^{2}$ then

$$
\begin{aligned}
& 16 R r-5 r^{2} \geq 10 R r-2 r^{2} \Leftrightarrow 6 R r \geq 3 r^{2} \Leftrightarrow 2 R \geq r \Leftrightarrow R \geq \frac{r}{2} \text { true, because } \\
& R \geq 2 r>\frac{r}{2} . \text { Proved. }
\end{aligned}
$$

$$
\begin{aligned}
& \text { (3) } \\
& \Leftrightarrow r s^{2}\left(28 R^{3}-87 R^{2} r+74 R r^{2}-17 r^{3}\right) \geq 20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}-3 r^{6} \\
& \because 28 R^{3}-87 R^{2} r+74 R r^{2}-17 r^{3}=(R-2 r)\left(28 R^{2}-31 R r+12 r^{2}\right)+7 r^{3}>0 \\
& \therefore \text { LHS of }(3) \stackrel{\text { Gerretsen }}{\geq} \\
& \left(28 R^{3}-87 R^{2} r+74 R r^{2}-17 r^{3}\right)\left(16 R r-5 r^{2}\right) r s^{2} \stackrel{?}{\sim} \underset{\sim}{\geq} 20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}-3 r^{6} \\
& \Leftrightarrow 112 t^{4}-388 t^{3}+402 t^{2}-159 t+22 \stackrel{?}{\geq} 0\left(w h e r e t=\frac{R}{r}\right) \\
& \Leftrightarrow(t-2)\left\{(t-2)\left(112 t^{2}+60 t+194\right)+377\right\} \stackrel{?}{\geq} 0 \rightarrow \text { true } \because t \stackrel{\text { Euler }}{\geq} 2 \\
& \Rightarrow(3) \Rightarrow(2) \text { is true }: \sum \frac{4 m_{a}^{2}}{b^{2}+c^{2}} \geq \frac{12 r}{R} \sum \sin ^{2} \frac{A}{2} \Rightarrow \sum \frac{m_{a}^{2}}{b^{2}+c^{2}} \\
& \geq \frac{3 r}{R} \sum \sin ^{2} \frac{A}{2} \text { (Proved) }
\end{aligned}
$$

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$(R-2 r)^{2}\left(16 R r-5 r^{2}\right) s^{2}+\left(20 R^{3} r-6 R^{2} r^{2}+6 r^{4}\right) s^{2}$
$\geq 20 R^{3} r^{3}+11 R^{2} r^{4}-6 R r^{5}-3 r^{6}+\left(2 R r+r^{2}\right)\left(4 R^{2}+4 R r+3 r^{2}\right) s^{2}$


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1554. In $\triangle A B C$ the following relationship holds:

$$
\frac{1}{2} \sum_{c y c} \cot \frac{A}{4} \geq \sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{m_{c}}{r_{c}}}+\frac{a+b+c}{4 r}
$$

Proposed by Bogdan Fuştei-Romania
Solution 1 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& b+c-a=4 R \cos \frac{A}{2} \cos \frac{B-C}{2}-4 R \cos \frac{A}{2} \sin \frac{A}{2} \\
= & 4 R \cos \frac{A}{2}\left(\cos \frac{B-C}{2}-\cos \frac{B+C}{2}\right)=8 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
\end{aligned}
$$

$$
\Rightarrow s-a \stackrel{(1)}{=} 4 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}
$$

$$
\text { Now, } A I=\frac{r}{\sin \frac{A}{2}}=\frac{4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}}=4 R \sin \frac{B}{2} \sin \frac{C}{2} \stackrel{\text { by (1) }}{=} \frac{s-a}{\cos \frac{A}{2}}
$$

$$
\Rightarrow \cos \frac{A}{2} \stackrel{(2)}{=} \frac{s-a}{A I}
$$

$$
\text { Also, } \cot \frac{A}{4}=\frac{\sin \frac{A}{2}}{1-\cos \frac{A}{2}}=\frac{\left(1+\cos \frac{A}{2}\right) \sin \frac{A}{2}}{1-\cos ^{2} \frac{A}{2}}=\frac{\left(1+\cos \frac{A}{2}\right) \sin \frac{A}{2}}{\sin ^{2} \frac{A}{2}}
$$

$$
=\frac{1+\cos \frac{A}{2} \frac{b y(2)}{\mathscr{} i n} \frac{1}{2}}{\stackrel{1+\frac{s-a}{A I}}{\frac{r}{A I}}=\frac{A I+(s-a)}{r} \Rightarrow \cot \frac{A}{4}=\frac{A I+(s-a)}{r}, \frac{1}{r}}
$$

and analogs $\Rightarrow \Rightarrow \frac{1}{2} \sum \cot \frac{A}{4}=\frac{\sum A I+(3 s-2 s)}{2 r} \stackrel{(3)}{=} \frac{\sum A I}{2 r}+\frac{a+b+c}{4 r}$

$$
\begin{aligned}
& \text { Also, by Panaitopol, } \sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{m_{c}}{r_{c}}} \leq \sum \sqrt{\frac{R h_{a}}{2 r r_{a}}} \\
& =\sum \sqrt{\frac{R .2 r s}{2 r .4 R \sin \frac{A}{2} \cos \frac{A}{2} \cdot \operatorname{stan} \frac{A}{2}}}=\frac{1}{r} \sum \frac{r}{2 \sin \frac{A}{2}}=\frac{\sum A I}{2 r}
\end{aligned}
$$



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$$
\Rightarrow \sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{m_{c}}{r_{c}}}+\frac{a+b+c}{4 r} \leq \frac{\sum A I}{2 r}+\frac{a+b+c}{4 r} \stackrel{b y(3)}{=} \frac{1}{2} \sum \cot \frac{A}{4}(\text { Proved })
$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\tan \frac{A}{4}=\frac{\sin \frac{A}{2}}{1+\cos \frac{A}{2}}=\frac{1-\cos \frac{A}{2}}{\sin \frac{A}{2}} \Rightarrow \cot \frac{A}{4}=\frac{1+\cos \frac{A}{2}}{\sin \frac{A}{2}}=\frac{1}{\sin \frac{A}{2}}+\cot \frac{A}{2} \Rightarrow \\
\sum_{c y c} \cot \frac{A}{4}=\sum_{c y c} \frac{1}{\sin \frac{A}{2}}+\sum_{c y c} \cot \frac{A^{\sum_{c y c} \cot \frac{A}{2}=\frac{s}{r}}}{=} \sum_{c y c} \frac{1}{\sin \frac{A}{2}}+\frac{s}{r} \Rightarrow L H S=\frac{1}{2}\left(\sum_{c y c} \frac{1}{\sin \frac{A}{2}}+\frac{s}{r}\right) \\
m_{a} \leq \frac{R}{2 r} \cdot h_{a} \Rightarrow \frac{m_{a}}{r_{a}} \leq \frac{R}{2 r} \cdot \frac{h_{a}}{r_{a}}=\frac{R}{2 r} \cdot \frac{4 R \sin \frac{A}{2} \cos \frac{A}{2}}{\sin \frac{A}{2}}=\frac{1}{4 \sin ^{2} \frac{A}{2}} \Rightarrow \\
R H S=\sum_{c y c}^{\frac{m_{a}}{r_{a}}} \leq \sqrt{\frac{1}{4 \sin ^{2} \frac{A}{2}}}=\frac{1}{2 \sin \frac{A}{2}} \cdot \operatorname{sos} \\
\sqrt{\frac{m_{a}}{r_{a}}}+\frac{a+b+c}{4 r}=\sum_{c y c} \sqrt{\frac{m_{a}}{r_{a}}}+\frac{2 s}{4 r} \leq \frac{1}{2} \sum_{c y c} \frac{1}{\sin \frac{A}{2}}+\frac{s}{2 r}=\frac{1}{2}\left(\sum_{c y c}^{\sin \frac{A}{2}}+\frac{1}{r}\right) \\
=L H S
\end{gathered}
$$

1555. In $\triangle A B C$ the following relationship holds:

$$
\frac{1}{2} \sum_{c y c} \tan \frac{A}{4} \geq \sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{m_{c}}{r_{c}}}-\frac{a+b+c}{4 r}
$$

Proposed by Bogdan Fuştei-Romania
Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\tan \frac{A}{4}=\frac{1-\cos \frac{A}{2}}{\sin \frac{A}{2}}=\frac{1}{\sin \frac{A}{2}}-\cot \frac{A}{2} \Rightarrow
$$



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$$
\begin{gathered}
\sum_{c y c} \tan \frac{A}{4}=\sum_{c y c} \frac{1}{\sin \frac{A}{2}}-\sum_{c y c} \cot \frac{A}{2} \stackrel{\sum_{c y c} \cot \frac{A}{2}=\frac{s}{r}}{=} \sum_{c y c} \frac{1}{\sin \frac{A}{2}}-\frac{s}{r} \Rightarrow L H S=\frac{1}{2}\left(\sum_{c y c} \frac{1}{\sin \frac{A}{2}}-\frac{s}{r}\right) \\
m_{a} \leq \frac{R}{2 r} \cdot h_{a} \Rightarrow \frac{m_{a}}{r_{a}} \leq \frac{R}{2 r} \cdot \frac{h_{a}}{r_{a}}=\frac{R}{2 r} \cdot \frac{\frac{2 \sin }{4 R \sin \frac{A}{2} \cos \frac{A}{2}}}{\sin \frac{A}{2}}=\frac{1}{4 \sin ^{2} \frac{A}{2}} \Rightarrow \\
\cos \frac{A}{2}
\end{gathered}
$$

$$
\begin{aligned}
& R H S=\sum_{c y c} \sqrt{\frac{m_{a}}{r_{a}}}-\frac{a+b+c}{4 r}=\sum_{c y c} \sqrt{\frac{m_{a}}{r_{a}}}-\frac{2 s}{4 r} \leq \frac{1}{2} \sum_{c y c} \frac{1}{\sin \frac{A}{2}}-\frac{s}{2 r}=\frac{1}{2}\left(\sum_{c y c} \frac{1}{\sin \frac{A}{2}}-\frac{s}{r}\right) \\
& =L H S
\end{aligned}
$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& b+c-a=4 R \cos \frac{A}{2} \cos \frac{B-C}{2}-4 R \cos \frac{A}{2} \sin \frac{A}{2}=4 R \cos \frac{A}{2}\left(\cos \frac{B-C}{2}-\cos \frac{B+C}{2}\right) \\
& =8 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \Rightarrow s-a \stackrel{(1)}{=} 4 R \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\
& \text { Now, } A I=\frac{r}{\sin \frac{A}{2}}=\frac{4 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{\sin \frac{A}{2}}=4 R \sin \frac{B}{2} \sin \frac{C^{b y(1)}}{\stackrel{m}{=}} \frac{s-a}{\cos \frac{A}{2}} \Rightarrow \cos \frac{A}{2} \stackrel{(2)}{=} \frac{s-a}{A I} \\
& \text { Again, } \tan \frac{A}{4}=\frac{1-\cos \frac{A}{2}}{\sin \frac{A}{2}} \stackrel{(2)(2)}{=} \frac{1-\frac{s-a}{A I}}{\frac{r}{A I}}=\frac{A I-(s-a)}{r} \\
& \Rightarrow \tan \frac{A}{4}=\frac{A I-(s-a)}{r} \text { and analogs } \Rightarrow \frac{1}{2} \sum \tan \frac{A}{4}=\frac{\sum A I-(3 s-2 s)}{2 r} \\
& \stackrel{(3)}{\leftrightarrows} \frac{\sum A I}{2 r}-\frac{a+b+c}{4 r}
\end{aligned}
$$



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Also, by Panaitopol, $\sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{\boldsymbol{m}_{c}}{r_{c}}} \leq \sum \sqrt{\frac{R h_{a}}{2 r r_{a}}}$
$=\sum \sqrt{\frac{R \cdot 2 r s}{2 r \cdot 4 R \sin \frac{A}{2} \cos \frac{A}{2} \cdot \operatorname{stan} \frac{A}{2}}}=\frac{1}{r} \sum \frac{r}{2 \sin \frac{A}{2}}=\frac{\sum A I}{2 r}$

$$
\Rightarrow \sqrt{\frac{m_{a}}{r_{a}}}+\sqrt{\frac{m_{b}}{r_{b}}}+\sqrt{\frac{m_{c}}{r_{c}}}-\frac{a+b+c}{4 r} \leq \frac{\sum A I}{2 r}-\frac{a+b+c^{b y}}{4 r} \stackrel{(3)}{=} \frac{1}{2} \sum \tan \frac{A}{4} \text { (Proved) }
$$

1556. In $\triangle A B C$ the following relationship holds:

$$
\operatorname{coa}(A-B)+\cos (B-C)+\cos (C-A) \leq \frac{3}{2}+\frac{3 r}{R}
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution by Daniel Sitaru-Romania

$$
\begin{gathered}
\sum_{c y c} \cos (A-B)=\frac{s^{2}+r^{2}+2 R r}{2 R^{2}}-1 \leq \frac{3}{2}+\frac{3 r}{R} \leftrightarrow \\
\leftrightarrow \frac{s^{2}+r^{2}+2 R r}{2 R^{2}} \leq \frac{5}{2}+\frac{3 r}{R} \leftrightarrow \\
\leftrightarrow s^{2}+r^{2}+2 R r \leq 5 R^{2}+6 R r \leftrightarrow s^{2} \leq 5 R^{2}+4 R r-r^{2} \\
s^{2} \begin{array}{c}
\text { GERRETSEN } \\
\leftrightarrows
\end{array} 4 R^{2}+4 R r+3 r^{2} \leq 5 R^{2}+4 R r-r^{2} \leftrightarrow \\
\leftrightarrow R^{2} \geq 4 r^{2} \leftrightarrow R \geq 2 r
\end{gathered}
$$

1557. In $\triangle A B C$ the following relationship holds:

$$
3\left(a^{2}+b^{2}+c^{2}\right)+4\left(h_{a}^{2}+h_{b}^{2}+h_{c}^{2}\right) \geq 24 \sqrt{3} S
$$

Proposed by Daniel Sitaru-Romania
Solution by Ravi Prakash-New Delhi-India

$$
\begin{gathered}
\frac{1}{2} a h_{a}=S \Rightarrow h_{a}=\frac{2 S}{a} . \text { Now } \\
3 a^{2}+4 h_{a}^{2}=3 a^{2}+\frac{16 S^{2}}{a^{2}} \geq 2 \sqrt{3 a^{2} \cdot \frac{16 S^{2}}{a^{2}}}=8 \sqrt{3} S
\end{gathered}
$$



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1558. In $\triangle A B C$ the following relationship holds:

$$
\frac{r_{a} r_{b} r_{c}}{w_{a} w_{b} w_{c}} \geq 1+\frac{r(R-2 r)}{16 R(R-r)}
$$

Proposed by Adil Abdulayev-Baku-Azerbaijan

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& \therefore r_{a} r_{b} r_{c}=s^{2} r \\
& \therefore w_{a} w_{b} w_{c}=\frac{16 R r^{2} s^{2}}{s^{2}+2 R r+r^{2}} \\
& \frac{r_{a} r_{b} r_{c}}{w_{a} w_{b} w_{c}} \geq 1+\frac{r(R-2 r)}{16 R(R-r)} \\
& \Leftrightarrow \frac{s^{2} r\left(s^{2}+2 R r+r^{2}\right)}{16 R r^{2} s^{2}} \geq 1+\frac{r(R-2 r)}{16 R(R-r)} \\
& \frac{s^{2}+2 R r+r^{2}}{16 R r} \geq 1+\frac{r(R-2 r)}{16 R(R-r)} \\
& \Leftrightarrow(R-r)\left(s^{2}+2 R r+r^{2}\right) \geq 16 R r(R-r)+r^{2}(R-2 r) \\
& \therefore s^{2} \geq 16 R r-5 r^{2} \therefore s^{2} \geq 16 R r-5 r^{2} \xrightarrow{R \geq 2 r \geq r} \\
& L H S \geq(R-r)\left(16 R r-5 r^{2}+2 R r+r^{2}\right)=(R-r)\left(18 R r-4 r^{2}\right) \\
& =r(R-r)(18 R-4 r) \text {. We must show: } \\
& r(R-r)(18 R-4 r) \geq 16 R r(R-r)+r^{2}(R-2 r) \\
& \Leftrightarrow(R-r)(18 R-4 r) \geq 16 R(R-r)+r(R-2 r) \\
& \Leftrightarrow 18 R^{2}-7 R r+6 r^{2} \geq 0 \\
& \Leftrightarrow 2(R-2 r)\left(R-\frac{3 r}{2}\right) \geq 0 \text {...true because }: R \geq 2 r \geq \frac{3 r}{2}
\end{aligned}
$$

1559. $\triangle P Q R$ is Vecten's triangle of $\triangle A B C$. Prove that:

$$
\left(\frac{A Q}{a}\right)^{2}+\left(\frac{B R}{b}\right)^{2}+\left(\frac{C P}{c}\right)^{2} \leq \frac{1}{2}\left((3+\sqrt{3})\left(\frac{R}{r}\right)^{2}-6-\sqrt{3}\right)
$$



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## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
& \text { - } \widehat{P A C}=\widehat{C A B}+\widehat{P A B}=\widehat{A}+45^{\circ} \\
& \rightarrow C P^{2}=A C^{2}+A P^{2}-2 \cdot A C \cdot A P \cdot \cos \widehat{P A C} \\
& =b^{2}+\left(\frac{c \sqrt{2}}{2}\right)^{2}-2 \cdot b \cdot \frac{c \sqrt{2}}{2} \cdot \cos \left(\widehat{A}+45^{0}\right) \\
& =b^{2}+\frac{c^{2}}{2}-2 \cdot b \cdot \frac{c \sqrt{2}}{2} \cdot \frac{1}{\sqrt{2}}(\cos A-\sin A)=b^{2}+\frac{c^{2}}{2}-b c(\cos A-\sin A) \\
& =b^{2}+\frac{c^{2}}{2}-b c \cos A+b c \sin A=b^{2}+\frac{c^{2}}{2}-b c . \frac{b^{2}+c^{2}-a^{2}}{2 b c}+2 S \\
& =b^{2}+\frac{c^{2}}{2}-\frac{b^{2}}{2}-\frac{c^{2}}{2}+\frac{a^{2}}{2}+2 S=\frac{b^{2}+a^{2}}{2}+2 S ; \\
& \rightarrow\left(\frac{C P}{c}\right)^{2}=\frac{C P^{2}}{c^{2}}=\frac{1}{2} \cdot \frac{b^{2}+a^{2}}{c^{2}}+\frac{2 S}{c^{2}} ; \text { Similary: } \\
& \left(\frac{A Q}{a}\right)^{2}=\frac{1}{2} \cdot \frac{b^{2}+c^{2}}{a^{2}}+\frac{2 S}{a^{2}} ;\left(\frac{B R}{b}\right)^{2}=\frac{1}{2} \cdot \frac{a^{2}+c^{2}}{b^{2}}+\frac{2 S}{b^{2}} ; S O, \\
& L H S=\left(\frac{A Q}{a}\right)^{2}+\left(\frac{B R}{b}\right)^{2}+\left(\frac{C P}{c}\right)^{2} \\
& =\frac{1}{2}\left(\frac{b^{2}+c^{2}}{a^{2}}+\frac{a^{2}+c^{2}}{b^{2}}+\frac{b^{2}+a^{2}}{c^{2}}\right)+2 S\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{c^{2}}\right) \\
& =\left(\frac{\left(a^{2}+b^{2}+c^{2}\right)}{2}+2 S\right)\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{c^{2}}\right)-\frac{3}{2} \\
& =\left(p^{2}-4 R r-r^{2}+2 p r\right)\left(\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{c^{2}}\right)-\frac{3}{2} \stackrel{\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{c^{2}} \leq \frac{1}{4 r^{2}}}{\leq} \\
& \left(p^{2}+2(p-2 R) r-r^{2}\right) \cdot \frac{1}{4 r^{2}}-\frac{3}{2} \stackrel{(*)}{\sim} \frac{1}{2}\left((3+\sqrt{3})\left(\frac{R}{r}\right)^{2}-6-\sqrt{3}\right) \\
& (*) \leftrightarrow p^{2}+2(p-2 R) r-r^{2}-6 r^{2} \leq(6+2 \sqrt{3}) R^{2}-2(6+\sqrt{3}) r^{2} \\
& \leftrightarrow p^{2} \leq(6+2 \sqrt{3}) R^{2}+2(2 R-p) r-(5+2 \sqrt{3}) r^{2} ; \\
& \text { Other: } \quad p^{2} \leq 4 R^{2}+4 R r+3 r^{2} \text {. We need to prove: } \\
& (6+2 \sqrt{3}) R^{2}+2(2 R-p) r-(5+2 \sqrt{3}) r^{2} \geq 4 R^{2}+4 R r+3 r^{2} ; \\
& \leftrightarrow(1+\sqrt{3}) R^{2}-r p-(4+\sqrt{3}) r^{2} \geq 0 ;
\end{aligned}
$$



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But: $p \leq \frac{3 \sqrt{3} R}{2} \rightarrow(1+\sqrt{3}) R^{2}-\frac{3 \sqrt{3}}{2} R r-(4+\sqrt{3}) r^{2} \stackrel{(* *)}{\geq} 0 ;$

$$
\begin{gathered}
(* *) \leftrightarrow 2(1+\sqrt{3}) R^{2}-3 \sqrt{3} R r-(8+2 \sqrt{3}) r^{2} \geq 0 ; \quad \therefore t=\frac{R}{r} \geq 2 \\
\leftrightarrow(t-2)(2(1+\sqrt{3}) t+\sqrt{3}+4) \geq 0 ;
\end{gathered}
$$

Which is clearly true by: $t \geq 2 \rightarrow(* *)$ is true $\rightarrow\left(^{*}\right)$ is true. Proved.
1560. In acute $\triangle A B C, G$ - centroid, $G D \perp B C, G E \perp C A, G F \perp A B, D \in(B C)$, $E \in(C A), F \in(A B)$. Prove that:

$$
\left(\frac{B C}{E F}\right)^{2}+\left(\frac{C A}{F D}\right)^{2}+\left(\frac{A B}{D E}\right)^{2} \leq \frac{3}{2}\left(\frac{R}{r}\right)^{2}
$$

Proposed by Mehmet Șahin-Ankara-Turkey
Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
G F \perp A B, C H \perp A B \Rightarrow C F \| C H \Rightarrow \frac{G F}{C H}=\frac{G M}{C M}=\frac{1}{3} \Rightarrow G F=\frac{1}{3} C H=\frac{1}{3} h_{c} \\
\text { Similary: } G E=\frac{1}{3} h_{b} ; G D=\frac{1}{3} h_{a} \\
A E G F-c y c l i c \text { quadrilateral } \Rightarrow \angle B A C+\angle F G E=180^{\circ} \\
\Rightarrow E F^{2}=F G^{2}+E G^{2}-2 F G \cdot E G \cdot \cos (F G E)=\left(\frac{1}{3} h_{c}\right)^{2}+\left(\frac{1}{3} h_{b}\right)^{2}+2 \cdot \frac{1}{3} h_{b} \cdot \frac{1}{3} h_{c} \cos A \\
=\frac{4 S^{2}}{9 c^{2}}+\frac{4 S^{2}}{9 b^{2}}+2 \cdot \frac{2 S}{3 c} \cdot \frac{2 S}{3 b} \cdot \frac{b^{2}+c^{2}-a^{2}}{2 b c} \\
=\frac{4 S^{2}}{9 b^{2} c^{2}}\left(b^{2}+c^{2}+b^{2}+c^{2}-a^{2}\right)=\frac{4 S^{2}}{9 b^{2} c^{2}}\left(2 b^{2}+2 c^{2}-a^{2}\right) \\
=\frac{16 S^{2}}{9 b^{2} c^{2}} \cdot \frac{2 b^{2}+2 c^{2}-a^{2}}{4}=\frac{16 S^{2}}{9 b^{2} c^{2}} \cdot m_{a}^{2} \\
\Rightarrow \frac{B C^{2}}{E F^{2}}=\frac{a^{2}}{\frac{16 S^{2}}{9 b^{2} c^{2}} \cdot m_{a}^{2}}=\frac{9 a^{2} b^{2} c^{2}}{16 S^{2} m_{a}^{2}} \frac{m_{a} \geq h_{a}}{\leq} \frac{9 a^{2} b^{2} c^{2}}{16 S^{2} h_{a}^{2}} \\
S i m i l a r y: \frac{C A^{2}}{F D^{2}} \leq \frac{9 a^{2} b^{2} c^{2}}{16 S^{2} h_{b}^{2}} ; \frac{A B^{2}}{D E^{2}} \leq \frac{9 a^{2} b^{2} c^{2}}{16 s^{2} h_{c}^{2}} \Rightarrow
\end{gathered}
$$



$$
\begin{aligned}
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& L H S \leq \frac{9 a^{2} b^{2} c^{2}}{16 S^{2}}\left(\frac{1}{h_{a}^{2}}+\frac{1}{h_{b}^{2}}+\frac{1}{h_{c}^{2}}\right)=9 R^{2}\left(\frac{1}{h_{a}^{2}}+\frac{1}{h_{b}^{2}}+\frac{1}{h_{c}^{2}}\right)=\frac{9 R^{2}\left(s^{2}-r^{2}-4 R r\right)}{2 s^{2} r^{2}} \leq \frac{3}{2}\left(\frac{R}{r}\right)^{3} \\
& \Leftrightarrow 3 r\left(s^{2}-r^{2}-4 R r\right) \leq \operatorname{Rs}^{2}(*) \\
& 16 R r-5 r^{2} \leq s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \\
& \text { So,we need to prove: } \\
& 3 r\left(4 R^{2}+2 r^{2}\right) \leq R\left(16 R r-5 r^{2}\right) \Leftrightarrow 4 R^{2}-5 R r-6 r^{2} \geq 0 \\
& \Leftrightarrow(R-2 r)(4 R+3 r) \geq 0 \text { true by } R \geq 2 r \text { (Euler), so (*) true. Proved. }
\end{aligned}
$$

1561. In $\triangle A B C$ the following relationship holds:

$$
\frac{12\left(a^{2}+b^{2}+c^{2}\right)}{(a+b+c)^{2}} \leq 3+\frac{R}{2 r}
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gather*}
\frac{12\left(a^{2}+b^{2}+c^{2}\right)}{(a+b+c)^{2}} \leq 3+\frac{R}{2 r} \Leftrightarrow \frac{24\left(s^{2}-4 R r-r^{2}\right)}{4 s^{2}} \leq 3+\frac{R}{2 r} \\
\Leftrightarrow 12 r\left(s^{2}-4 R r-r^{2}\right) \leq 6 r s^{2}+R s^{2} \Leftrightarrow(6 r-R) s^{2} \leq 48 R r^{2}+12 r^{3} \quad(*)  \tag{*}\\
\text { If } 6 r-R \leq 0 \Leftrightarrow 6 r \leq R \text { then } 48 R r^{2}+12 r^{3} \geq(6 r-R) s^{2} \Rightarrow(*) \text { is true. }
\end{gather*}
$$

$$
\text { If } 6 r \geq R \geq 2 r \text { then }:
$$

$$
s^{2} \leq 2 R^{2}+10 R r-r^{2}+2(R-2 r) \sqrt{R^{2}-2 R r}
$$

Let: $t=\frac{R}{r},(2 \leq t<6)$, we need to prove:

$$
\begin{gathered}
(6-t)\left(2 t^{2}+10 t-1+(2 t-4) \sqrt{t^{2}-2 t}\right) \leq 48 t+12 \Leftrightarrow \\
2 t^{3}-2 t^{2}-13 t+18 \geq 2(6-t)(t-2) \sqrt{t^{2}-2 t} \Leftrightarrow \\
\left(2 t^{3}-2 t^{2}-13 t+18\right)^{2} \geq 4(6-t)^{2}(t-2)^{2}\left(t^{2}-2 t\right) \Leftrightarrow \\
(t-2)^{2}(4 t-9)^{2}(4 t+1) \geq 0 \text { which is true by } 2 \leq t<6 . \text { Proved. }
\end{gathered}
$$

1562. In $\triangle A B C$ the following relationship holds:

$$
\prod_{c y c} \csc ^{2} \frac{A}{2}+\sum_{c y c} \csc ^{2} \frac{A}{2} \sec ^{2} \frac{B}{2} \sec ^{2} \frac{C}{2}=16 \prod_{c y c} \csc ^{2} A \sum_{c y c} \sin ^{2} A
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan


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## Solution by Izumi Ainsworth-Lima-Peru

$$
\begin{gathered}
\prod_{c y c} \csc ^{2} \frac{A}{2}+\sum_{c y c} \csc ^{2} \frac{A}{2} \sec ^{2} \frac{B}{2} \sec ^{2} \frac{C}{2}=16 \prod_{c y c} c s c^{2} A \sum_{c y c} \sin ^{2} A \Rightarrow \\
\left(\frac{4 R}{r}\right)^{2}+\sum_{c y c}\left(\frac{4 R}{r_{a}}\right)^{2}=16 \sum_{c y c} \frac{1}{\sin ^{2} B \cdot \sin ^{2} C}=16 \sum_{c y c}\left(\frac{2 R}{b}\right)^{2}\left(\frac{2 R}{c}\right)^{2} \Rightarrow \\
16 R^{2} \cdot \frac{1}{r^{2}}+16 R^{2} \sum_{c y c} \frac{1}{r_{a}^{2}}=\left(16 R^{2}\right)^{2} \sum_{c y c}\left(\frac{1}{b c}\right)^{2} \stackrel{: 16 R^{2}}{\Longrightarrow} \\
\left(\frac{s}{S}\right)^{2}+\sum_{c y c}\left(\frac{s-a}{S}\right)^{2}=16 R^{2} \sum_{c y c}\left(\frac{\operatorname{sinA}}{2 S}\right)^{2} \stackrel{s^{2}}{\Rightarrow} \\
s^{2}+\sum_{c y c}(s-a)^{2}=\sum_{c y c}(2 R s \sin A)^{2} \Rightarrow \\
s^{2}+3 s^{2}-2 s(a+b+c)+a^{2}+b^{2}+c^{2}=\sum_{c y c} a^{2} \Rightarrow \\
a^{2}+b^{2}+c^{2}=\sum_{c y c} a^{2}
\end{gathered}
$$

Proved.
1563. In $\triangle A B C$ the following relationship holds:

$$
\frac{7 R-2 r}{2 R} \leq \frac{m_{a}}{h_{a}}+\frac{m_{b}}{h_{b}}+\frac{m_{c}}{h_{c}} \leq \frac{4 R+r}{3 r}
$$

Proposed by Marin Chirciu-Romania
Solution by Bogdan Fuştei-Romania

$$
\begin{gather*}
\therefore \frac{1}{h_{a}}+\frac{1}{h_{b}}+\frac{1}{h_{c}}=\frac{1}{r} \\
\frac{m_{a}}{h_{a}}+\frac{m_{b}}{h_{b}}+\frac{m_{c}}{h_{c}} \stackrel{\text { Chebyshev's }}{\leq} \frac{m_{a}+m_{b}+m_{c}}{3 r} \\
m_{a}+m_{b}+m_{c} \leq r_{a}+r_{b}+r_{c}=4 R+r \\
\text { So } \frac{m_{a}}{h_{a}}+\frac{m_{b}}{h_{b}}+\frac{m_{c}}{h_{c}} \leq \frac{m_{a}+m_{b}+m_{c}}{3 r} \leq \frac{4 R+r}{3 r} \tag{1}
\end{gather*}
$$



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$$
\begin{align*}
& \text { www.ssmrmh.ro } \\
& m_{a} \geq \frac{b^{2}+c^{2}}{4 R} \Rightarrow \frac{m_{a}}{h_{a}} \geq \frac{b^{2}+c^{2}}{2 b c} ;\left(2 R h_{a}=b c\right) \\
& \sum_{c y c} \frac{m_{a}}{h_{a}} \geq \sum_{c y c}\left(\frac{b}{c}+\frac{c}{b}\right)=\frac{1}{2} \sum_{c y c} \frac{b+c}{a} \\
& \frac{h_{a}}{r}=1+\frac{b+c}{a} \text { and analogs } \\
& \frac{h_{a}-r}{r}=\frac{b+c}{a} \Rightarrow \frac{h_{a}+h_{b}+h_{c}-3 r}{r}=\sum_{c y c} \frac{b+c}{a} \\
& a b+b c+c a=s^{2}+4 R r+r^{2} \\
& \frac{h_{a}+h_{b}+h_{c}}{r}=\frac{s^{2}+4 R r+r^{2}-6 R r}{2 R r} \\
& \sum_{c y c} \frac{m_{a}}{h_{a}} \geq \frac{s^{2}+4 R r+r^{2}}{4 R r} \underbrace{\sum_{s^{2} \geq 16 R r-5 R^{2}}^{\sum}}_{\text {Gerretsen }} \frac{16 R r-5 r^{2}+4 R r+r^{2}-6 R r}{4 R r} \\
& \sum_{c y c} \frac{m_{a}}{h_{a}} \geq \frac{14 R r-4 r^{2}}{4 R r}=\frac{7 R-2 r}{2 R} \tag{2}
\end{align*}
$$

From (1)+(2) we have: $\frac{7 R-2 r}{2 R} \leq \frac{m_{a}}{h_{a}}+\frac{m_{b}}{h_{b}}+\frac{m_{c}}{h_{c}} \leq \frac{4 R+r}{3 r}$
1564. In $\triangle A B C$ the following relationship holds:

$$
\frac{\sqrt{\sin 2 A}+\sqrt{\sin 2 B}+\sqrt{\sin 2 C}}{\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\tan C}} \geq \frac{\sqrt{2 r^{2}+4 R r-2 R^{2}}}{R}
$$

Proposed by Marian Ursărescu-Romania

## Solution by Soumava Chakraborty-Kolkata-India

Let $x, y, z$ be sides of a triangle.

$$
\begin{gathered}
\text { Now, } \sqrt{(x+y-z)(y+z-x)} \stackrel{A-G}{\sim} \frac{x+y-z+y+z-x}{2}=y \\
\Rightarrow 2 \sqrt{(x+y-z)(y+z-x)}+2 y \leq 4 y \\
\Rightarrow 2 \sqrt{(x+y-z)(y+z-x)}+\{(x+y-z)+(y+z-x)\} \leq 4 y \\
\Rightarrow 4 y \geq(\sqrt{x+y-z}+\sqrt{y+z-x})^{2} \Rightarrow 2 \sqrt{y} \geq \sqrt{x+y-z}+\sqrt{y+z-x} \text { and analogs } \\
\Rightarrow 2 \sum \sqrt{x} \geq \sum(\sqrt{\mathrm{x}+\mathrm{y}-\mathrm{z}}+\sqrt{\mathrm{y}+\mathrm{z}-\mathrm{x}})
\end{gathered}
$$



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$$
\Rightarrow \mathbf{2} \sum \sqrt{\mathbf{x}} \geq \mathbf{2} \sum \sqrt{\mathbf{y}+\mathbf{z - x}} \Rightarrow \sum \sqrt{\mathbf{x}} \stackrel{(1)}{\underset{2}{2} \sum \sqrt{y+z-\mathbf{x}}}
$$

Now, $\sin 2 A+\sin 2 B-\sin 2 C=2 \sin (A+B) \cos (A-B)-2 \sin (A+B) \cos C$

$$
=2 \sin (A+B)\{\cos (A-B)+\cos (A+B)\}
$$

$=4 \sin C \cos A \cos B>0(\because \triangle A B C$ is acute $) \Rightarrow \sin 2 A+\sin 2 B>\sin 2 C$ and analogs

$$
\Rightarrow \sin 2 A, \sin 2 B, \sin 2 C \text { are sides of a triangle }
$$

$$
\text { and } \therefore \sum \sqrt{\sin 2 A} \stackrel{\text { by }(1)}{\geq} \sum \sqrt{\sin 2 B+\sin 2 C-\sin 2 A}=\sum \sqrt{4 \sin C \cos A \cos B}
$$

$$
=2 \sqrt{\prod \cos A} \sum \sqrt{\tan \mathrm{C}} \Rightarrow \frac{\sum \sqrt{\sin 2 \mathrm{~A}}}{\sum \sqrt{\tan \mathrm{~A}}} \geq 2 \sqrt{\prod \cos A}
$$

$$
=2 \sqrt{\frac{s^{2}-4 \mathbf{R}^{2}-4 R r-\mathbf{r}^{2}}{4 R^{2}}}=\frac{\sqrt{s^{2}-4 R^{2}-4 R r-\mathbf{r}^{2}}}{\mathbf{R}} \underset{\sim}{?} \frac{\sqrt{2 r^{2}+4 R r-2 R^{2}}}{\mathbf{R}}
$$

$$
\Leftrightarrow s^{2}-4 R^{2}-4 R r-r^{2} \stackrel{?}{2} 2^{2}+4 R r-2 R^{2}
$$

$\Leftrightarrow \mathbf{s}^{2} \stackrel{\stackrel{?}{\sim}}{\geq} \mathbf{2} R^{2}+\mathbf{8 R r}+3 r^{2} \rightarrow$ true, by Walker $\Rightarrow \frac{\sqrt{\sin 2 A}+\sqrt{\sin 2 B}+\sqrt{\sin 2 C}}{\sqrt{\tan A}+\sqrt{\tan B}+\sqrt{\operatorname{tanC}}}$

$$
\geq \frac{\sqrt{2 r^{2}+4 R r-2 R^{2}}}{R}(\text { Proved })
$$

1565. In $\triangle A B C, \boldsymbol{n}_{\boldsymbol{a}}$-Nagel's cevian, $\boldsymbol{g}_{\boldsymbol{a}}$-Gergonne's cevian, the following relationship holds:

$$
\frac{2 n_{a}}{r_{b}+r_{c}}+\frac{2 n_{b}}{r_{c}+r_{a}}+\frac{2 n_{c}}{r_{a}+r_{b}} \geq \frac{h_{a}}{g_{a}}+\frac{\mathbf{h}_{b}}{g_{b}}+\frac{h_{c}}{g_{c}}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \quad \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& = \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$



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$$
\begin{aligned}
& \quad \therefore \mathbf{a n}_{\mathbf{a}}{ }^{2} \cdot \mathbf{a g} \mathbf{a}^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{(\mathbf{a})}{\geq} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$

Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c $=x+y$

Using these substitutions, (a)
$\Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathbf{x})^{2}+\mathbf{y}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathbf{x})^{2}+\mathrm{z}(\mathbf{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathbf{y}\right.$
$+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$
$\Leftrightarrow x^{2}+x z^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow$ true

$$
\Rightarrow(\mathrm{a}) \text { is true } \Rightarrow \mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}} \overbrace{\geq} \mathrm{s}(\mathrm{~s}-\mathrm{a})
$$

$$
\text { Also, } r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{sos}^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2}
$$

$\therefore r_{b}+r_{c} \stackrel{(i i)}{=} 4 \operatorname{Rcos}^{2} \frac{A}{2}$

$$
\begin{aligned}
N o w, \frac{2 n_{a}}{r_{b}+r_{c}} & \geq \frac{h_{a}}{g_{a}} \Leftrightarrow 2 n_{a} g_{a} \geq h_{a}\left(r_{b}+r_{c}\right) \stackrel{\text { by (ii) }}{\cong}\left(\frac{2 r s}{4 R \cos \frac{A}{2} \sin \frac{A}{2}}\right) 4 \operatorname{Rcos}^{2} \frac{A}{2}=\frac{2 r s^{2}}{\operatorname{stan} \frac{A}{2}} \\
& =\frac{2 r_{a} r_{b} r_{c}}{r_{a}}=2 r_{b} r_{c}=2 s(s-a) \\
\Leftrightarrow n_{a} g_{a} & \geq s(s-a) \rightarrow \operatorname{true} b y(i) \therefore \frac{2 n_{a}}{r_{b}+r_{c}} \geq \frac{h_{a}}{g_{a}} \operatorname{and} \operatorname{analog} s \Rightarrow \sum \frac{2 n_{a}}{r_{b}+r_{c}} \\
& \geq \sum \frac{h_{a}}{g_{a}}(\text { Proved })
\end{aligned}
$$

1566. In $\triangle A B C$ the following relationship holds:

$$
8 \sum_{c y c} \csc A \cdot \tan \frac{B+C}{2} \leq \sum_{c y c}\left(\csc \frac{B}{2} \cdot \csc \frac{C}{2}\right)^{2}
$$

Proposed by Gheorghe Alexe and George Florin Şerban-Romania


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Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{align*}
& \sum_{c y c} \frac{1}{\sin ^{2} \frac{A}{2}}=\frac{s^{2}+r^{2}-8 R r}{r^{2}} \\
& \sum_{c y c}\left(\frac{1}{\sin ^{2} \frac{A}{2}} \cdot \frac{1}{\sin ^{2} \frac{B}{2}}\right)=\frac{8 R(2 R-r)}{r^{2}} \\
& L H S=8 \sum_{c y c} \csc A \cdot \tan \frac{B+C}{2}=8 \sum_{c y c}\left(\frac{1}{\sin A} \cdot \tan \left(\frac{\pi}{2}-\frac{A}{2}\right)\right) \\
& =4 \sum_{c y c}\left(\frac{1}{\sin \frac{A}{2} \cos \frac{A}{2}} \cdot \frac{\cos \frac{A}{2}}{\sin \frac{A}{2}}\right)=4 \sum_{c y c} \frac{1}{\sin ^{2} \frac{A}{2}}=4 \cdot \frac{s^{2}+r^{2}-8 R r}{r^{2}}  \tag{1}\\
& R H S=\sum_{c y c}\left(\csc \frac{B}{2} \cdot \csc \frac{C}{2}\right)^{2}=\sum_{c y c}\left(\csc \frac{A}{2} \cdot \csc \frac{B}{2}\right)^{2} \\
& =\sum_{c y c}\left(\frac{1}{\sin ^{2} \frac{A}{2}} \cdot \frac{1}{\sin ^{2} \frac{B}{2}}\right)=\frac{8 R(2 R-r)}{r^{2}}  \tag{2}\\
& \text { From (1)+(2) we must show: } \\
& 4 \cdot \frac{s^{2}+r^{2}-8 R r}{r^{2}} \leq \frac{8 R(2 R-r)}{r^{2}} \Leftrightarrow s^{2}+r^{2}-8 R r \leq 4 R^{2}-2 R r \Leftrightarrow \\
& s^{2}+r^{2} \leq 4 R^{2}+6 R r \xrightarrow{s^{2} \leq 4 R^{2}+4 R r+3 r^{2}} s^{2}+r^{2} \leq 4 R^{2}+4 R r+4 r^{2}
\end{align*}
$$

We just check: $4 R^{2}+6 R r \geq 4 R^{2}+4 R r+4 r^{2} \Leftrightarrow 2 R r \geq 4 r^{2} \Leftrightarrow R \geq 2 r$ (Euler) Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
8 \sum \csc A \tan \frac{B+C}{2}=8 \sum\left\{\frac{2 R}{4 R \sin \frac{A}{2} \cos \frac{A}{2}} \tan \left(\frac{\pi}{2}-\frac{A}{2}\right)\right\} \\
=8 \sum \frac{2 R}{4 R \sin \frac{A}{2} \tan \frac{A}{2} \cos \frac{A}{2}}=4 \sum \frac{1}{\sin ^{2} \frac{A}{2}}
\end{gathered}
$$



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$=4 \sum \frac{b c(s-a)}{(s-a)(s-b)(s-c)}=\frac{4}{s r^{2}}\left(s \sum a b-12 R r s\right)=\frac{4\left(s^{2}-8 R r+r^{2}\right)}{r^{2}}$
$\therefore 8 \sum \csc A \tan \frac{B+C}{2} \stackrel{(1)}{=} \frac{4\left(s^{2}-8 R r+r^{2}\right)}{r^{2}}$
$\begin{aligned} & \sum\left(\csc \frac{B}{2} \csc \frac{C}{2}\right)^{2}=\sum \frac{c a \cdot a b}{(s-c)(s-a) \cdot(s-a)(s-b)}=\frac{4 R r s}{s r^{2}} \sum \frac{a-s+s}{s-a} \\ &= \frac{4 R}{r}\left(-3+\frac{s \sum(s-b)(s-c)}{s r^{2}}\right) \\ &=\frac{4 R}{r}\left(-3+\frac{4 R r+r^{2}}{r^{2}}\right)=\frac{4 R}{r}\left(\frac{4 R-2 r}{r}\right) \therefore \sum\left(\csc \frac{B}{2} \csc \frac{C}{2}\right)^{2(2)} \stackrel{4 R(4 R-2 r)}{=}\end{aligned}$
$\therefore$ (1), (2) $\Rightarrow$ proposed inequality $\Leftrightarrow$
$s^{2}-8 R r+r^{2} \leq R(4 R-2 r) \Leftrightarrow s^{2} \leq 4 R^{2}+6 R r-r^{2}$
$\Leftrightarrow s^{2}-4 R^{2}-4 R r-3 r^{2}-2 r(R-2 r) \leq 0 \rightarrow$ true
$\because s^{2}-4 R^{2}-4 R r-3 r^{2} \underset{\sim}{\text { Gerretsen }} 0$ and $-2 r(R-2 r) \stackrel{\text { Euler }}{\leftrightarrows} 0 . \therefore 8 \sum \csc \operatorname{Atan} \frac{B+C}{2}$
$\leq \sum\left(\csc \frac{B}{2} \csc \frac{C}{2}\right)^{2}($ Proved $)$
1567. Let $\Delta A^{\prime} B^{\prime} C^{\prime}$ be the circumcevian triangle of $H$-ortocenter in acute $\triangle A B C$.

Prove that:

$$
\frac{\left[A^{\prime} B^{\prime} C^{\prime}\right]}{[A B C]} \geq 4\left(\left(\frac{r}{R}\right)^{2}+\frac{2 r}{R}-1\right)
$$

Proposed by Marian Ursărescu-Romania
Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\frac{\left[A^{\prime} B^{\prime} C^{\prime}\right]}{[A B C]}=\frac{A^{\prime} B^{\prime} \cdot B^{\prime} C^{\prime} \cdot C^{\prime} A^{\prime}}{4 R} \cdot \frac{4 R}{a b c}=\frac{A^{\prime} B^{\prime} \cdot B^{\prime} C^{\prime} \cdot C^{\prime} A^{\prime}}{a b c} \\
\left\{\begin{array}{l}
\angle B A H=\angle B B^{\prime} A^{\prime} \Rightarrow \Delta H A B \sim \Delta H B^{\prime} A^{\prime} \Rightarrow \frac{H A}{H B^{\prime}}=\frac{A B}{A^{\prime} B^{\prime}} \Rightarrow \\
\angle A B B^{\prime}=\angle A A^{\prime} B^{\prime} \Rightarrow \Delta \\
A^{\prime} B^{\prime}= \\
H B^{\prime} \cdot A B \\
H A
\end{array} \frac{c \cdot H B \cdot H B^{\prime}}{H A \cdot H B}=\frac{c \cdot \rho(H)}{H A \cdot H B}\right.
\end{gathered}
$$



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$$
\begin{gathered}
\text { Similarly: } A^{\prime} C^{\prime}=\frac{b \cdot \rho(H)}{H A \cdot H C} ; B^{\prime} C^{\prime}=\frac{a \cdot \rho(H)}{H B \cdot H C} \\
\rho(H)=R^{2}-O H^{2}=R^{2}-\left(9 R^{2}-\left(a^{2}+b^{2}+c^{2}\right)\right)=a^{2}+b^{2}+c^{2}-8 R^{2} \\
\Delta A B C-a c u t e \Rightarrow H A=2 R \cos A ; H B=2 R \cos B ; H C=2 R \cos C \\
\Rightarrow \frac{\left[A^{\prime} B^{\prime} C^{\prime}\right]}{[A B C]}=\frac{a b c(\rho(H))^{3}}{a b c(H A \cdot H B \cdot H C)^{2}}=\frac{\left(a^{2}+b^{2}+c^{2}-8 R^{2}\right)^{3}}{64 R^{6}(\cos A \cdot \cos B \cdot \cos C)^{2}} \\
=\frac{8\left(s^{2}-4 R^{2}-4 R r-r^{2}\right)^{3}}{4 R^{2}\left(s^{2}-4 R^{2}-4 R r-r^{2}\right)^{2}}=\frac{2\left(s^{2}-4 R^{2}-4 R r-r^{2}\right)}{R^{2}} \stackrel{(*)}{\geq} 4\left(\left(\frac{r}{R}\right)^{2}+\frac{2 r}{R}-1\right)
\end{gathered}
$$

$(*) \Leftrightarrow s^{2}-4 R^{2}-4 R r-r^{2} \geq 2\left(r^{2}+2 R r-R^{2}\right) \Leftrightarrow s^{2} \geq 2 R^{2}+* R r+3 r^{2}$ true by Walker's inequality, then (*) true.Proved.
1568. In $\triangle A B C, n_{a}$-Nagel's cevian, $\boldsymbol{g}_{a}$-Gergonne's cevian, the following relationship holds:

$$
\frac{R}{r} \geq 1+\sqrt[3]{\frac{m_{\mathrm{a}} m_{\mathrm{b}} \mathrm{~m}_{\mathrm{c}} W_{\mathrm{a}} w_{\mathrm{b}} w_{\mathrm{c}}}{\mathrm{~h}_{\mathrm{a}} h_{\mathrm{b}} h_{\mathrm{c}} g_{\mathrm{a}} g_{\mathrm{b}} g_{\mathrm{c}}}} \geq 2
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})=\mathbf{a n}_{\mathbf{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=a n_{a}^{2}+a\left(s^{2}-s(2 s-a)+b c\right) \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c \\
& =\mathbf{a n} \mathbf{a}^{2}+\mathbf{a}\left(\mathbf{a s}-\mathbf{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+c^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{a}^{2}-\mathbf{a s}^{2} \Rightarrow \mathbf{a n}_{a}^{2}=\mathbf{a s}^{2}+s(2 b \cos A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin} 2 \frac{A^{(i)}}{2} \stackrel{( }{\cong} \mathbf{a s}^{2}-4 s(s-b)(s-c) \\
& =a s^{2}-\frac{\mathbf{a s}(\mathbf{c}+\mathbf{a}-\mathbf{b})(\mathbf{a}+\mathbf{b}-\mathbf{c})}{\mathbf{a}}=\mathbf{a s ^ { 2 }}-\mathbf{a s}\left(\frac{\mathbf{a}^{2}-(b-c)^{2}}{a}\right) \Rightarrow \mathbf{n}_{\mathbf{a}}^{2} \\
& =s\left(s-\frac{\mathbf{a}^{2}-(\mathbf{b}-\mathbf{c})^{2}}{a}\right) \Rightarrow \mathbf{n}_{\mathbf{a}}^{2} \stackrel{(1)}{\cong} \mathbf{s}\left(\mathrm{s}-\mathbf{a}+\frac{(\mathbf{b}-\mathbf{c})^{2}}{\mathrm{a}}\right)
\end{aligned}
$$



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$$
\begin{aligned}
& \quad \text { www.ssmrmh.ro } \\
& \quad \text { Also,Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& =\mathbf{a n}_{\mathbf{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& = \\
& \mathbf{a g}_{\mathbf{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}),
\end{aligned}
$$

and adding the above two, we get : $\left(\mathrm{b}^{2}+\mathrm{c}^{2}\right)(2 \mathrm{~s}-\mathrm{b}-\mathrm{c})$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathrm{a}}^{2}+\mathbf{a g}_{\mathrm{a}}^{2}+2 \mathbf{a}(\mathrm{~s}-\mathrm{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(n_{a}^{2}+g_{a}^{2}\right)+\mathbf{a}(\mathbf{a}+\mathbf{b}-\mathbf{c})(\mathbf{c}+\mathbf{a}-\mathbf{b}) \Rightarrow 2\left(b^{2}+\mathbf{c}^{2}\right) \\
& =2\left(n_{a}^{2}+g_{a}^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \\
& \Rightarrow(b-c)^{2}+4 s(s-a)+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \\
& \text { (ii) } \\
& \Rightarrow \mathbf{n a}_{\mathrm{a}}^{2}+\mathrm{g}_{\mathrm{a}}^{2} \xlongequal{\cong}(\mathrm{~b}-\mathrm{c})^{2}+\mathbf{2 s}(\mathrm{s}-\mathrm{a}) \therefore \text { (i), (ii) } \Rightarrow \mathrm{g}_{\mathrm{a}}^{2} \\
& =(b-c)^{2}+2 s(s-a)-s^{2}+\frac{4 s(s-b)(s-c)}{a} \\
& =s^{2}-2 s a+a^{2}+(b-c)^{2}-a^{2}+\frac{4 s(s-b)(s-c)}{a} \\
& =(s-\mathbf{a})^{2}+(\mathbf{b}-\mathbf{c}+\mathbf{a})(\mathbf{b}-\mathbf{c}-\mathbf{a})+\frac{4 \mathbf{s}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{a}} \\
& =(s-a)^{2}-4(s-b)(s-c)+\frac{4 s(s-b)(s-c)}{a}=(s-a)^{2}+4(s-b)(s-c)\left(\frac{s}{a}-1\right) \\
& =(\mathbf{s}-\mathbf{a})^{2}+\frac{4(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{a}} \\
& =(s-a)\left(s-a+\frac{\mathbf{a}^{2}-(b-c)^{2}}{a}\right) \Rightarrow \mathbf{g a}_{a}^{2} \stackrel{(2)}{=}(s-a)\left(s-\frac{(b-c)^{2}}{a}\right) \therefore(1),(2) \Rightarrow \mathbf{n}_{a}^{2} g_{a}^{2} \\
& =s(s-a)\left(s-a+\frac{(b-c)^{2}}{a}\right)\left(s-\frac{(b-c)^{2}}{a}\right) \\
& =s(s-a)\left(s(s-a)+s \frac{(b-c)^{2}}{a}-\frac{(b-c)^{2}}{a}(s-a)-\frac{(b-c)^{4}}{a^{2}}\right) \\
& \Rightarrow \mathbf{n}_{\mathbf{a}}^{2} \mathbf{g}_{\mathbf{a}} \stackrel{(\mathbf{a})}{\stackrel{m}{m}} \mathbf{s}(\mathbf{s}-\mathbf{a})\left(\mathbf{s}(\mathbf{s}-\mathbf{a})+(\mathbf{b}-\mathbf{c})^{2}-\frac{(\mathbf{b}-\mathbf{c})^{4}}{\mathbf{a}^{2}}\right)
\end{aligned}
$$



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$$
\begin{aligned}
& \text { Again, } \mathrm{m}_{\mathrm{a}}^{2} \mathrm{w}_{\mathrm{a}}^{2}=\frac{(\mathrm{b}-\mathrm{c})^{2}+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a})}{4} \cdot \frac{4 \mathrm{bcs}(\mathrm{~s}-\mathrm{a})}{(\mathrm{b}+\mathrm{c})^{2}} \\
& \Rightarrow \mathbf{m}_{\mathbf{a}}^{2} \mathbf{w}_{\mathbf{a}} \stackrel{(\mathbf{b})}{\cong} \mathbf{s}(\mathbf{s}-\mathbf{a}) \frac{\mathbf{b c}}{(\mathbf{b}+\mathbf{c})^{2}}\left((\mathbf{b}-\mathbf{c})^{2}+4 \mathbf{s}(\mathbf{s}-\mathbf{a})\right) \therefore(\mathbf{a}),(\mathbf{b}) \\
& \Rightarrow \mathbf{n}_{\mathrm{a}}^{2} \mathbf{g}_{\mathrm{a}}^{2}-\mathbf{m a}_{\mathrm{a}}^{2} \mathbf{w}_{\mathrm{a}}^{2} \\
& =\mathbf{s}(\mathbf{s}-\mathbf{a})\left(\mathbf{s}(\mathbf{s}-\mathbf{a})+(\mathbf{b}-\mathbf{c})^{2}-\frac{(\mathbf{b}-\mathbf{c})^{4}}{\mathbf{a}^{2}}-\frac{\mathbf{b c}}{(\mathbf{b}+\mathbf{c})^{2}}\left((\mathbf{b}-\mathbf{c})^{2}+4 \mathbf{s}(\mathbf{s}-\mathbf{a})\right)\right) \\
& =\mathbf{s}(\mathbf{s}-\mathbf{a})\left(\mathbf{s}(\mathbf{s}-\mathbf{a})+(\mathbf{b}-\mathbf{c})^{2}\left(\frac{\mathbf{a}^{2}-(\mathbf{b}-\mathbf{c})^{2}}{\mathbf{a}^{2}}\right)-\frac{\mathbf{b c}}{(\mathbf{b}+\mathbf{c})^{2}}\left((\mathbf{b}-\mathbf{c})^{2}+(\mathbf{b}+\mathbf{c})^{2}-\mathbf{a}^{2}\right)\right) \\
& =s(s-a)\left(s(s-a)-b c+\left(\mathbf{a}^{2}-(b-\mathbf{c})^{2}\right)\left(\frac{(b-\mathbf{c})^{2}}{\mathbf{a}^{2}}+\frac{\mathbf{b c}}{(b+\mathbf{c})^{2}}\right)\right) \\
& =\frac{s(s-a)}{4}\left(\left((b+c)^{2}-\mathbf{a}^{2}-4 \mathbf{b c}\right)+\left(\mathbf{a}^{2}-(\mathbf{b}-\mathbf{c})^{2}\right)\left(\frac{4(\mathbf{b}-\mathbf{c})^{2}}{\mathbf{a}^{2}}+\frac{4 \mathbf{b c}}{(\mathbf{b}+\mathbf{c})^{2}}\right)\right) \\
& =\frac{\mathbf{s}(\mathbf{s}-\mathbf{a})}{4}\left((\mathbf{b}-\mathbf{c})^{2}-\mathbf{a}^{2}+\left(\mathbf{a}^{2}-(\mathbf{b}-\mathbf{c})^{2}\right)\left(\frac{4(\mathbf{b}-\mathbf{c})^{2}}{\mathbf{a}^{2}}+\frac{\mathbf{4 b c}}{(\mathbf{b}+\mathbf{c})^{2}}\right)\right) \\
& =\frac{s(s-a)}{4}\left(a^{2}-(b-c)^{2}\right)\left(\frac{4(b-c)^{2}}{a^{2}}+\frac{4 b c}{(b+c)^{2}}-1\right) \\
& =\frac{s(s-a)}{4} .4(s-b)(s-c)\left(\frac{4(b-c)^{2}}{a^{2}}-\frac{(b-c)^{2}}{(b+c)^{2}}\right)=r^{2} s^{2}(b-c)^{2}\left(\frac{4}{a^{2}}-\frac{1}{(b+c)^{2}}\right) \\
& =\mathbf{r}^{2} \mathbf{s}^{2}(b-c)^{2}\left(\frac{2}{a}+\frac{1}{b+c}\right)\left(\frac{2 b+2 c-a}{a(b+c)}\right) \geq 0 \\
& \Rightarrow n_{a}^{2} g_{a}^{2} \geq \mathbf{m}_{\mathrm{a}}^{2} \mathbf{w}_{\mathrm{a}}^{2} \Rightarrow \mathbf{m}_{\mathrm{a}} \mathbf{w}_{\mathrm{a}} \leq \mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}} \text { and analogs } \Rightarrow 1+\sqrt[3]{\frac{\mathbf{m}_{\mathrm{a}} m_{b} m_{\mathrm{c}} \mathbf{w}_{\mathrm{a}} \mathbf{w}_{\mathrm{b}} \mathbf{w}_{\mathrm{c}}}{\mathbf{h}_{\mathrm{a}} \mathbf{h}_{\mathrm{b}} \mathbf{h}_{\mathrm{c}} \mathbf{g}_{\mathrm{a}} \mathrm{~g}_{\mathrm{b}} \mathrm{~g}_{\mathrm{c}}}} \\
& \leq 1+\sqrt[3]{\frac{n_{\mathrm{a}} n_{b} n_{c} g_{a} g_{b} g_{c}}{\mathbf{h}_{\mathrm{a}} \mathbf{h}_{\mathrm{b}} \mathbf{h}_{\mathrm{c}} \mathrm{~g}_{\mathrm{a}} g_{\mathrm{b}} g_{\mathrm{c}}}}
\end{aligned}
$$



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$$
\text { Also, } \begin{aligned}
\mathrm{an}_{\mathrm{a}}^{2} & =\mathrm{as}^{2}-4 \operatorname{sbcsin}^{2} \frac{\mathrm{~A}}{2}=\mathrm{as}^{2}-\frac{4 \operatorname{sbc}(s-b)(s-c)(s-a)}{\mathbf{b c}(s-a)}=\mathrm{as}^{2}-\frac{4 \Delta^{2}}{s-a} \\
& =\mathrm{as}^{2}-2 \mathrm{a}\left(\frac{2 \Delta}{\mathrm{a}}\right)\left(\frac{\Delta}{s-a}\right)=\mathrm{as}^{2}-2 \mathrm{ah}_{\mathrm{a}} \mathrm{r}_{\mathrm{a}}
\end{aligned}
$$

(iv)

$$
\therefore \mathbf{n}_{\mathbf{a}}^{2} \stackrel{M}{\cong} s^{2}-2 h_{a} r_{a}
$$

$$
=\frac{\mathbf{a}^{2}}{4 \mathbf{r}^{2}}-\left(\frac{2 r s}{s-a}\right)\left(\frac{\mathbf{a}}{2 \mathbf{r} s}\right)=\frac{\mathbf{a}^{2}}{4 \mathbf{r}^{2}}-\frac{(\mathbf{a}-\mathbf{s})+\mathbf{s}}{\mathbf{s}-\mathbf{a}}
$$

$$
=\frac{a^{2}}{4 r^{2}}+1-\frac{s}{s-a}=1+\frac{a^{2}(s-a)-4\left(s r^{2}\right)}{4(s-a) r^{2}}=1+\frac{a^{2}(s-a)-4(s-a)(s-b)(s-c)}{4(s-a) r^{2}}
$$

$$
=1+\frac{\mathbf{a}^{2}-\left(\mathbf{a}^{2}-(\mathbf{b}-\mathbf{c})^{2}\right)}{4 \mathbf{r}^{2}}=1+\frac{(\mathbf{b}-\mathbf{c})^{2}}{4 \mathbf{r}^{2}}
$$

$$
\Leftrightarrow \frac{\mathbf{R}^{2}}{\mathbf{r}^{2}}-\frac{2 \mathbf{R}}{\mathbf{r}} \geq \frac{(\mathbf{b}-\mathbf{c})^{2}}{4 \mathbf{r}^{2}} \Leftrightarrow \frac{\mathbf{R}(\mathbf{R}-2 \mathbf{r})}{\mathbf{r}^{2}} \geq \frac{\mathbf{b}^{2}+\mathbf{c}^{2}-2 \mathbf{b c}}{4 \mathbf{r}^{2}} \Leftrightarrow \mathbf{R}-2 \mathbf{r} \geq \frac{\mathbf{b}^{2}+\mathbf{c}^{2}}{4 \mathbf{R}}-\frac{\mathbf{b c}}{2 \mathbf{R}}
$$

$$
\Leftrightarrow R\left(1-\frac{2 r}{R}\right) \geq \frac{4 R^{2}\left(\sin ^{2} B+\sin ^{2} C\right)}{4 R}-\frac{4 R^{2} \sin B \sin C}{2 R} \Leftrightarrow 1-\frac{8 R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}}{R}
$$

$$
\geq \sin ^{2} B+\sin ^{2} C-2 \sin B \sin C=(\sin B-\sin C)^{2}
$$

$$
\Leftrightarrow 1-4 \sin \frac{A}{2}\left(2 \sin \frac{B}{2} \sin \frac{C}{2}\right) \geq\left(2 \cos \frac{B+C}{2} \sin \frac{B-C}{2}\right)^{2}
$$

$$
\Leftrightarrow 1-4 \sin \frac{A}{2}\left(\cos \frac{B-C}{2}-\cos \frac{B+C}{2}\right) \geq 4 \sin ^{2} \frac{A}{2}\left(1-\cos ^{2} \frac{B-C}{2}\right)
$$

$$
\Leftrightarrow 1-4 \sin \frac{A}{2} \cos \frac{B-C}{2}+4 \sin ^{2} \frac{A}{2} \geq 4 \sin ^{2} \frac{A}{2}-4 \sin ^{2} \frac{A}{2} \cos ^{2} \frac{B-C}{2}
$$

$$
\Leftrightarrow 4 \sin ^{2} \frac{A}{2} \cos ^{2} \frac{B-C}{2}-4 \sin \frac{A}{2} \cos \frac{B-C}{2}+1 \geq 0
$$

$$
\Leftrightarrow\left(2 \sin \frac{A}{2} \cos \frac{B-C}{2}-1\right)^{2} \geq 0 \rightarrow \operatorname{true} \Rightarrow \frac{n_{a}}{h_{a}} \leq \frac{R}{r}-1 \text { and analogs } \Rightarrow \frac{n_{a} n_{b} n_{c}}{h_{a} h_{b} h_{c}}
$$

$$
\leq\left(\frac{\mathbf{R}}{\mathbf{r}}-\mathbf{1}\right)^{3} \Rightarrow \sqrt[3]{\frac{\mathbf{n}_{\mathrm{a}} \mathbf{n}_{\mathrm{b}} \mathbf{n}_{\mathrm{c}}}{\mathbf{h}_{\mathrm{a}} \mathbf{h}_{\mathrm{b}} \mathbf{h}_{\mathrm{c}}}} \leq \frac{\mathbf{R}}{\mathbf{r}}-\mathbf{1} \Rightarrow \mathbf{1}+\sqrt[3]{\frac{\mathbf{n}_{\mathrm{a}} \mathbf{n}_{\mathrm{b}} \mathbf{n}_{\mathrm{c}}}{\mathbf{h}_{\mathrm{a}} \mathbf{h}_{\mathrm{b}} \mathbf{h}_{\mathrm{c}}}} \stackrel{(v)}{\leq} \frac{\mathbf{R}}{\mathbf{r}}
$$



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$$
\begin{aligned}
& \Leftrightarrow \frac{r}{\sin \frac{A}{2}}+r \stackrel{?}{\ddot{\ddot{ }}} \frac{8 R r s \cos \frac{A}{2}}{4 R(b+c) \sin \frac{A}{2} \cos \frac{A}{2}}
\end{aligned}
$$

$$
\begin{aligned}
& \Leftrightarrow 4 R \cos \frac{A}{2} \cos \frac{B-C}{2} \sin \frac{A}{2} \stackrel{?}{\leftrightarrows} 4 R \sin \frac{A}{2} \cos \frac{A}{2} \\
& \Leftrightarrow \cos \frac{B-C}{2} \underset{\leq}{\sim} 1 \rightarrow \text { true } \therefore w_{a} \geq g_{a} \text { and analogs } \Rightarrow 1+\sqrt[3]{\frac{m_{a} m_{b} m_{c} w_{a} w_{b} w_{c}}{h_{a} h_{b} h_{c} g_{a} g_{b} g_{c}}}
\end{aligned}
$$

$$
\begin{aligned}
& \Rightarrow \mathbf{1}+\sqrt[3]{\frac{m_{a} m_{b} m_{c} w_{a} w_{b} w_{c}}{h_{a} h_{b} h_{c} g_{a} g_{b} g_{c}}} \stackrel{(n)}{n} 2 \because(m),(n) \Rightarrow \frac{R}{r} \geq 1+\sqrt[3]{\frac{m_{a} m_{b} m_{c} w_{a} w_{b} w_{c}}{h_{a} h_{b} h_{c} g_{a} g_{b} g_{c}}} \\
& \geq 2 \text { (Proved) }
\end{aligned}
$$

1569. If $\triangle A B C$ the following relationship holds:

$$
\prod_{c y c} \frac{m_{a}}{r_{a}} \leq \prod_{c y c} \frac{a^{2}+b^{2}}{(a+b) c}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{aligned}
w_{a}=\frac{2 \sqrt{b c}}{b+c} \cdot \sqrt{r_{a} r_{b}} ; w_{b} & =\frac{2 \sqrt{a c}}{a+c} \cdot \sqrt{r_{a} r_{c}} ; w_{c}=\frac{2 \sqrt{a b}}{a+b} \cdot \sqrt{r_{a} r_{c}} \Rightarrow \\
\frac{w_{a} w_{b} w_{c}}{r_{a} r_{b} r_{c}} & =\frac{8 a b c}{(a+b)(b+c)(c+a)}
\end{aligned}
$$



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$$
\begin{aligned}
& \frac{m_{a} m_{b} m_{c}}{r_{a} r_{b} r_{c}}=\frac{\left(a^{2}+b^{2}\right)\left(b^{2}+c^{2}\right)\left(c^{2}+a^{2}\right)}{8(a b c)^{2}} \cdot \frac{s_{a} s_{b} s_{c}}{r_{a} r_{b} r_{c}} \leq \frac{\left(a^{2}+b^{2}\right)\left(b^{2}+c^{2}\right)\left(c^{2}+a^{2}\right)}{(a+b)(b+c)(c+a) a b c} \Leftrightarrow \\
& \frac{s_{a} s_{b} s_{c}}{r_{a} r_{b} r_{c}} \leq \frac{8 a b c}{(a+b)(b+c)(c+a)}=\frac{w_{a} w_{b} w_{c}}{r_{a} r_{b} r_{c}}
\end{aligned}
$$

Which is true because: $s_{a} \leq w_{a} ; s_{b} \leq w_{b} ; s_{c} \leq w_{c}$. Proved.
1570. In $\triangle A B C, g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\sum \frac{\mathbf{h}_{a}}{\left|m_{a}+w_{b}+w_{c}+\sqrt{3}\left(g_{a}-a\right)\right|} \geq \frac{g_{a}+g_{b}+g_{c}-s}{2 \sqrt{3} r}
$$

## Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{align*}
& \text { Stewart's theorem } \Rightarrow b^{2}(s-c)+\mathbf{c}^{2}(s-b) \\
& =\mathbf{a n}_{\mathrm{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathbf{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \text { Adding the above two, we get : }\left(b^{2}+c^{2}\right)(2 s-b-c) \\
& =\mathbf{a n}_{\mathrm{a}}^{2}+\mathbf{a g}_{\mathrm{a}}^{2}+\mathbf{2 a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(n_{a}^{2}+g_{a}^{2}\right)+\mathbf{a}(a+b-c)(c+a-b) \Rightarrow 2\left(b^{2}+c^{2}\right) \\
& =2\left(n_{a}^{2}+g_{a}^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \\
& \Rightarrow(b-c)^{2}+4 s(s-a)+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \\
& \Rightarrow \mathbf{n}_{\mathrm{a}}^{2}+\mathrm{g}_{\mathrm{a}}^{2} \stackrel{m}{=}(\mathrm{b}-\mathrm{c})^{2}+2 \mathrm{~s}(\mathrm{~s}-\mathrm{a})  \tag{i}\\
& \text { Also, Stewart's theorem } \Rightarrow b^{2}(s-c)+\mathbf{c}^{2}(s-b)=\mathbf{a n}_{\mathrm{a}}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=\mathbf{a n}_{a}^{2}+a\left(s^{2}-s(2 s-a)+b c\right) \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c \\
& =\mathbf{a n}_{\mathrm{a}}^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+c^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{a}^{2}-\mathbf{a s}^{2} \Rightarrow \mathbf{a n}_{\mathrm{a}}^{2}=\mathbf{a s}{ }^{2}+\mathbf{s}(2 b c \cos A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \operatorname{sbc}(s-b)(s-c)(s-a)}{\mathbf{b c}(s-a)}
\end{align*}
$$



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$$
\text { Now, Lessel }- \text { Pelling } \Rightarrow \sum \frac{\mathbf{h}_{\mathrm{a}}}{\left|\mathrm{~m}_{\mathrm{a}}+\mathrm{w}_{\mathrm{b}}+\mathbf{w}_{\mathrm{c}}+\sqrt{3}\left(\mathrm{~g}_{\mathrm{a}}-\mathbf{a}\right)\right|} \geq \sum \frac{\mathbf{h}_{\mathrm{a}}}{\left|s \sqrt{3}+\sqrt{3}\left(\mathbf{g}_{\mathrm{a}}-\mathbf{a}\right)\right|}
$$

$$
=\frac{1}{\sqrt{3}} \sum \frac{h_{a}}{g_{a}+s-a} \stackrel{m}{m}^{\text {(iii) }} \frac{g_{a}+g_{b}+g_{c}-s}{2 \sqrt{3} r}(\text { Proved })
$$

1571. In $\triangle \mathrm{ABC}$ the following relationship holds:

$$
\frac{\tan ^{2} \frac{A}{2}+\tan ^{2} \frac{B}{2}}{\tan \frac{A}{2}+\tan \frac{B}{2}}+\frac{\tan ^{2} \frac{B}{2}+\tan ^{2} \frac{C}{2}}{\tan \frac{B}{2}+\tan \frac{C}{2}}+\frac{\tan ^{2} \frac{C}{2}+\tan ^{2} \frac{A}{2}}{\tan \frac{C}{2}+\tan \frac{A}{2}} \leq \frac{3\left(\tan ^{2} \frac{A}{2}+\tan ^{2} \frac{B}{2}+\tan ^{2} \frac{C}{2}\right)}{\tan \frac{A}{2}+\tan \frac{B}{2}+\tan \frac{C}{2}}
$$

## Solution by Rahim Shahbazov-Baku-Azerbaijan

$$
\begin{gather*}
x, y, z>0 \text { then: } \\
\frac{x^{2}+y^{2}}{x+y}+\frac{y^{2}+z^{2}}{y+z}+\frac{z^{2}+x^{2}}{z+x} \leq \frac{3\left(x^{2}+y^{2}+z^{2}\right)}{x+y+z} \tag{1}
\end{gather*}
$$

$$
\begin{aligned}
& =\mathbf{a s}^{2}-\frac{4 \mathbf{a s}(s-b)(s-c)}{a} \Rightarrow n_{a}^{2} \stackrel{(i i)}{\cong} \mathbf{s}^{2}-\frac{4 \mathbf{s}(s-b)(s-c)}{a} \therefore \text { (i), (ii) } \Rightarrow g_{a}^{2} \\
& =(b-c)^{2}+2 s(s-a)-s^{2}+\frac{4 s(s-b)(s-c)}{a} \\
& =s^{2}-2 s a+a^{2}+(b-c)^{2}-a^{2}+\frac{4 s(s-b)(s-c)}{a} \\
& =(s-a)^{2}+(b-c+a)(b-c-a)+\frac{4 s(s-b)(s-c)}{a} \\
& =(s-a)^{2}-4(s-b)(s-c)+\frac{4 s(s-b)(s-c)}{a}=(s-a)^{2}+4(s-b)(s-c)\left(\frac{s}{a}-1\right) \\
& =(s-a)^{2}+\frac{4(s-a)(s-b)(s-c)}{a} \\
& \Rightarrow \mathbf{g}_{a}^{2}=(\mathbf{s}-\mathbf{a})^{2}+\frac{4(\mathbf{s}-\mathbf{a})(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})}{\mathbf{a}} \Rightarrow\left(\mathbf{g}_{a}+\mathbf{s}-\mathbf{a}\right)\left(\mathbf{g}_{a}-\mathbf{s}+\mathbf{a}\right) \\
& =\frac{4(s-a)(s-b)(s-c)}{a}=2 r\left(\frac{2 r s}{a}\right)=2 r h_{a} \\
& \Rightarrow \frac{h_{a}}{g_{a}+s-a}=\frac{g_{a}-s+a}{2 r} \text { and analogs } \Rightarrow \sum \frac{h_{a}}{g_{a}+s-a} \\
& =\frac{\sum\left(g_{a}-s+\mathbf{a}\right)}{2 r} \stackrel{(\mathrm{iii})}{\cong} \frac{g_{a}+g_{b}+g_{c}-s}{2 r}
\end{aligned}
$$



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$$
\begin{gather*}
\sum\left(x^{2}+y^{2}+\frac{z\left(x^{2}+y^{2}\right)}{x+y}\right) \leq 3\left(x^{2}+y^{2}+z^{2}\right) \\
\frac{z\left(x^{2}+y^{2}\right)}{x+y}+\frac{x\left(y^{2}+z^{2}\right)}{y+z}+\frac{y\left(z^{2}+x^{2}\right)}{z+x} \leq 3\left(x^{2}+y^{2}+z^{2}\right) \tag{2}
\end{gather*}
$$

$$
\frac{z\left(x^{2}+y^{2}\right)-2 x y z}{x+y}+\frac{x\left(y^{2}+z^{2}\right)-2 x y z}{y+z}+\frac{y\left(z^{2}+x^{2}\right)-2 x y z}{z+x} \leq x^{2}+y^{2}+z^{2}
$$

$$
x^{2}+y^{2}+z^{2}+2 x y z\left(\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}\right) \geq 2(x y+y z+z x)
$$

$$
\text { But: } \frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x} \geq \frac{9}{2(x+y+z)} \text { then }
$$

$$
x^{2}+y^{2}+z^{2}+\frac{9 x y z}{x+y+z} \geq 2(x y+y z+z x) \text { Schur's inequality. }
$$

$$
\text { Let: } x=\tan \frac{A}{2}, y=\tan \frac{B}{2}, z=\tan \frac{C}{2} . \text { Proved. }
$$

1572. In $\triangle A B C$ the following relationship holds:

$$
r^{3}(1+2 \sqrt{3})^{3} \leq(a+r)(b+r)(c+r) \leq R^{3}\left(\frac{1+2 \sqrt{3}}{2}\right)^{3}
$$

## Proposed by Marin Chirciu-Romania

## Solution by Marian Ursărescu-Romania

From Huygens inequality we have:

$$
\sqrt[3]{(a+r)(b+r)(c+r)} \geq \sqrt[3]{a b c}+\sqrt[3]{r^{3}}
$$

We must show: $\sqrt[3]{a b c}+r \geq r(1+2 \sqrt{3}) \Leftrightarrow \sqrt[3]{a b c} \geq 2 r \sqrt{3} \Leftrightarrow \sqrt[3]{4 s R r} \geq 2 \sqrt{3} r$

$$
\Leftrightarrow 4 s R r \geq 8 \cdot 3 \sqrt{3} r^{3} \Leftrightarrow R s \geq 6 \sqrt{3} r^{2}
$$

true because $R \geq 2 r$ (Euler) and $s \geq 3 \sqrt{3} r$ (Mitrinovic)

$$
\sqrt[3]{(a+r)(b+r)(c+r)} \leq \frac{a+b+c+3 r}{3}
$$

We must show: $\frac{2 s+3 r}{3} \leq \frac{R(1+2 \sqrt{3})}{2} \Leftrightarrow 2 s+3 r \leq \frac{R(3+6 \sqrt{3})}{2}$

$$
\begin{gather*}
3 r \leq \frac{3 R}{2} \text { (Euler) and } 2 s \leq 3 \sqrt{3} R \text { (Mitrinovic) }  \tag{1}\\
2 s+3 r \leq \frac{3 R}{2}+3 \sqrt{3} R=\frac{R(3+6 \sqrt{3})}{2} \Rightarrow(1) \text { is true. }
\end{gather*}
$$



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1573. Let $\Delta A^{\prime} B^{\prime} C^{\prime}$ be the circumcevian triangle of orthocenter in acute $\triangle A B C$. Prove that:

$$
\frac{A A^{\prime}}{B A^{\prime} \cdot C A^{\prime}}+\frac{B B^{\prime}}{C B^{\prime} \cdot A C^{\prime}}+\frac{C C^{\prime}}{A C^{\prime} \cdot B C^{\prime}} \geq \frac{1}{R}\left(2+\frac{R^{2}}{r^{2}}\right)
$$

## Proposed by Marian Ursărescu - Romania

## Solution by Tran Hong-Dong Thap-Vietnam


$A B A^{\prime} C$-cyclic quadrilateral $\Rightarrow A K \cdot K A^{\prime}=K B \cdot K C$

$$
\begin{gathered}
\Rightarrow K A^{\prime}=\frac{K B \cdot K C}{A K}=\frac{K B \cdot K C}{h_{a}}=\frac{c \cos B \cdot b \cdot \cos C}{\frac{2 S}{a}}=\frac{a b c \cdot \cos B \cdot \cos C}{2 S}=2 R \cos B \cos C(\therefore \Delta A B C- \\
\text { acute })
\end{gathered}
$$

$$
\begin{gathered}
\Rightarrow A A^{\prime}=h_{a}+K A^{\prime}=h_{a}+2 R \cdot \cos B \cos C \\
S_{\Delta A^{\prime} B C}=\frac{1}{2} \cdot A^{\prime} K \cdot B C=\frac{1}{2} \cdot B A^{\prime} \cdot C A^{\prime} \sin \left(B A^{\prime} C\right)=\frac{1}{2} B A^{\prime} C A^{\prime} \sin \left(180^{\circ}-A\right) \\
=\frac{1}{2} \cdot B A^{\prime} \cdot C A^{\prime} \cdot \sin A \\
\Rightarrow B A^{\prime} \cdot C A^{\prime}=\frac{A^{\prime} K \cdot B C}{\sin A}=\frac{2 R \cos B \cos C a}{\sin A}=\frac{2 R \cos B \cos C \cdot 2 R \sin A}{\sin A} \\
=4 R^{2} \cos B \cos C \\
\Rightarrow \frac{A A^{\prime}}{B A^{\prime} \cdot C A^{\prime}}=\frac{h_{a}+2 R \cos B \cos C}{4 R^{2} \cos B \cos C}=\frac{h_{a}}{4 R^{2} \cdot \cos B \cos C}+\frac{1}{2 R}(e t c) \\
\Rightarrow L H S=\frac{A A^{\prime}}{B A^{\prime} \cdot C A^{\prime}}+\frac{B B^{\prime}}{C B^{\prime} \cdot A B^{\prime}}+\frac{C C^{\prime}}{A C^{\prime} \cdot B C^{\prime}} \\
=\frac{1}{4 R^{2}}\left(\frac{h_{a}}{\cos B \cos C}+\frac{h_{b}}{\cos A \cos C}+\frac{h_{c}}{\cos A \cos B}\right)+\frac{3}{2 R}
\end{gathered}
$$



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$=\frac{2 S}{4 R^{2}}\left(\frac{b c \cdot \cos A+a c \cdot \cos B+a b \cdot \cos C}{a b c \cos A \cos B \cos C}\right)+\frac{3}{2 R}$

$$
=\frac{2 S}{4 R^{2}}\left(\frac{\frac{a^{2}+b^{2}+c^{2}}{2}}{4 R \cdot S \cdot \frac{s^{2}-(2 R r+r)^{2}}{4 R^{2}}}\right)+\frac{3}{2 R}=\frac{a^{2}+b^{2}+c^{2}}{4 R\left(s^{2}-(2 R+r)^{2}\right)}+\frac{3}{2 R}=
$$

$$
=\frac{\left(s^{2}-4 R r-r^{2}\right)}{2 R\left(s^{2}-(2 R+r)^{2}\right)}+\frac{3}{2 R} \stackrel{(*)}{\geq} \frac{1}{R}\left(2+\frac{R^{2}}{r^{2}}\right)
$$

$$
(*) \Leftrightarrow \frac{s^{2}-4 R r-r^{2}}{s^{2}-(2 R+r)^{2}}+3 \geq 4+\frac{2 R^{2}}{r^{2}}
$$

$$
\Leftrightarrow \frac{s^{2}-4 R r-r^{2}}{s^{2}-(2 R+r)^{2}} \geq 1+\frac{2 R^{2}}{r}\left(\because \Delta A B C-\text { acute } \Rightarrow s^{2}>(2 R+r)^{2}\right)
$$

$$
\Leftrightarrow r^{2}\left(s^{2}-4 R r-r^{2}\right) \geq\left(r^{2}+2 R^{2}\right)\left(s^{2}-4 R^{2}-4 R r-r^{2}\right)
$$

$$
\Leftrightarrow r^{2} s^{2}-4 R r^{3}-r^{4} \geq s^{2} s^{2}-4 R^{2} r^{2}-4 R r^{3}-r^{4}+2 R^{2} s^{2}-8 R^{4}-8 R^{3} r-2 R^{2} r^{2}
$$

$$
\Leftrightarrow 8 R^{4}+8 R^{3} r+6 R^{2} r^{2} \geq 2 R^{2} s^{2}
$$

$$
\Leftrightarrow s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \quad(\because \text { true })(\text { proved })
$$

1574. In acute $\triangle A B C, D, E, F$ - midpoints of (BC), (CA), $(A B), X \in(B C)$,
$Y \in(C A), Z \in(C A), X E \perp A C, Y D \perp B C, Z F \perp A B$. Prove that:

$$
\frac{Y D}{h_{a}}+\frac{X E}{h_{b}}+\frac{Z F}{h_{c}} \geq 12\left(\frac{r}{R}\right)^{2}
$$

Proposed by Mehmet Șahin - Ankara - Turkey

## Solution by Tran Hong-Dong Thap-Vietnam




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$$
\begin{aligned}
& Y D \perp B C \stackrel{\Delta A B C-\text { acute }}{\Rightarrow} \tan B=\frac{Y D}{B D}=\frac{Y D}{\frac{B C}{2}}=\frac{Y D}{\frac{a}{2}}=\frac{2 Y D}{a} \\
& \Rightarrow Y D=\frac{a \tan B}{2} \Rightarrow \frac{Y D}{h_{a}}=\frac{a \tan B}{2 h_{a}}=\frac{a^{2} \tan B}{4 S} \text {; } \\
& \text { Similarly: } \frac{X E}{h_{b}}=\frac{b^{2} \tan C}{4 S} ; \frac{Z F}{h_{c}}=\frac{c^{2} \tan A}{4 S} \\
& \Rightarrow L H S=\frac{Y D}{h_{a}}+\frac{X E}{h_{b}}+\frac{Z F}{h_{c}}=\frac{1}{4 S}\left(a^{2} \tan B+b^{2} \tan C+c^{2} \tan A\right) \stackrel{A M-G M}{\geq} \\
& \frac{3}{4 S} \sqrt[3]{(a b c)^{2}(\tan A \tan B \tan C)}=\frac{3}{4 S} \sqrt[3]{(4 R r s)^{2} \cdot \frac{2 s r}{s^{2}-(2 R+r)^{2}}} \stackrel{(*)}{\geq} 12\left(\frac{r}{R}\right)^{2} \\
& \left(^{*}\right) \Leftrightarrow \frac{1}{(s r)^{3}} \cdot \frac{(4 R r s)^{2} \cdot 2 s r}{s^{2}-(2 R+r)^{2}} \geq(16)^{3}\left(\frac{r}{R}\right)^{6} \\
& \Leftrightarrow \frac{32 R^{2}}{s^{2}-(2 R+r)^{2}} \geq 16^{3}\left(\frac{r}{R}\right)^{6} \quad\left(\because \Delta A B C-\text { acute } \Rightarrow s^{2}>(2 R+r)^{2}\right) \\
& \Leftrightarrow R^{8} \geq\left(64 r^{6}\right) \cdot 2 \cdot\left(s^{2}-(2 R+r)^{2}\right) \quad\left({ }^{* *}\right) \\
& \text { Because: } R \geq 2 r \Leftrightarrow R^{6} \geq 64 r^{6} \\
& R^{2} \geq 2\left(s^{2}-(2 R+r)^{2}\right) \Leftrightarrow R^{2} \geq 2\left(s^{2}-4 R^{2}-4 R r-r^{2}\right) \\
& \Leftrightarrow 9 R^{2}+8 R r+2 r^{2} \geq 2 s^{2} \\
& \text { But } s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \Rightarrow 2 s^{2} \leq 8 R^{2}+8 R r+6 r^{2} \\
& \text { We need to prove: } 8 R^{2}+8 R r+6 r^{2} \leq 9 R^{2}+8 R r+2 r^{2} \\
& \Leftrightarrow R^{2} \geq \mathbf{4 r} \mathbf{r}^{\mathbf{2}} \Leftrightarrow R \geq \mathbf{2 r} \text { (Euler) } \Rightarrow \text { ( }^{* *} \text { ) true } \Rightarrow \text { (*) true. Proved. }^{*}
\end{aligned}
$$

1575. In $\triangle A B C, \boldsymbol{n}_{\boldsymbol{a}}$-Nagel's cevian, $\boldsymbol{g}_{\boldsymbol{a}}$-Gergonne's cevian, the following relationship holds:

$$
\sqrt{n_{a} g_{a} h_{a}}+\sqrt{n_{b} g_{b} h_{b}}+\sqrt{n_{c} g_{c} h_{c}} \geq s\left(\sqrt{h_{a}-2 r}+\sqrt{h_{b}-2 r}+\sqrt{h_{c}-2 r}\right)
$$

## Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& \qquad=\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$



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$$
\begin{aligned}
& \quad \therefore \mathbf{a n}_{\mathbf{a}}{ }^{2} \cdot \mathbf{a g}_{\mathbf{a}^{2}}^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{(\mathbf{a})}{\geq} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$

Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, ( a )

$$
\Leftrightarrow\left\{\mathbf{z}(\mathbf{z}+\mathbf{x})^{2}+\mathbf{y}(\mathbf{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathbf{y}+\mathbf{z})\right\}\left\{\mathbf{y}(\mathbf{z}+\mathbf{x})^{2}+\mathbf{z}(\mathbf{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathbf{y}\right.
$$

$$
+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}
$$

$$
\Leftrightarrow x y^{2}+x z^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow \text { true }
$$

$$
\Rightarrow(\mathbf{a}) \text { is } \text { true } \Rightarrow \mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}} \geq \mathbf{s}(\mathbf{s}-\mathbf{a})
$$

$$
\begin{aligned}
\Rightarrow \sqrt{n_{a} g_{a} h_{a}} & \geq \sqrt{s(s-a)\left(\frac{2 r s}{a}\right)}=s \sqrt{\frac{2 r(s-a)}{a}}=s \sqrt{h_{a}-2 r} \text { and analogs } \\
& \Rightarrow \sum \sqrt{n_{a} g_{a} h_{a}} \geq s \sum \sqrt{h_{a}-2 r} \text { (Proved) }
\end{aligned}
$$

1576. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} \frac{\left(r_{a}^{2}+r_{b}^{2}+r_{c}^{2}+2 r_{a} r_{b}+2 r_{a} r_{c}\right) a^{2}}{r_{b} r_{c}} \geq 28 \sqrt{3} S
$$

## Proposed by Daniel Sitaru-Romania

## Solution 1 by George Florin Şerban-Romania

$$
\begin{gathered}
\text { Let: } r_{a}=x ; r_{b}=y ; r_{c}=z ; x+y+z=4 R+r \\
\sum_{c y c} \frac{\left(r_{a}^{2}+r_{b}^{2}+r_{c}^{2}+2 r_{a} r_{b}+2 r_{a} r_{c}\right)}{r_{b} r_{c}} \cdot a^{2}=\sum_{c y c} \frac{x^{2}+y^{2}+z^{2}+2 x y+2 x z}{y z} \cdot a^{2} \\
=\sum_{c y c} \frac{(x+y+z)^{2}-2 y z}{y z} \cdot a^{2}=\sum_{c y c} \frac{(x+y+z)^{2}}{y z} \cdot a^{2}-2 \sum_{c y c} a^{2}= \\
=(4 R+r)^{2} \sum_{c y c} \frac{a^{2}}{y z}-4\left(s^{2}-r^{2}-4 R r\right) \geq 4(4 R+r)^{2}-4 s^{2}+4 r(r+4 R) \\
\text { Because } \sum_{c y c} \frac{a^{2}}{y z}=\sum_{c y c} \frac{a^{2}}{r_{r b} r_{c}}=\frac{4(R-r)}{r} \geq \frac{4(2 r-r)}{r}=4 ; R \geq 2 r-\text { Euler }
\end{gathered}
$$



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$$
\begin{aligned}
& \text { www.ssmrmh.ro } \\
& 4(4 R+r)^{2}-4 s^{2}+4 r(r+4 R) \stackrel{?}{\dot{\sim}} 28 \sqrt{3} S \\
& 4(4 R+r)^{2}+4 r(r+4 R) \stackrel{?}{\underset{\sim}{n}} 4 s^{2}+28 \sqrt{3} S \\
& (4 R+r)^{2}+r(r+4 R) \geq s^{2}+7 \sqrt{3} r s
\end{aligned}
$$

$$
\begin{aligned}
& =4 R^{2}+4 R r+3 r^{2}+\frac{63 R r}{2} \stackrel{\text { ? }}{\leq} 4(4 R+r)^{2}+4 r(r+4 R) \\
& 8 R^{2}+71 R r+6 r^{2} \leq 32 R^{2}+24 R r+4 r^{2} \\
& 24 R^{2}-47 R r-2 r^{2} \geq 0 \mid: r^{2} \\
& 24\left(\frac{R}{r}\right)^{2}-47 \frac{R}{r}-2 \geq 0 \stackrel{\stackrel{R}{r}=t}{\Rightarrow} 24 t^{2}-47 t-2 \geq 0
\end{aligned}
$$

$(t-2)(24 t+1) \geq 0$ true, because $t \geq 2$ (Euler) and $24 t+1 \geq 24 \cdot 2+1$ Solution 2 by Tran Hong-Dong Thap-Vietnam

For all $x, y, z>0$, in any $\triangle A B C$ :

$$
x a^{2}+y b^{2}+z c^{2} \geq 4 \sqrt{x y+y z+z x} \cdot S
$$

## Choosing

$$
x=\frac{\sum_{c y c} r_{a}^{2}+2 r_{a}\left(r_{b}+r_{c}\right)}{r_{b} r_{c}} ; y=\frac{\sum_{c y c} r_{a}^{2}+2 r_{c}\left(r_{b}+r_{a}\right)}{r_{b} r_{a}} ; z=\frac{\sum_{c y c} r_{a}^{2}+2 r_{b}\left(r_{a}+r_{c}\right)}{r_{a} r_{c}}
$$

We must show that: $4 \sqrt{x y+y z+z x} \geq 28 \sqrt{3} \Leftrightarrow x y+y z+z x \geq 147 ; ~(*)$

$$
\text { Let: } \alpha=r_{a} ; \beta=r_{b} ; \gamma=r_{c} \rightarrow \alpha, \beta, \gamma>0
$$

$$
\begin{gathered}
(*) \leftrightarrow \sum_{c y c}\left(\frac{(\alpha+\beta+\gamma)^{2}-2 \beta \gamma}{\beta \gamma}\right)\left(\frac{(\alpha+\beta+\gamma)^{2}-2 \alpha \beta}{\alpha \beta}\right) \geq 147 \\
\leftrightarrow \sum_{c y c}\left[\alpha \gamma\left((\alpha+\beta+\gamma)^{2}-2 \beta \gamma\right)\left((\alpha+\beta+\gamma)^{2}-2 \alpha \beta\right)\right] \geq 147(\alpha \beta \gamma)^{2} \\
\stackrel{k=\alpha+\beta+\gamma}{\Longleftrightarrow} \sum_{c y c}\left[\alpha \gamma\left(k^{2}-2 \beta \gamma\right)\left(k^{2}-2 \alpha \beta\right)\right] \geq 147(\alpha \beta \gamma)^{2}
\end{gathered}
$$



$$
\begin{aligned}
& \text { ROMANIAN MATHEMATICAL MAGAZINE } \\
& \text { www.ssmrmh.ro } \\
& \leftrightarrow \sum_{c y c}\left[\alpha \beta\left(k^{4}-2 \beta(\alpha+\gamma) k^{2}+4 \alpha \gamma \beta^{2}\right] \geq 147(\alpha \beta \gamma)^{2}\right. \\
& \leftrightarrow(\alpha \beta+\beta \gamma+\gamma \alpha) k^{4}-4 \alpha \beta \gamma k^{3}+12(\alpha \beta \gamma)^{2} \geq 147(\alpha \beta \gamma)^{2} \\
& \leftrightarrow k^{3}[(\alpha \beta+\beta \gamma+\gamma \alpha) k-4 \alpha \beta \gamma] \geq 135(\alpha \beta \gamma)^{2} ; \\
& k^{3}=(\alpha+\beta+\gamma)^{3} \stackrel{A m-G m}{\geq} 27 \alpha \beta \gamma \\
& (\alpha \beta+\beta \gamma+\gamma \alpha)(\alpha+\beta+\gamma) \stackrel{A m-G m}{\geq} 3 \cdot \sqrt[3]{(\alpha \beta \gamma)^{2}} \cdot 3 \cdot \sqrt[3]{\alpha \beta \gamma}=9 \alpha \beta \gamma \\
& \rightarrow(\alpha \beta+\beta \gamma+\gamma \alpha)(\alpha+\beta+\gamma)-4 \alpha \beta \gamma \geq 9 \alpha \beta \gamma-4 \alpha \beta \gamma=5 \alpha \beta \gamma \\
& \rightarrow \text { LHS }_{(*)} \geq 27 \alpha \beta \gamma \cdot 5 \alpha \beta \gamma=135(\alpha \beta \gamma)^{2} \rightarrow(*) \text { is true.Proved. }
\end{aligned}
$$

1577. In $\triangle A B C$ the following relationship holds:

$$
\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leq \frac{3 R}{2 r}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution 1 by George Florin Şerban-Romania

$$
\begin{aligned}
& \sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leq \frac{3 R}{2 r} \Leftrightarrow 2 s \cdot \frac{s^{2}+r^{2}+4 R r}{4 R r s} \leq \frac{9 R^{2}}{4 r^{2}} \\
& \Leftrightarrow \frac{s^{2}+r^{2}+4 R r}{2 R r} \leq \frac{9 R^{2}}{4 r^{2}} \Leftrightarrow \frac{s^{2}+r^{2}+4 R r}{R} \leq \frac{9 R^{2}}{2 r} \\
& \Leftrightarrow 2 s^{2} r+2 r^{3}+8 R r^{2} \leq 9 R^{3} \Leftrightarrow 2 s^{2} r \leq 9 R^{3}-8 R r^{2}-2 r^{3} \\
& \Leftrightarrow s^{2} \stackrel{?}{=} \frac{9 R^{3}-8 R r^{2}-2 r^{3}}{2 r} \\
& \Leftrightarrow s^{2} \stackrel{\text { Gerretsen }}{\leq} 4 R^{2}+4 R r+3 r^{2} \stackrel{?}{\leq} \frac{9 R^{3}-8 R r^{2}-2 r^{3}}{2 r} \\
& \Leftrightarrow 8 R^{2} r+8 R r^{2}+6 r^{3} \leq 9 R^{3}-8 R r^{2}-2 r^{3} \\
& 9 R^{3}-16 R r^{2}-8 R^{2} r-8 r^{3} \geq 0 \mid: r^{3} \\
& 9\left(\frac{R}{r}\right)^{3}-16 \frac{R}{r}-8\left(\frac{R}{r}\right)^{2}-8 \geq 0 \xlongequal{\frac{R}{r}=t \geq 2(\text { Euler ) }} \\
& 9 t^{3}-8 t^{2}-16 t-8 \geq 0 \Leftrightarrow 9 t^{3}-18 t^{2}+10 t^{2}-20 t+4 t-8 \geq 0
\end{aligned}
$$



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$$
\begin{gathered}
\Leftrightarrow 9 t^{2}(t-2)+10 t(t-2)+4(t-2) \geq 0 \Leftrightarrow(t-2)\left(9 t^{2}+10 t+4\right) \geq 0, \text { true by } \\
t \geq 2 \text { and } 9 t^{2}+10 t+4 \geq 9 \cdot 2^{2}+10 \cdot 2+4=60>0
\end{gathered}
$$

## Solution 2 by Marian Ursărescu-Romania

$$
\begin{gather*}
\text { We must show: } 3+\frac{a}{b}+\frac{b}{a}+\frac{c}{a}+\frac{b}{c}+\frac{c}{b} \leq \frac{9 R^{2}}{4 r^{2}} ;  \tag{1}\\
\text { From Panaitopol inequality: } \frac{a}{b}+\frac{b}{a} \leq \frac{R}{r} ; \text { (2) } \\
\text { From (1)+(2) we must show that: } \\
3+\frac{3 R}{r} \leq \frac{9 R^{2}}{4 r^{2}} \Leftrightarrow \frac{3 R^{2}}{4 r^{2}}-\frac{R}{r}-1 \geq 0 \Leftrightarrow 3 R^{2}-4 R r-4 r^{2} \geq 0 \Leftrightarrow \\
(R-2 r)(3 R+2 r) \geq 0 \text { true from } R \geq 2 r \text { Euler }
\end{gather*}
$$

## Solution 3 by Tran Hong-Dong Thap-Vietnam

In any $\triangle A B C$ :

$$
\frac{2 R}{r} \geq \frac{a^{3}+b^{3}+c^{3}+a b c}{a b c} \Leftrightarrow \frac{3 R}{2 r} \geq \frac{3\left(a^{3}+b^{3}+c^{3}+a b c\right)}{4 a b c}
$$

So, we need to prove:

$$
\sqrt{(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)} \leq \frac{3\left(a^{3}+b^{3}+c^{3}+a b c\right)}{4 a b c} \Leftrightarrow
$$

$$
6 a b c(a+b+c)(a b+b c+c a) \leq 9\left(a^{3}+b^{3}+c^{3}+a b c\right)^{2}
$$

$$
9\left(a^{6}+b^{6}+c^{6}\right)+18\left(a^{3} b^{3}+b^{3} c^{3}+c^{3} a^{3}\right)+18 a b c\left(a^{3}+b^{3}+c^{3}\right)
$$

$$
\geq 16 a b c\left(a b^{2}+b c^{2}+c a^{2}+b a^{2}+a c^{2}+c b^{2}\right)+39(a b c)^{2} ;(*)
$$

$$
9\left(a^{6}+b^{6}+c^{6}\right) \stackrel{A m-G m}{\geq} 9 \cdot 3 \cdot \sqrt[3]{(a b c)^{6}}=27(a b c)^{2} ;(1)
$$

$$
a^{3}+b^{3}+c^{3} \stackrel{A m-G m}{\geq} 3 a b c \Leftrightarrow 4 a b c\left(a^{3}+b^{3}+c^{3}\right) \geq 12(a b c)^{2} ;(2)
$$

$$
X^{3}+Y^{3}+Z^{3} \geq X Y^{2}+Y Z^{2}+Z X^{2} ;(\forall X, Y, Z \geq 0)
$$

$$
a^{3} b^{3}+b^{3} c^{3}+c^{3} a^{3} \geq a b c\left(a b^{2}+b c^{2}+c a^{2}\right)
$$

$$
\begin{equation*}
16\left(a^{3} b^{3}+b^{3} c^{3}+c^{3} a^{3}\right) \geq 16 a b c\left(a b^{2}+b c^{2}+c a^{2}\right) \tag{3}
\end{equation*}
$$

$$
X^{3}+Y^{3}+Z^{3} \geq Y X^{2}+Z Y^{2}+X Z^{2} ;(\forall X, Y, Z \geq 0)
$$

$$
a^{3} b^{3}+b^{3} c^{3}+c^{3} a^{3} \geq a b c\left(a b^{2}+b c^{2}+c a^{2}\right)
$$

$$
\begin{equation*}
2\left(a^{3} b^{3}+b^{3} c^{3}+c^{3} a^{3}\right) \geq 2 a b c\left(a b^{2}+b c^{2}+c a^{2}\right) \tag{4}
\end{equation*}
$$



## ROMANIAN MATHEMATICAL MAGAZINE <br> $a^{3}+b^{3}+\boldsymbol{c}^{3} \geq \boldsymbol{a} \boldsymbol{b}^{2}+\boldsymbol{b} \boldsymbol{c}^{2}+\boldsymbol{c} \boldsymbol{a}^{2},(\forall a, b, c>0)$ <br> $14 a b c\left(a^{3}+b^{3}+c^{3}\right) \geq 14 a b c\left(a b^{2}+b c^{2}+c a^{2}\right),(\forall a, b, c>0) ;(5)$

From (1)+(2)+(3)+(4)+(5) we have:
$9 \sum a^{6}+4 a b c\left(a^{3}+b^{3}+c^{3}\right)+16 \sum a^{3} b^{3}+2 \sum a^{3} b^{3}+14 a b c\left(a b^{2}+b c^{2}+c a^{2}\right) \geq$

$$
16 a b c \sum\left(a b^{2}+b a^{2}\right)+27(a b c)^{2}+12(a b c)^{2} \Rightarrow(*) \text { true } \Rightarrow \text { proved. }
$$

1578. In $\triangle A B C, n_{a}$-Nagel's cevian, $\boldsymbol{g}_{\boldsymbol{a}}$-Gergonne's cevian the following relationship holds:

$$
\mathbf{s}^{2} \mathbf{g}_{\mathrm{a}} \mathbf{g}_{\mathrm{b}} \mathbf{g}_{\mathrm{c}} \geq \mathbf{r}^{2} \prod\left(\mathbf{n}_{\mathrm{a}}+\frac{2 \mathbf{r}_{\mathrm{a}} \mathbf{h}_{\mathrm{a}}}{\mathbf{n}_{\mathrm{a}}}\right)
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow b^{2}(s-c)+c^{2}(s-b)=\mathbf{a n}_{a}{ }^{2}+\mathbf{a}(s-b)(s-c) \\
& \Rightarrow \mathbf{s}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)-\mathbf{b c}(2 s-a)=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}\left(\mathbf{s}^{2}-\mathbf{s}(2 s-a)+b c\right) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow s\left(b^{2}+c^{2}-a^{2}-2 b c\right)=\mathbf{a n}_{\mathrm{a}}{ }^{2}-\mathbf{a s}^{2} \Rightarrow \mathrm{an}_{\mathrm{a}}{ }^{2}=\mathrm{as}^{2}+\mathrm{s}(2 \mathrm{bc} \cos \mathrm{~A}-2 \mathrm{bc}) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{\mathrm{~A}}{2}=\mathbf{a s}^{2}-\frac{4 \mathbf{s b c}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})(\mathbf{s}-\mathbf{a})}{\mathbf{b c}(\mathbf{s}-\mathbf{a})} \\
& =a s^{2}-\frac{4 \Delta^{2}}{s-a}=a s^{2}-2 a\left(\frac{2 \Delta}{a}\right)\left(\frac{\Delta}{s-a}\right)=a s^{2}-2 a_{a} r_{a} \therefore n_{a} \stackrel{(1)}{=} s^{2}-2 h_{a} r_{a} \\
& \text { Again, Stewart's theorem } \Rightarrow \mathbf{b}^{2}(s-c)+\mathbf{c}^{2}(s-b) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathrm{s}-\mathrm{b})(\mathrm{s}-\mathrm{c}) \text { and } \mathbf{b}^{2}(\mathrm{~s}-\mathrm{b})+\mathbf{c}^{2}(\mathrm{~s}-\mathrm{c}) \\
& =\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \therefore \mathbf{a n}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}_{\mathrm{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{\mathbf{2}}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(s-c)+\mathbf{c}^{2}(s-b)-\mathbf{a}(s-b)(s-c)\right\}\left\{\mathbf{b}^{2}(s-b)+\mathbf{c}^{2}(s-c)\right. \\
& \text { (a) } \\
& -\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \sum^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$



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Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathrm{s}-\mathbf{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}$

$$
=\mathrm{z}+\mathrm{x} \text { and } \mathrm{c}=\mathrm{x}+\mathrm{y}
$$

Using these substitutions, (a)
$\Leftrightarrow\left\{\mathbf{z}(\mathbf{z}+\mathbf{x})^{\mathbf{2}}+\mathbf{y}(\mathbf{x}+\mathbf{y})^{\mathbf{2}}-\mathbf{y z}(\mathbf{y}+\mathbf{z})\right\}\left\{\mathbf{y}(\mathbf{z}+\mathbf{x})^{\mathbf{2}}+\mathbf{z}(\mathbf{x}+\mathbf{y})^{\mathbf{2}}\right.$
$-\mathbf{y z}(y+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$

$$
\Leftrightarrow x y^{2}+x z^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0
$$

$\rightarrow$ true $\Rightarrow(a)$ is true $\Rightarrow n_{a} g_{a} \geq s(s-a)$ and analogs

$$
\Rightarrow \prod\left(n_{a} g_{a}\right) \geq s^{3} \prod(s-a)=r^{2} s^{4} \Rightarrow \prod\left(n_{a} g_{a}\right) \stackrel{(2)}{\geq} r^{2} s^{4}
$$

$$
N o w, s^{2} g_{a} g_{b} g_{c} \geq \mathbf{r}^{2} \prod\left(n_{a}+\frac{2 r_{a} h_{a}}{n_{a}}\right) \Leftrightarrow s^{2} g_{a} g_{b} g_{c}
$$

$$
\geq r^{2} \prod\left(\frac{n_{a}^{2}+2 r_{a} h_{a}}{n_{a}}\right) \stackrel{b y(1)}{\cong} r^{2}\left(\frac{s^{6}}{\prod n_{a}}\right) \Leftrightarrow \prod\left(n_{a} g_{a}\right) \geq r^{2} s^{4}
$$

$$
\rightarrow \text { true, by (2) (Proved) }
$$

1579. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\sqrt{\frac{n_{a} g_{a} h_{a}}{h_{a}-2 r}}+\sqrt{\frac{n_{b} g_{b} h_{b}}{h_{b}-2 r}}+\sqrt{\frac{n_{c} g_{c} h_{c}}{h_{c}-2 r}} \geq 3 s
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow b^{2}(s-c)+c^{2}(s-b) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathrm{b})(\mathrm{s}-\mathrm{c}) \text { and } \mathbf{b}^{2}(\mathrm{~s}-\mathrm{b})+\mathbf{c}^{2}(\mathrm{~s}-\mathrm{c}) \\
& =\mathbf{a g} \mathbf{a}^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \therefore \mathrm{an}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}_{\mathrm{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathrm{~s}-\mathrm{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& \text { (a) } \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\} \xrightarrow{〔} \mathbf{a}^{2} \mathbf{s}^{\mathbf{2}}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$



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Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c$

$$
=x+y
$$

Using these substitutions, (a)
$\Leftrightarrow\left\{\mathbf{z}(\mathrm{z}+\mathrm{x})^{2}+\mathbf{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right.$
$+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$
$\Leftrightarrow \mathrm{xy}^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq \mathbf{2 x y z}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq \mathbf{0} \rightarrow$ true $\Rightarrow(a)$ is true $\Rightarrow \mathbf{n}_{\mathrm{a}} \mathrm{g}_{\mathrm{a}} \geq \mathbf{s}(\mathbf{s}-\mathrm{a})$

$$
\begin{gathered}
\Rightarrow \sqrt{n_{a} g_{a} h_{a}} \geq \sqrt{s(s-a)\left(\frac{2 r s}{a}\right)}=s \sqrt{\frac{2 r(s-a)}{a}}=s \sqrt{h_{a}-2 r} \Rightarrow \sqrt{\frac{n_{a} g_{a} h_{a}}{h_{a}-2 r}} \\
\geq s \text { and analogs } \Rightarrow \sum \sqrt{\frac{n_{a} g_{a} h_{a}}{h_{a}-2 r}} \geq 3 s \text { (Proved) }
\end{gathered}
$$

1580. In $\triangle A B C$ the following relationship holds:

$$
\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{3}{2}+\frac{3 r(R-2 r)}{8 R^{2}}
$$

Proposed by Adil Abdullayev-Baku-Azerbaijan
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
\sum \frac{a}{b+c}=\frac{\sum a(c+a)(a+b)}{\prod(b+c)}=\frac{\sum a\left(a^{2}+\sum a b\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)} \\
=\frac{2 s\left(s^{2}-6 R r-3 r^{2}\right)+2 s\left(s^{2}+4 R r+r^{2}\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)}=\frac{2 s\left(2 s^{2}-2 R r-2 r^{2}\right)}{2 s\left(s^{2}+2 R r+r^{2}\right)} \\
\Rightarrow \sum \frac{a}{b+c}-\frac{3}{2}=\frac{2 s^{2}-2 R r-2 r^{2}}{s^{2}+2 R r+r^{2}}-\frac{3}{2}=\frac{4 s^{2}-4 R r-4 r^{2}-3\left(s^{2}+2 R r+r^{2}\right)}{2\left(s^{2}+2 R r+r^{2}\right)} \\
=\frac{s^{2}-10 R r-7 r^{2}}{2\left(s^{2}+2 R r+\mathbf{r}^{2}\right)} \geq \frac{3 r(R-2 r)}{8 R^{2}} \\
\Leftrightarrow 4 R^{2}\left(s^{2}-10 R r-7 r^{2}\right)-3 r(R-2 r)\left(s^{2}+2 R r+r^{2}\right) \geq 0
\end{gathered}
$$

$$
\Leftrightarrow\left(4 R^{2}-3 R r+6 r^{2}\right) s^{2}-4 R^{2}\left(10 R r+7 r^{2}\right)-3 r(R-2 r)\left(2 R r+r^{2}\right) \stackrel{(1)}{\underset{n}{\gtrless}} 0
$$



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Now, LHS of $(1) \stackrel{\text { Gerretsen }}{\geq}\left(4 R^{2}-3 R r+6 r^{2}\right)\left(16 R r-5 r^{2}\right)-4 R^{2}\left(10 R r+7 r^{2}\right)$

$$
-3 r(R-2 r)\left(2 R r+r^{2}\right) \geq 0
$$

$$
\Leftrightarrow 4 t^{3}-17 t^{2}+20 t-4 \stackrel{?}{\underset{\sim}{0}} 0\left(\text { where } t=\frac{R}{r}\right) \Leftrightarrow(4 t-1)(t-2)^{2} \stackrel{?}{\underset{\sim}{n}} 0 \rightarrow \text { true } \because t \stackrel{\text { Euler }}{\geq} 2
$$

$\Rightarrow(1)$ is true

$$
\Rightarrow \sum \frac{a}{b+c} \geq \frac{3}{2}+\frac{3 r(\mathbf{R}-2 \mathbf{r})}{8 \mathbf{R}^{2}}(\text { Proved })
$$

1581. In acute $\triangle A B C$ the following relationship holds:

$$
\left(\sum_{c y c} \sin ^{2} A\right)\left(\sum_{c y c} \cos ^{2} A\right) \geq\left(\sum_{c y c} \sin ^{2} \frac{A}{2}\right)\left(\sum_{c y c} \cos ^{2} \frac{A}{2}\right)
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution 1 by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\sum_{c y c} \cos ^{2} A=\frac{6 R^{2}+4 R r+r^{2}-s^{2}}{2 R^{2}} ; \sum_{c y c} \sin ^{2} A=\frac{s^{2}-4 R r-r^{2}}{2 R^{2}} \\
\sum_{c y c} \sin ^{2} \frac{A}{2}=\frac{2 R-r}{2 R} ; \sum_{c y c} \cos ^{2} \frac{A}{2}=\frac{4 R+r}{2 R} \\
\left(\sum_{c y c} \sin ^{2} A\right)\left(\sum_{c y c} \cos ^{2} A\right) \geq\left(\sum_{c y c} \sin ^{2} \frac{A}{2}\right)\left(\sum_{c y c} \cos ^{2} \frac{A}{2}\right) \\
\Leftrightarrow \frac{6 R^{2}+4 R r+r^{2}-s^{2}}{2 R^{2}} \cdot \frac{s^{2}-4 R r-r^{2}}{2 R^{2}}=\frac{2 R-r}{2 R} \cdot \frac{4 R+r}{2 R} \\
\Leftrightarrow\left(6 R^{2}+4 R r+r^{2}-s^{2}\right)\left(s^{2}-4 R r-r^{2}\right) \geq R^{2}(2 R-r)(2 R+r) \\
\Leftrightarrow-s^{4}+\left(6 R^{2}+8 R r+2 r^{2}\right) s^{2} \geq \\
\geq R^{2}(2 R-r)(2 R+r)+\left(4 R r+r^{2}\right)\left(6 R^{2}+4 R r+r^{2}\right) \\
\Leftrightarrow-s^{4}+\left(6 R^{2}+8 R r+2 r^{2}\right) s^{2}- \\
-R^{2}(2 R-r)(2 R+r)-\left(4 R r+r^{2}\right)\left(6 R^{2}+4 R r+r^{2}\right) \geq 0 \\
\Leftrightarrow\left[\left(4 R^{2}+5 R r+r^{2}\right)-s^{2}\right]\left[s^{2}-\left(2 R^{2}+3 R r+r^{2}\right)\right] \geq 0
\end{gathered}
$$



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From $s^{2} \geq 4 R^{2}+4 R r+3 r^{2}$ (Mitrinovic), we get:
$\left(4 R^{2}+5 R r+r^{2}\right)-\left(4 R^{2}+4 R r+3 r^{2}\right)=R r-2 r^{2}=r(R-2 r) \stackrel{\text { Euler }}{\geq} 0$
$\triangle A B C$-acute, then

$$
\begin{gathered}
s^{2}>(2 R+r)^{2} \Rightarrow s^{2}-\left(2 R^{2}+3 R r+r^{2}\right)>(2 R+r)^{2}-\left(2 R^{2}+3 R r+r^{2}\right)=R r> \\
0 \Rightarrow(*) \text { is true.Proved. }
\end{gathered}
$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$
\begin{equation*}
\Leftrightarrow s^{4}+8 R^{4}+22 R^{3} r+21 R^{2} r^{2}+8 R r^{3}+r^{4}-\left(6 R^{2}+8 R r+2 r^{2}\right) s^{2} \underset{\leq}{ } \tag{ii}
\end{equation*}
$$

Now, $s^{2}-2 R^{2}-8 R r-3 r^{2} \underbrace{\text { Walker }}_{(a)} 0(\because \Delta A B C$ is acute - angled $)$ and $s^{2}-4 R^{2}-4 R r$

$$
-3 \mathbf{r}^{2} \underbrace{\stackrel{\text { Gerretsen }}{\Sigma}}_{(b)} \mathbf{x}
$$

$\therefore(\mathbf{a}) .(b) \Rightarrow\left(s^{2}-2 R^{2}-8 R r-3 r^{2}\right)\left(s^{2}-4 R^{2}-4 R r-3 r^{2}\right) \leq 0$
$\Rightarrow s^{4}+8 R^{4}+40 R^{3} r+50 R^{2} r^{2}+36 R r^{3}+9 r^{4}-\left(6 R^{2}+12 R r+6 r^{2}\right) s^{2} \stackrel{(1)}{\leq} 0$
(1) $\Rightarrow$ in order to prove (ii), it suffices to prove

$$
: s^{4}+8 R^{4}+22 R^{3} r+21 R^{2} r^{2}+8 R r^{3}+r^{4}-\left(6 R^{2}+8 R r+2 r^{2}\right) s^{2}
$$

$$
\begin{aligned}
& \left(\sum \sin ^{2} \mathbf{A}\right)\left(\sum \cos ^{2} \mathbf{A}\right)=\left(\sum \sin ^{2} \mathbf{A}\right)\left(3-\sum \sin ^{2} \mathbf{A}\right) \\
& =\left(\frac{s^{2}-4 R r-r^{2}}{2 R^{2}}\right)\left(3-\frac{s^{2}-4 R r-r^{2}}{2 R^{2}}\right) \\
& \stackrel{(1)}{\cong} \frac{\left(s^{2}-4 R r-r^{2}\right)\left(6 R^{2}+4 R r+r^{2}-s^{2}\right)}{4 R^{4}} \text { and }\left(\sum \sin ^{2} \frac{A}{2}\right)\left(\sum \cos ^{2} \frac{A}{2}\right) \\
& =\frac{\left\{\sum(1-\cos \mathrm{A})\right\}\left\{\sum(1+\cos \mathrm{A})\right\}}{4}=\frac{\left(3-1-\frac{r}{\mathrm{R}}\right)\left(3+1+\frac{\mathrm{r}}{\mathrm{R}}\right)}{4} \\
& \begin{array}{c}
\stackrel{(2)}{=} \frac{(2 R-r)(4 R+r)}{4 R^{2}} \therefore(1),(2) \Rightarrow(i) \Leftrightarrow \frac{\left(s^{2}-4 R r-r^{2}\right)\left(6 R^{2}+4 R r+r^{2}-s^{2}\right)}{4 R^{4}} \\
\geq \frac{(2 R-r)(4 R+r)}{4 R^{2}}
\end{array}
\end{aligned}
$$



$$
\begin{aligned}
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& \leq \mathbf{s}^{\mathbf{4}+\mathbf{8} \mathbf{R}^{\mathbf{4}}+\mathbf{4 0} \mathbf{R}^{\mathbf{3}} \mathbf{r}+\mathbf{5 0} \mathbf{R}^{\mathbf{2}} \mathbf{r}^{\mathbf{2}}+\mathbf{3 6} \mathbf{R r}^{3}+\mathbf{9} \mathbf{r}^{\mathbf{4}}-\left(\mathbf{6} \mathbf{R}^{\mathbf{2}}+\mathbf{1 2 R r}+\mathbf{6 r}^{\mathbf{2}}\right) \mathbf{s}^{\mathbf{2}}}
\end{aligned}
$$

$$
\Leftrightarrow(4 R+4 r) s^{2} \stackrel{M}{\leq} 18 R^{3}+29 R^{2} r+28 R r^{2}+8 r^{3}
$$

Now, LHS of (iii) $\stackrel{\text { Gerretsen }}{\sim}\left(4 R^{2}+4 R r+3 r^{2}\right)(4 R+4 r) \stackrel{?}{\sim} \mathbf{n}^{2} 18 R^{3}+29 R^{2} r+28 R r^{2}$

$$
+8 r^{3} \Leftrightarrow 2 t^{3}-3 t^{2}-4 \stackrel{?}{n} \mathbf{0}\left(\text { where } t=\frac{R}{r}\right)
$$

$$
\Leftrightarrow(t-2)\left(2 \mathbf{t}^{2}+\mathbf{t}+2\right) \stackrel{?}{\stackrel{?}{n}} \mathbf{0} \rightarrow \text { true } \because \mathbf{t} \stackrel{\text { Euler }}{\geq} 2 \Rightarrow(\text { iii }) \Rightarrow(\mathrm{ii}) \Rightarrow(\mathrm{i}) \text { is true (Proved) }
$$

1582. In acute $\triangle A B C, P \in \operatorname{Int}(\triangle A B C)$. Prove that:

$$
\left(\frac{P A}{a \cos A}\right)^{n}+\left(\frac{P B}{b \cos B}\right)^{n}+\left(\frac{P C}{c \cos C}\right)^{n}>\frac{(\sqrt{3})^{n}}{3^{n-1}}, n \in \mathbb{N}, n \geq 2
$$

Proposed by Radu Diaconu-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

Lemma: In any $\triangle A B C, P \in \operatorname{Int}(\triangle A B C)$ :

$$
\frac{P A}{a}+\frac{P B}{b}+\frac{P C}{c} \geq \sqrt{3}
$$

Equality $\Leftrightarrow a=b=c$. Now, because: $\triangle A B C-a c u t e$

$$
\begin{gathered}
\Rightarrow 0<\cos A, \cos B, \cos C<1 \Rightarrow \frac{1}{\cos A}, \frac{1}{\cos B}, \frac{1}{\cos C}>1 \\
\Rightarrow\left(\frac{P A}{\operatorname{acos} A}\right)^{n}+\left(\frac{P B}{b \cos B}\right)^{n}+\left(\frac{P C}{\operatorname{ccos} C}\right)^{n} \stackrel{n \geq 2}{\geq} \frac{\left(\frac{P A}{a \cos A}+\frac{P B}{b \cos B}+\frac{P C}{\cos C}\right)^{n}}{3^{n-1}} \\
>\frac{\left(\frac{P A}{a}+\frac{P B}{b}+\frac{P C}{C}\right)^{n}}{3^{n-1}} \geq \frac{(\sqrt{3})^{n}}{3^{n-1}} \\
\left(\frac{P A}{a \cos A}\right)^{n}+\left(\frac{P B}{b \cos B}\right)^{n}+\left(\frac{P C}{c \cos C}\right)^{n}>\frac{(\sqrt{3})^{n}}{3^{n-1}}, n \in \mathbb{N}, n \geq 2 . \text { Proved. }
\end{gathered}
$$

1583. In $\triangle A B C$ prove that:

$$
\frac{1}{3 \sqrt{2 S}}\left(\frac{S}{R}\right)^{2} \leq \frac{w_{a} m_{a}}{\sqrt{b h_{b}}}+\frac{w_{b} m_{b}}{\sqrt{c h_{c}}}+\frac{w_{c} h_{c}}{\sqrt{a h_{a}}} \leq \frac{27 R^{2}}{4 \sqrt{2 S}}
$$



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## Solution by Bogdan Fuştei-Romania

$$
\begin{gathered}
2 S=a \cdot h_{a}=b \cdot h_{b}=c \cdot h_{c} \text { so, the inequality becomes } \\
\frac{1}{3} \cdot\left(\frac{S}{R}\right)^{2} \leq w_{a} m_{a}+w_{b} m_{b}+w_{c} m_{c} \leq \frac{27 R^{2}}{4} \\
\therefore a^{2}+b^{2}+c^{2} \leq 9 R^{2}-\text { Leibniz Inequality } \\
m_{a}^{2}+m_{b}^{2}+m_{c}^{2}=\frac{3}{4}\left(a^{2}+b^{2}+c^{2}\right) \leq \frac{3}{4} \cdot 9 R^{2}=\frac{27 R^{2}}{4} \\
w_{a} \leq \sqrt{s(s-a)} \leq m_{a} \text { and analogs, then } \\
w_{a} \cdot m_{a} \leq m_{a}^{2} \Rightarrow w_{a} m_{a}+w_{b} m_{b}+w_{c} m_{c} \leq \frac{27 R^{2}}{4} ;(1) \\
\frac{1}{3} \cdot\left(\frac{S}{R}\right)^{2}=\frac{1}{3} \cdot \frac{s^{2} r^{2}}{R^{2}}=\frac{1}{3} \cdot s^{2} \cdot \frac{r^{2}}{R^{2}} \leq \frac{R 2 r-E u l e r}{\leq} \frac{s^{2}}{12} \\
\left.m_{a} \geq \frac{b+c}{2} \cos \frac{A}{2} \right\rvert\, \cdot w_{a} \\
\Rightarrow w_{a} m_{a} \geq \frac{b+c}{2} \cos \frac{A}{2} \cdot \frac{2 b c}{b+c} \cos \frac{A}{2}=b c \cdot \cos ^{2} \frac{A}{2}=b c \cdot \frac{s(s-a)}{b c}=s(s-a) \text { and analogs } \\
\text { So, we have: } s^{2} \leq w_{a} m_{a}+w_{b} m_{b}+w_{c} m_{c} ; \frac{s^{2}}{12}<s^{2}
\end{gathered}
$$

$$
\begin{equation*}
\frac{1}{3} \cdot\left(\frac{S}{R}\right)^{2} \leq w_{a} m_{a}+w_{b} m_{b}+w_{c} m_{c} \tag{2}
\end{equation*}
$$

From (1) and (2) we have:

$$
\frac{1}{3 \sqrt{2 S}}\left(\frac{S}{R}\right)^{2} \leq \frac{w_{a} m_{a}}{\sqrt{b h_{b}}}+\frac{w_{b} m_{b}}{\sqrt{c h_{c}}}+\frac{w_{c} h_{c}}{\sqrt{a h_{a}}} \leq \frac{27 R^{2}}{4 \sqrt{2 S}}
$$

1584. In acute $\triangle A B C$ the following relationship holds:

$$
\left(\sum_{c y c} \sqrt{\sin A} \cdot \cos ^{4} \frac{A}{2}\right)\left(\sum_{c y c} a^{2} \cos ^{2} A\right)<\frac{9 R\left(s^{2}-r^{2}-4 R r\right)}{16 r} \cdot \sqrt[4]{\frac{3}{4}}
$$

Proposed by Radu Diaconu-Romania


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Solution 1 by George Florin Şerban-Romania
Suppose: $a \leq b \leq c \Rightarrow \sqrt{\sin A} \leq \sqrt{\sin B} \leq \sqrt{\sin C} ;$

$$
\cos ^{4} \frac{A}{2} \leq \cos ^{4} \frac{B}{2} \leq \cos ^{4} \frac{C}{2} ; a^{2} \leq b^{2} \leq c^{2} ; \cos ^{2} A \geq \cos ^{2} B \geq \cos ^{2} C
$$

Applying Chebyshev's inequality:

$$
\begin{gathered}
L H S=\left(\sum_{c y c} \sqrt{\sin A} \cdot \cos ^{4} \frac{A}{2}\right)\left(\sum_{c y c} a^{2} \cos ^{2} A\right) \\
\leq \frac{1}{3}\left(\sum_{c y c} \sqrt{\sin A}\right)\left(\sum_{c y c} \cos ^{4} \frac{A}{2}\right) \frac{1}{3}\left(\sum_{c y c} a^{2}\right)\left(\sum_{c y c} \cos ^{2} A\right) \\
\sum_{c y c} \sqrt{\sin A} c-B-S \sqrt{\leq \sum_{c y c} \sin A}=\sqrt{\frac{3 s}{R}} \stackrel{\text { Mitrinovic }}{\leq} \sqrt{\frac{3 \cdot 3 \sqrt{3} R}{2 R}}=3 \cdot \sqrt[4]{\frac{3}{4}} \\
\sum_{c y c} \cos ^{4} \frac{A}{2}=\frac{(4 R+r)^{2}-s^{2}}{8 R^{2}} \stackrel{\Delta A B C-}{a c u t e} \leq \frac{16 R^{2}+8 R r+r^{2}-\left(2 R^{2}+8 R r+3 r^{2}\right)}{8 R^{2}} \\
=\frac{14 R^{2}-2 r^{2}}{8 R^{2}}=\frac{7}{4}-\frac{1}{4}\left(\frac{r}{R}\right)^{2}=\frac{7}{4}-\frac{1}{4 t^{2}}=\frac{7 t^{2}-1}{4 t^{2}} ; t=\frac{R}{r} \geq 2(\text { Euler }) \\
\sum_{c y c} \cos ^{2} A=\frac{6 R^{2}+4 R r+r^{2}-s^{2}}{2 R^{2}} \leq
\end{gathered}
$$

$$
\stackrel{\stackrel{\Delta A B C-}{\text { acute }} \leq}{\leq} \frac{6 R^{2}+4 R r+r^{2}-\left(2 R^{2}+8 R r+3 r^{2}\right)}{2 R^{2}}=\frac{4 R^{2}-4 R r-2 r^{2}}{2 R^{2}}
$$

$$
=2-2 \cdot \frac{r}{R}-\left(\frac{r}{R}\right)^{2}=2-\frac{2}{t}+\frac{1}{t^{2}}=\frac{2 t^{2}-2 t+1}{t^{2}}
$$

$$
L H S \leq \frac{1}{9} \cdot 3 \cdot \sqrt[4]{\frac{3}{4}} \cdot \frac{7 t^{2}-1}{4 t^{2}} \cdot \frac{2 t^{2}-2 t-1}{t^{2}} \cdot 2\left(s^{2}-r^{2}-4 R r\right)<
$$

$$
<\frac{9 R\left(s^{2}-r^{2}-4 R r\right)}{16 r} \cdot \sqrt[4]{\frac{3}{4}}=\frac{9 t}{16} \cdot\left(s^{2}-r^{2}-4 R r\right) \cdot \sqrt[4]{\frac{3}{4}}
$$

$$
s^{2}-r^{2}-4 R r=\frac{1}{2} \sum a^{2}>0
$$



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We must show: $\frac{\left(7 t^{2}-1\right)\left(2 t^{2}-2 t-1\right)}{6 t^{4}}<\frac{9 t}{16}$

## Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\text { In acute } \triangle A B C \text {, suppose: } a \geq b \geq c \Rightarrow \sin A \geq \sin B \geq \sin C
$$

$\cos A \leq \cos B \leq \cos C ; \cos \frac{A}{2} \leq \cos \frac{B}{2} \leq \cos \frac{C}{2}$
$\Rightarrow\left(\sum_{c y c} \sqrt{\sin A} \cdot \cos ^{4} \frac{A}{2}\right) \stackrel{C^{\text {Chebyshev's }}}{\leq}$
$\leq \frac{1}{3}(\sqrt{\sin A}+\sqrt{\sin B}+\sqrt{\sin C})\left(\cos ^{4} \frac{A}{2}+\cos ^{4} \frac{B}{2}+\cos ^{4} \frac{C}{2}\right)$

$$
\stackrel{B-C-S}{\leq} \frac{\sqrt{3}}{3}(\sqrt{\sin A+\sin B+\sin C})\left(\cos ^{4} \frac{A}{2}+\cos ^{4} \frac{B}{2}+\cos ^{4} \frac{C}{2}\right)
$$

$$
\sum \sin A \leq \frac{3 \sqrt{3}}{2} \frac{\sqrt[4]{3}}{\sqrt{2}} \cdot \frac{(4 R+r)^{2}-s^{2}}{8 R^{2}}
$$

$$
\sum_{c y c} a^{2} \cos ^{2} A \stackrel{\text { Chebyshev's }}{\leq} \frac{1}{3}\left(\sum_{c y c} a^{2}\right)\left(\sum_{c y c} \cos ^{2} A\right)
$$

$$
=\frac{2}{3} \cdot\left(s^{2}-r^{2}-4 R r\right) \cdot \frac{6 R^{2}+4 R r+r^{2}-s^{2}}{2 R^{2}}=\frac{\left(s^{2}-r^{2}-4 R r\right)\left(6 R^{2}+4 R r+r^{2}-s^{2}\right)}{3 R^{2}}
$$

We must show that:

$$
\frac{\sqrt[4]{3}\left(s^{2}-r^{2}-4 R r\right)\left(6 R^{2}+4 R r+r^{2}-s^{2}\right)}{3 \cdot 8 \sqrt{2} R^{4}}<\frac{9 R\left(s^{2}-r^{2}-4 R r\right)}{16 r} \cdot \frac{\sqrt[4]{3}}{\sqrt{2}}
$$

$$
\begin{aligned}
& 8\left(14 t^{4}-14 t^{3}-9 t^{2}+2 t+1\right)<27 t^{5} \\
& \Leftrightarrow \mathbf{2 7} \boldsymbol{t}^{5}-\mathbf{1 1 2 t} \mathbf{t}^{4}+\mathbf{1 1 2 t}{ }^{3}+\mathbf{7 2} \boldsymbol{t}^{2} \mathbf{- 1 6 t - 8}>0 \\
& \Leftrightarrow(t-2)\left(27 t^{4}-58 t^{3}-4 t^{2}+64 t+112\right)+216>0 \\
& \Leftrightarrow 27 t^{4}-58 t^{3}-4 t^{2}+64 t+112 \\
& \Leftrightarrow(t-2)\left(27 t^{3}-4 t^{2}-12 t+40\right)+192>0 \\
& \Leftrightarrow 27 t^{3}-4 t^{2}-12 t+40>0 \\
& \Leftrightarrow(t-2)\left(27 t^{2}+50 t+88\right)+216>0 \text { true.Proved. }
\end{aligned}
$$



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$$
\begin{aligned}
& \Leftrightarrow 2 r\left[(4 R+r)^{2}-s^{2}\right]\left(6 R^{2}+4 R r+r^{2}-s^{2}\right) \stackrel{(*)}{<} 27 R^{5} \\
& \text { But: } s^{2} \geq 16 R r-5 r^{2} \text { (Mitrinovic) } \Rightarrow \\
& 2 r\left[(4 R+r)^{2}-s^{2}\right]\left(6 R^{2}+4 R r+r^{2}-s^{2}\right) \\
& \leq 2 r\left(16 R^{2}-8 R r+6 r^{2}\right)\left(6 R^{2}-12 R r+6 r^{2}\right) \stackrel{(* *)}{\leq} 27 R^{5} \\
& (* *) \stackrel{t=\frac{R}{r} \geq 2}{\Longleftrightarrow} 27 t^{5} \geq 2\left(16 t^{2}-8 t+6\right)\left(6 t^{2}-12 t+6\right) \\
& \Leftrightarrow 3\left(9 t^{5}-32 t^{4}+80 t^{3}-76 t^{2}+40 t-12\right)>0 \\
& \text { Let: } \varphi(t)=9 t^{5}-32 t^{4}+80 t^{3}-76 t^{2}+40 t-12, \forall t \geq 2 \\
& \varphi^{\prime}(t)=45 t^{4}-128 t^{3}+240 t^{2}-152 t+40 \\
& \varphi^{\prime \prime}(t)=180 t^{3}-384 t^{2}+480 t-152 \\
& \boldsymbol{\varphi}^{\prime \prime \prime}(\boldsymbol{t})=\mathbf{5 4 0} \boldsymbol{t}^{\mathbf{2}}-\mathbf{7 6 8} \boldsymbol{t}+\mathbf{4 8 0}>0, \forall t \geq 2 \\
& \Rightarrow \varphi^{\prime \prime}(t) \uparrow[2, \infty) \Rightarrow \varphi^{\prime \prime}(t) \geq \varphi^{\prime \prime}(2)=712>0 \\
& \Rightarrow \varphi^{\prime}(t) \uparrow[2, \infty) \Rightarrow \varphi^{\prime}(t) \geq \varphi^{\prime}(2)=392>0 \\
& \Rightarrow \varphi(t) \uparrow[2, \infty) \Rightarrow \varphi(t) \geq \varphi(2)=180>0 \\
& \Rightarrow(* *) \text { is true } \Rightarrow(*) \text { is true.Proved. }
\end{aligned}
$$

1585. In acute $\triangle A B C$ the following relationship holds:

$$
\left(\sum_{c y c} \frac{r_{a}}{\sqrt{\sin A}}\right)\left(\sum_{c y c} \frac{1}{9+\left(h_{a}+h_{b}\right)^{2}}\right) \leq \frac{1}{16} \cdot \sqrt[4]{\frac{27}{4}} \cdot\left(\frac{R}{r}\right)^{3}
$$

## Proposed by Radu Diaconu-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

In acute $\triangle A B C$, suppose $a \geq b \geq c \Rightarrow r_{a} \geq r_{b} \geq r_{c}, \sin A \geq \sin B \geq \sin C$

$$
\begin{gathered}
\Rightarrow 0<\frac{1}{\sqrt{\sin A}} \leq \frac{1}{\sqrt{\sin B}} \leq \frac{1}{\sqrt{\sin C}} \\
\sum_{c y c} \frac{r_{a}}{\sqrt{\sin A}} \stackrel{\text { Chebyshev }}{\leq} \frac{1}{3} \cdot\left(\sum_{c y c} r_{a}\right)\left(\sum_{c y c} \frac{1}{\sqrt{\sin A}}\right)=\frac{1}{3} \cdot(4 R+r) \cdot \sum_{c y c} \sqrt{\frac{1}{\sin A}}
\end{gathered}
$$



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$$
\begin{gathered}
\stackrel{B-C-S}{\leq} \frac{1}{3} \cdot(4 R+r) \sqrt{3} \cdot \sqrt{\sum_{c y c} \frac{1}{\sin A}}=\frac{\sqrt{3}}{3} \cdot(4 R+r) \cdot \sqrt{\frac{s^{2}+r^{2}+4 R r}{2 s r}} \\
\quad r \leq \frac{R}{2} \sqrt{3} \\
\leq \frac{9 R}{2} \cdot \frac{s^{2}+r^{2}+4 R r}{2} \cdot \sqrt{2 s r}
\end{gathered}
$$

$9+\left(h_{a}+h_{b}\right)^{2} \stackrel{A m-G m}{\geq} 9+4 h_{a} h_{b} \stackrel{A m-G m}{\geq} 12 \sqrt{h_{a} h_{b}}$ and analogs.
$\sum_{c y c} \frac{1}{9+\left(h_{a}+h_{b}\right)^{2}} \leq \frac{1}{12} \sum_{c y c} \frac{1}{\sqrt{h_{a} h_{b}}} \stackrel{B-c-S}{\leq} \frac{1}{12} \cdot \sqrt{3} \cdot \sqrt{\frac{1}{h_{a} h_{b}}+\frac{1}{h_{b} h_{c}}+\frac{1}{h_{c} h_{a}}}$

$$
=\frac{\sqrt{3}}{12} \cdot \sqrt{\frac{s^{2}+r^{2}+4 R r}{4 s^{2} r^{2}}}=\frac{\sqrt{3} \cdot \sqrt{s^{2}+r^{2}+4 R r}}{24 s r}
$$

$$
\Rightarrow L H S=\left(\sum_{c y c} \frac{r_{a}}{\sqrt{\sin A}}\right)\left(\sum_{c y c} \frac{1}{9+\left(h_{a}+h_{b}\right)^{2}}\right)
$$

$$
\leq \frac{3 \sqrt{3} R}{2} \sqrt{\frac{s^{2}+r^{2}+4 R r}{2 s r}} \cdot \frac{\sqrt{3} \cdot \sqrt{s^{2}+r^{2}+4 R r}}{24 s r}=\frac{3}{16} \cdot \frac{R\left(s^{2}+r^{2}+4 R r\right)}{s r \sqrt{2 s r}}
$$

$$
\stackrel{(*)}{\leq} \frac{1}{16} \cdot \frac{\sqrt[4]{27}}{\sqrt{2}} \cdot\left(\frac{R}{r}\right)^{3}
$$

$(*) \Leftrightarrow \frac{3\left(s^{2}+r^{2}+4 R r\right)}{s \sqrt{s r}} \leq \sqrt[4]{27} \cdot\left(\frac{R}{r}\right)^{2} \Leftrightarrow 3 r^{3}\left(s^{2}+r^{2}+4 R r\right)^{2} \leq 3 \sqrt{3} \cdot s^{3} \cdot R^{4}$

$$
\text { Because: } s \geq 3 \sqrt{3} r \Rightarrow s^{3} \geq 81 \sqrt{3} r^{3} \Rightarrow \sqrt{3} \cdot s^{3} R^{4} \geq\left(3 r^{3}\right)\left(81 R^{4}\right)
$$

So, we must show that:

$$
\begin{gathered}
\left(s^{2}+r^{2}+4 R r\right)^{2} \leq 81 R^{4} \Leftrightarrow s^{2}+r^{2}+4 R r \leq 9 R^{2} \Leftrightarrow s^{2} \leq 9 R^{2}-4 R r-r^{2} \\
\text { But }: s^{2} \leq 4 R^{2}+4 R r+3 r^{2} \stackrel{(* *)}{\leq} 9 R^{2}-4 R r-r^{2}
\end{gathered}
$$

$$
\begin{gathered}
(* *) \Leftrightarrow 5 R^{2}-8 R r-4 r^{2} \geq 0 \Leftrightarrow(R-2 r)(5 R+2 r) \geq 0 \text { true from } R \geq 2 r \text { (Euler) } \\
\Rightarrow(* *) \text { is true } \Rightarrow(*) \text { is true.Proved. }
\end{gathered}
$$



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1586. In $\triangle A B C$ the following relationship holds:

$$
\frac{\left(m_{b}+m_{c}\right) \sin A}{m_{a} \sin B \sin C}+\frac{\left(m_{c}+m_{a}\right) \sin B}{m_{b} \sin C \sin A}+\frac{\left(m_{a}+m_{b}\right) \sin C}{m_{c} \sin A \sin B} \geq 4 \sqrt{3}
$$

Proposed by Daniel Sitaru-Romania

## Solution 1 by Catinca Alexandru-Romania

$$
\begin{gather*}
\sum_{c y c} \frac{\left(m_{b}+m_{c}\right) \sin A}{m_{a} \sin B \sin C} \geq 4 \sqrt{3} \Leftrightarrow \sum_{c y c} \frac{\left(m_{b}+m_{c}\right) \cdot \frac{a}{2 R}}{m_{a} \cdot \frac{b c}{4 R^{2}}} \geq 4 \sqrt{3} \\
\sum_{c y c} \frac{\left(m_{b}+m_{c}\right)}{m_{a}} \cdot \frac{a \cdot 2 R}{b c} \geq 4 \sqrt{3} \Leftrightarrow 2 R \sum_{c y c} a^{2} \cdot \frac{m_{b}+m_{c}}{m_{a}} \geq 4 \sqrt{3} a b c \\
\Leftrightarrow \sum_{c y c} a^{2} \cdot \frac{m_{b}+m_{c}}{m_{a}} \geq \frac{2 \sqrt{3} a b c}{R}=8 \sqrt{3} S ;(1)  \tag{1}\\
\text { WLOG } a \geq b \geq c \Rightarrow m_{a} \leq m_{b} \leq m_{c} \Rightarrow \frac{m_{b}+m_{c}}{m_{a}} \geq \frac{m_{c}+m_{a}}{m_{b}} \geq \frac{m_{a}+m_{b}}{m_{c}} \\
\sum_{\text {cyc }} a^{2} \cdot \frac{m_{b}+m_{c}}{m_{a}} \stackrel{\text { Cebyshevis }}{\geq} \frac{1}{3}\left(\sum_{c y c} a^{2}\right)\left(\sum_{c y c} \frac{m_{a}+m_{b}}{m_{c}}\right) \geq 2\left(\sum_{c y c} a^{2}\right)
\end{gather*}
$$

(1) $\Leftrightarrow 2\left(\sum_{c y c} a^{2}\right) \geq 8 \sqrt{3} S \Leftrightarrow\left(\sum_{c y c} a^{2}\right) \geq 4 \sqrt{3} S$ (Ionescu Weitzenbock)

$$
\begin{equation*}
\Leftrightarrow\left(\sum_{c y c} a^{2}\right)^{2} \geq \mathbf{4 8 S} S^{2} \tag{2}
\end{equation*}
$$

But: $\mathbf{1 6} S^{2} \stackrel{\text { Heron }}{=} \sum_{c y c} a^{2}\left(b^{2}+c^{2}-a^{2}\right) \stackrel{\text { Cebyshev's }}{\leq}$

$$
\begin{gathered}
\leq \frac{1}{3}\left(\sum_{c y c} a^{2}\right)\left(\sum_{c y c}\left(b^{2}+c^{2}-a^{2}\right)\right)=\frac{1}{3}\left(\sum_{c y c} a^{2}\right)^{2} \\
\Rightarrow 48 S^{2} \leq\left(\sum_{c y c} a^{2}\right)^{2} \Rightarrow(2) \text { is true. }
\end{gathered}
$$

Solution 2 by Tran Hong-Dong Thap-Vietnam

$$
\frac{\left(m_{b}+m_{c}\right) \sin A}{m_{a} \sin B \sin C}+\frac{\left(m_{c}+m_{a}\right) \sin B}{m_{b} \sin C \sin A}+\frac{\left(m_{a}+m_{b}\right) \sin C}{m_{c} \sin A \sin B}
$$



$$
\begin{gathered}
\text { ROMANIAN MATHEMATICAL MAGAZINE } \\
=\left(\frac{m_{b}}{m_{a}}+\frac{m_{c}}{m_{a}}\right) \frac{\sin A}{\sin B \sin C}+\left(\frac{m_{c}}{m_{b}}+\frac{m_{a}}{m_{b}}\right) \frac{\sin B}{\sin C \sin A}+\left(\frac{m_{a}}{m_{c}}+\frac{m_{b}}{m_{c}}\right) \frac{\sin C}{\sin A \sin B} \\
=\sum_{c y c}\left(\frac{m_{b}}{m_{a}} \cdot \frac{\sin A}{\sin B \sin C}+\frac{m_{a}}{m_{b}} \cdot \frac{\sin B}{\sin C \sin A}\right) \\
\begin{array}{c}
\text { Am-Gm } \\
\geq
\end{array} \sum_{c y c} \sqrt{\frac{m_{b} m_{a}}{m_{a} m_{b}} \cdot \frac{1}{\sin ^{2} C}}=2\left(\frac{1}{\sin A}+\frac{1}{\sin B}+\frac{1}{\sin C}\right) \\
\geq 2 \cdot \frac{18}{\operatorname{cin} A+\sin B+\sin C} \geq \frac{18}{\frac{3 \sqrt{3}}{2}}=4 \sqrt{3}
\end{gathered}
$$

$$
\text { Because: } \sin A+\sin B+\sin C=\frac{3 \sqrt{3}}{2}
$$

## Solution 3 by Avishek Mitra-West Bengal-India

$$
\begin{gathered}
\frac{\left(m_{b}+m_{c}\right) \sin A}{m_{a} \sin B \sin C}+\frac{\left(m_{c}+m_{a}\right) \sin B}{m_{b} \sin C \sin A}+\frac{\left(m_{a}+m_{b}\right) \sin C}{m_{c} \sin A \sin B} \geq 4 \sqrt{3} \\
\Leftrightarrow \sum_{c y c} \frac{m_{b}+m_{c}}{m_{a}} \cdot \frac{\sin A}{\sin B \sin C}=\sum_{c y c} \frac{m_{b}+m_{c}}{m_{a}} \cdot \frac{\frac{a}{2 R}}{\frac{b c}{4 R^{2}}}=2 R \sum_{c y c} \frac{m_{b}+m_{c}}{m_{a}} \cdot \frac{a}{b c} \\
\stackrel{A m-G m}{\geq} 2 R \cdot 3\left(\frac{\prod\left(m_{b}+m_{c}\right)}{\prod m_{a}} \cdot \frac{a b c}{a^{2} b^{2} c^{2}}\right)^{\frac{1}{3}} \\
\stackrel{A m-G m}{\geq} 3 \cdot 2 R\left(\frac{\prod 2 \sqrt{m_{b} \cdot m_{c}}}{\prod m_{a}} \cdot \frac{1}{a b c}\right)^{\frac{1}{3}}=6 R \cdot 2\left(\frac{\prod m_{a}}{\prod m_{a}} \cdot \frac{1}{a b c}\right)^{\frac{1}{3}}
\end{gathered}
$$

We need to show: $\frac{12 R}{\sqrt[3]{a b c}} \geq 4 \sqrt{3} \Rightarrow 12^{3} R^{3} \geq 64 \cdot 3 \sqrt{3} \cdot 4$ Rrs $\Rightarrow \frac{9}{\sqrt{3}} R^{2} \geq 4 r s \Rightarrow(3 \sqrt{3} R) R \geq(2 s)(2 r)$ true from $R \geq 2 r$ (Euler) and

$$
3 \sqrt{3} R \geq 2 s \text { (Mitrinovic) }
$$

1587. In $\triangle A B C, \boldsymbol{n}_{\boldsymbol{a}}$-Nagel's cevian, $\boldsymbol{g}_{\boldsymbol{a}}$-Gergonne's cevian, the following relationship holds:

$$
\frac{g_{a} r_{a}+g_{b} r_{b}+g_{c} r_{c}}{r} \geq \sum\left(n_{a}+\frac{2 r_{a} h_{a}}{\mathbf{n}_{a}}\right)
$$

Proposed by Bogdan Fuștei-Romania


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Solution by Soumava Chakraborty-Kolkata-India

Let $\mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathrm{c}=\mathrm{z} \therefore \mathrm{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, (a)
$\Leftrightarrow\left\{\mathbf{z}(\mathrm{z}+\mathrm{x})^{2}+\mathbf{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right.$
$+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$
$\Leftrightarrow \mathrm{xy}^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq \mathbf{2 x y z}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq 0 \rightarrow$ true $\Rightarrow(a)$ is true $\Rightarrow \mathbf{n}_{\mathrm{a}} \mathrm{g}_{\mathrm{a}} \geq \mathbf{s}(\mathbf{s}-\mathbf{a})$

$$
\Rightarrow n_{a} \mathbf{g}_{\mathrm{a}} \mathbf{r}_{\mathrm{a}} \geq \mathbf{s}(\mathbf{s}-\mathbf{a})\left(\frac{\mathbf{r s}}{\mathbf{s}-\mathbf{a}}\right)=\mathbf{r s}^{2} \Rightarrow \mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}} \mathbf{r}_{\mathrm{a}} \stackrel{(2)}{\stackrel{(2)}{\geq} \mathbf{r s}^{2} .}
$$

$$
N o w, \frac{g_{a} r_{a}}{r} \geq n_{a}+\frac{2 r_{a} h_{a}}{n_{a}} \Leftrightarrow \frac{\mathbf{g}_{a} r_{a}}{r} \geq \frac{n_{a}^{2}+2 r_{a} h_{a}}{n_{a}} \stackrel{\text { by }(1)}{=} \frac{s^{2}}{n_{a}} \Leftrightarrow n_{a} g_{a} r_{a} \geq r^{2}
$$

$\rightarrow \operatorname{true}(b y(2)) \therefore \frac{g_{a} r_{a}}{r} \geq n_{a}+\frac{2 r_{a} h_{a}}{n_{a}}$ and analogs

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow s\left(b^{2}+c^{2}\right)-b c(2 s-a)=\mathbf{a n}_{a}{ }^{2}+a\left(s^{2}-s(2 s-a)+b c\right) \Rightarrow s\left(b^{2}+c^{2}\right)-2 s b c \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{a}\left(\mathrm{as}-\mathrm{s}^{2}\right) \\
& \Rightarrow \mathbf{s}\left(\mathbf{b}^{2}+\mathbf{c}^{2}-\mathbf{a}^{2}-2 b c\right)=\mathbf{a n}_{\mathrm{a}}{ }^{2}-\mathbf{a s}^{2} \Rightarrow \mathrm{an}_{\mathrm{a}}{ }^{2}=\mathbf{a s}{ }^{2}+\mathbf{s}(2 \mathrm{bccos} A-2 b c) \\
& =\mathbf{a s}^{2}-4 \operatorname{sbcsin}^{2} \frac{A}{2}=\mathbf{a s}^{2}-\frac{4 \operatorname{sbc}(s-b)(s-c)(s-a)}{b c(s-a)} \\
& =\mathrm{as}^{2}-\frac{4 \Delta^{2}}{s-a}=\mathrm{as}^{2}-2 \mathrm{a}\left(\frac{2 \Delta}{\mathrm{a}}\right)\left(\frac{\Delta}{s-a}\right)=\mathrm{as}^{2}-2 \mathrm{ah}_{\mathrm{a}} \mathrm{r}_{\mathrm{a}} \therefore \mathrm{n}_{\mathrm{a}} \stackrel{\stackrel{(1)}{\stackrel{m}{m}} \mathrm{~s}^{2}-2 \mathrm{~h}_{\mathrm{a}} \mathrm{r}_{\mathrm{a}} .}{ } \\
& \text { Again, Stewart's theorem } \Rightarrow b^{2}(s-c)+c^{2}(s-b) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \therefore \mathrm{an}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}^{2} \geq \mathrm{a}^{2} \mathbf{s}^{2}(\mathrm{~s}-\mathrm{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(s-c)+\mathbf{c}^{2}(s-b)-\mathbf{a}(s-b)(s-c)\right\}\left\{\mathbf{b}^{2}(s-b)+\mathbf{c}^{2}(s-c)-\mathbf{a}(s\right. \\
& \text { (a) } \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\}{ }^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$



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$$
\Rightarrow \frac{\mathbf{g}_{\mathbf{a}} \mathbf{r}_{\mathbf{a}}+\mathbf{g}_{b} \mathbf{r}_{\mathbf{b}}+\mathbf{g}_{\mathbf{c}} \mathbf{r}_{\mathbf{c}}}{\mathbf{r}} \geq \sum\left(\mathbf{n}_{\mathrm{a}}+\frac{2 \mathbf{r}_{\mathbf{a}} \mathbf{h}_{\mathrm{a}}}{\mathbf{n}_{\mathbf{a}}}\right) \text { (Proved) }
$$

1588. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\frac{\mathbf{n}_{\mathrm{a}} \mathbf{h}_{\mathrm{a}}+\mathbf{n}_{\mathrm{b}} \mathbf{h}_{\mathrm{b}}+\mathbf{n}_{\mathbf{c}} \mathbf{h}_{\mathbf{c}}}{\mathbf{s}^{2}} \geq \frac{\mathbf{h}_{\mathrm{a}}-2 \mathbf{r}}{\mathbf{g}_{\mathrm{a}}}+\frac{\mathbf{h}_{\mathrm{b}}-2 \mathbf{r}}{\mathbf{g}_{b}}+\frac{\mathbf{h}_{\mathbf{c}}-2 \mathbf{r}}{\mathbf{g}_{\mathrm{c}}}
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathrm{~s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \therefore \mathbf{a n}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}_{\mathrm{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})\right. \\
& \text { (a) } \\
& -\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \sum^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \text { Let } \mathbf{s}-\mathrm{a}=\mathrm{x}, \mathrm{~s}-\mathrm{b}=\mathrm{y} \text { and } \mathrm{s}-\mathbf{c}=\mathrm{z} \cdot \therefore \mathrm{~s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z} \text {, } \\
& \mathrm{b}=\mathrm{z}+\mathrm{x} \text { and } \mathrm{c}=\mathrm{x}+\mathrm{y} \\
& \text { Using these substitutions, (a) } \\
& \Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathrm{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}\right. \\
& -\mathrm{yz}(\mathrm{y}+\mathrm{z})\} \geq \mathrm{x}^{2}(\mathrm{y}+\mathrm{z})^{2}(\mathrm{x}+\mathrm{y}+\mathrm{z})^{2} \\
& \Leftrightarrow x^{2}+x^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \\
& \rightarrow \text { true } \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \geq s(s-a) \\
& \Rightarrow n_{a} g_{a} h_{a} \geq s(s-a)\left(\frac{2 r s}{a}\right)=s^{2}\left\{\frac{2 r(s-a)}{a}\right\}=s^{2}\left(h_{a}-2 r\right) \Rightarrow \frac{n_{a} h_{a}}{s^{2}} \\
& \geq \frac{h_{a}-2 r}{g_{a}} \text { and analogs } \Rightarrow \frac{\sum n_{a} h_{a}}{s^{2}} \geq \sum \frac{h_{a}-2 r}{g_{a}} \text { (Proved) }
\end{aligned}
$$



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1589. In $\triangle A B C, \boldsymbol{n}_{a}$-Nagel's cevian, $\boldsymbol{g}_{a}$-Gergonne's cevian, the following relationship holds:

$$
\mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}}\left(\mathbf{r}_{\mathrm{a}}-\mathbf{r}\right)+\mathbf{n}_{\mathrm{b}} \mathbf{g}_{\mathrm{b}}\left(\mathbf{r}_{\mathrm{b}}-\mathbf{r}\right)+\mathbf{n}_{\mathrm{c}} \mathbf{g}_{\mathrm{c}}\left(\mathbf{r}_{\mathrm{c}}-\mathbf{r}\right) \geq 2 \mathbf{r}_{\mathrm{a}} \mathbf{r}_{\mathrm{b}} \mathbf{r}_{\mathrm{c}}
$$

Proposed by Bogdan Fuștei-Romania
Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \quad \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& =\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \quad \therefore \mathbf{a n}_{\mathbf{a}}{ }^{2} \cdot \mathbf{a g}_{\mathrm{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{(a)}{\stackrel{(a)}{ \pm} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}}
\end{aligned}
$$

Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathbf{c}=\mathrm{z} \therefore \mathbf{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, (a)
$\Leftrightarrow\left\{\mathbf{z}(\mathrm{z}+\mathrm{x})^{2}+\mathbf{y}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathbf{y}\right.$
$+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2}$
$\Leftrightarrow \mathrm{xy}^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq \mathbf{2 x y z}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq \mathbf{0} \rightarrow$ true $\Rightarrow(a)$ is true $\Rightarrow \mathbf{n}_{\mathrm{a}} \mathrm{g}_{\mathrm{a}} \geq \mathbf{s}(\mathrm{s}-\mathrm{a})$

$$
\begin{gathered}
\Rightarrow n_{a} g_{a}\left(r_{a}-r\right) \geq s(s-a)\left(\frac{r s}{s-a}-\frac{r s}{s}\right)=\left\{\frac{s(s-a)}{s(s-a)}\right\} r s a=r s a \text { and analogs } \\
\Rightarrow \sum n_{a} g_{a}\left(r_{a}-r\right) \geq r s(a+b+c)=2 r s^{2}=2 r_{a} r_{b} r_{c} \text { (Proved) }
\end{gathered}
$$

1590. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\sqrt{\mathbf{n}_{\mathrm{a}} \mathbf{g}_{\mathrm{a}}}+\sqrt{\mathbf{n}_{\mathrm{b}} \mathbf{g}_{\mathrm{b}}}+\sqrt{\mathbf{n}_{\mathbf{c}} \mathbf{g}_{\mathbf{c}}} \geq \mathbf{s}\left(\sqrt{\frac{\mathbf{r}}{\mathbf{r}_{\mathrm{a}}}}+\sqrt{\frac{\mathbf{r}}{\mathbf{r}_{\mathbf{b}}}}+\sqrt{\frac{\mathbf{r}}{\mathbf{r}_{\mathrm{c}}}}\right)
$$



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Solution by Soumava Chakraborty-Kolkata-India
Stewart's theorem $\Rightarrow b^{2}(s-c)+\mathbf{c}^{2}(s-b)$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$

$$
\therefore \mathbf{a n}_{\mathbf{a}}{ }^{2} \cdot \mathbf{a g}_{\mathbf{a}}^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
$$

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})\right. \\
& -\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{(a)}{\geq} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$

Let $\mathbf{s}-\mathbf{a}=\mathbf{x}, \mathbf{s}-\mathbf{b}=\mathbf{y}$ and $\mathbf{s}-\mathbf{c}=\mathrm{z} \therefore \mathbf{s}=\mathbf{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathbf{b}$

$$
=\mathrm{z}+\mathrm{x} \text { and } \mathrm{c}=\mathrm{x}+\mathrm{y}
$$

Using these substitutions, (a)
1591. In any $\triangle A B C$ :

$$
\frac{\sqrt{r_{a} r_{b}}+\sqrt{r_{b} r_{c}}+\sqrt{r_{c} r_{a}}}{\sqrt{h_{a} h_{b}}+\sqrt{h_{b} h_{c}}+\sqrt{h_{c} h_{a}}} \geq \sqrt{\frac{3}{2}} \cdot \frac{\sqrt{R}}{\sqrt[6]{s^{2} r}} \geq 1
$$

Proposed by Nguyen Van Canh-Ben Tre-Vietnam
Solution by Tran Hong-Dong Thap-Vietnam

$$
\sqrt{r_{a} r_{b}}+\sqrt{r_{b} r_{c}}+\sqrt{r_{c} r_{a}} \stackrel{A m-G m}{\geq} 3 \sqrt[3]{r_{a} r_{b} r_{c}}=3 \sqrt[3]{s^{2} r}
$$

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathbf{z}(\mathrm{z}+\mathbf{x})^{2}+\mathbf{y}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathbf{x})^{2}+\mathrm{z}(\mathrm{x}+\mathbf{y})^{\mathbf{2}}\right. \\
& -y z(y+z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2} \\
& \Leftrightarrow x^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq \mathbf{2 x y z}+\mathbf{y z}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq \mathbf{0} \\
& \rightarrow \text { true } \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \geq \mathbf{s}(s-a) \\
& \Rightarrow \sqrt{\mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}}} \geq \sqrt{\mathbf{s}(\mathbf{s}-\mathbf{a})} \geq \mathbf{s} \sqrt{\frac{\mathbf{s}-\mathrm{a}}{\mathrm{~s}}}=\mathbf{s} \sqrt{\frac{\mathbf{r}}{\mathbf{r}_{\mathrm{a}}}} \text { and analogs } \Rightarrow \sum \sqrt{\mathbf{n}_{\mathrm{a}} \mathrm{~g}_{\mathrm{a}}} \\
& \geq \mathbf{s} \sum \sqrt{\frac{\mathbf{r}}{\mathrm{r}_{\mathrm{a}}}} \text { (Proved) }
\end{aligned}
$$



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$$
\begin{gathered}
\sqrt{h_{a} h_{b}}+\sqrt{h_{b} h_{c}}+\sqrt{h_{c} h_{a}} \stackrel{B C s}{\leq} \sqrt{3\left(h_{a} h_{b}+h_{b} h_{c}+h_{c} h_{a}\right)}=\sqrt{3 \cdot \frac{2 s^{2} r}{R}} \\
\frac{\sqrt{r_{a} r_{b}}+\sqrt{r_{b} r_{c}}+\sqrt{r_{c} r_{a}}}{\sqrt{h_{a} h_{b}}+\sqrt{h_{b} h_{c}}+\sqrt{h_{c} h_{a}}} \geq \frac{3 \sqrt[3]{s^{2} r}}{\sqrt{3 \cdot \frac{2 s^{2} r}{R}}}=\sqrt{\frac{3}{2}} \cdot \frac{\sqrt{R}}{\sqrt[6]{s^{2} r}}
\end{gathered}
$$

$$
\text { Lastly, } \sqrt{\frac{3}{2}} \cdot \frac{\sqrt{R}}{\sqrt[6]{s^{2} r}} \geq 1 \Leftrightarrow \frac{3}{2} \cdot \frac{R}{\sqrt[3]{s^{2} r}} \geq 1 \Leftrightarrow \frac{27}{8} \cdot \frac{R^{3}}{s^{2} r} \Leftrightarrow 27 R^{3} \geq 8 s^{2} r
$$

$$
\Leftrightarrow 27 R^{2} \cdot R \geq 4 s^{2} \cdot 2 r, \text { which is true because: }
$$

$$
2 r \leq R ; s \leq \frac{3 \sqrt{3}}{2} \cdot R \Rightarrow s^{2} \leq \frac{27}{4} R^{2} \Rightarrow 4 s^{2} \leq 27 R^{2} \Rightarrow \text { Proved. }
$$

1592. In $\triangle \mathrm{ABC}$ the following relationship holds:

$$
\frac{h_{a}}{r_{a}-r}+\frac{h_{b}}{r_{b}-r}+\frac{h_{c}}{r_{c}-r} \geq \frac{1}{2}\left(\frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}\right)^{2} \geq \frac{9}{2}
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
\frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}=\sum_{c y c}\left(\frac{\frac{2 S}{a}}{\frac{S}{s-a}}\right)=2 \sum_{c y c} \frac{s-a}{a} \Rightarrow \\
\Omega=\frac{1}{2}\left(\frac{h_{a}}{r_{a}}+\frac{h_{b}}{r_{b}}+\frac{h_{c}}{r_{c}}\right)^{2}=\frac{1}{2}\left(2 \sum_{c y c} \frac{s-a}{a}\right)^{2}=2\left(\sum_{c y c} \frac{s-a}{a}\right)^{2}
\end{gathered}
$$

We must show that:

$$
2 s \sum_{c y c}\left(\frac{s-a}{a^{2}}\right) \geq 2\left(\sum_{c y c} \frac{s-a}{a}\right)^{2} \leftrightarrow s \sum_{c y c}\left(\frac{s-a}{a^{2}}\right) \stackrel{(*)}{\geq}\left(\sum_{c y c} \frac{s-a}{a}\right)^{2}
$$

Let: $x=s-a ; y=s-b ; z=s-c \rightarrow x+y+z=s ; a=y+z ; b=z+x ; c=x+y$

$$
\begin{aligned}
(*) \Leftrightarrow & (x+y+z)\left(\frac{x}{(y+z)^{2}}+\frac{y}{(z+x)^{2}}+\frac{z}{(x+y)^{2}}\right) \geq\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^{2} \\
& \Leftrightarrow\left(\sum x\right)\left(\sum x(x+y)^{2}(x+z)^{2}\right) \geq\left(\sum x(x+y)(x+z)\right)^{2} \\
& \Leftrightarrow x^{3}+y^{3}+z^{3}+3 x y z-x y(x+y)-y z(y+z)-z x(z+x) \geq 0
\end{aligned}
$$



## ROMANIAN MATHEMATICAL MAGAZINE www.ssmrmh.ro <br> $\Leftrightarrow x^{3}+y^{3}+z^{3}+3 x y z \geq x y(x+y)+y z(y+z)+z x(z+x)$

Which is true by Schur's inequality.

$$
\begin{gathered}
\text { Lastly, } \Omega \geq \frac{9}{2} \Leftrightarrow 2\left(\sum_{c y c} \frac{s-a}{a}\right)^{2} \geq \frac{9}{2} \Leftrightarrow\left(\sum_{c y c} \frac{s-a}{a}\right)^{2} \geq \frac{9}{4} \\
\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)^{2} \geq \frac{9}{4} \Leftrightarrow \frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2}
\end{gathered}
$$

True by Nesbitt's inequality.Proved.
1593. If in $\triangle A B C, g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\frac{g_{a}^{2}}{b c}+\frac{g_{b}^{2}}{c a}+\frac{g_{c}^{2}}{a b} \leq \frac{9}{4}
$$

Proposed by Rahim Shahbazov-Baku-Azerbaijan
Solution 1 by George Florin Şerban-Romania

$$
\begin{aligned}
& g_{a}^{2}=(b-c)^{2}+s^{2}-\frac{s}{a} \cdot \sum a^{2}+\frac{2 s}{a} \cdot b c \\
& g_{b}^{2}=(a-c)^{2}+s^{2}-\frac{s}{b} \cdot \sum b^{2}+\frac{2 s}{b} \cdot a c \\
& g_{c}^{2}=(a-b)^{2}+s^{2}-\frac{s}{c} \cdot \sum c^{2}+\frac{2 s}{c} \cdot a b \\
& \frac{g_{a}^{2}}{b c}+\frac{g_{b}^{2}}{c a}+\frac{g_{c}^{2}}{a b}= \\
& =\frac{(b-c)^{2}}{b c}+\frac{(a-c)^{2}}{a c}+\frac{(a-b)^{2}}{a b}+s^{2} \sum_{c y c} \frac{1}{b c}-\frac{3 s}{a b c} \sum_{c y c} a^{2}+2 s \sum_{c y c} \frac{1}{a} \\
& =\sum_{c y c}\left[\frac{(b-c)^{2}}{b c}+4-4\right]+\frac{s^{2}(a+b+c)}{a b c}-\frac{3 s}{a b c} \sum_{c y c} a^{2}+\frac{2 s}{a b c} \sum_{c y c} b c \\
& =\sum_{c y c} \frac{(b-c)^{2}}{b c}-12+\frac{2 s^{3}}{4 R r s}-\frac{3 s}{4 R r s} \sum_{c y c} a^{2}-\frac{2 s}{4 R r s} \sum_{c y c} b c \\
& =\frac{s^{2}+r^{2}+10 R r}{2 R r}-12+\frac{s^{2}}{2 R r}-\frac{6\left(s^{2}-r^{2}-4 R r\right)}{4 R r}+\frac{s^{2}+r^{2}+4 R r}{2 R r} \\
& =\frac{s^{2}+r^{2}+10 R r-24 R r+s^{2}-3 s^{2}+3 r^{2}+12 R r+s^{2}+r^{2}+4 R r}{2 R r}
\end{aligned}
$$



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$$
=\frac{5 r^{2}+2 R r}{2 R r} \stackrel{?}{\leq} \frac{9}{4} \Leftrightarrow
$$

$$
20 r^{2}+8 R r \leq 18 R r \Leftrightarrow 20 r^{2} \leq 10 R r \Leftrightarrow R r \geq 2 r^{2} \Leftrightarrow r(R-2 r) \geq 0 \text { true from }
$$

$$
R \geq 2 r \text { (Euler).Proved. }
$$

## Solution 2 by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})=\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})$
(a)

$$
\Rightarrow \mathbf{a g}_{\mathbf{a}}{ }^{\underline{M}}=\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
$$

(b)
and similarly, by Stewart's theorem, $\mathbf{b g}_{\mathbf{b}}^{2} \xlongequal[=]{m} \mathbf{c}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{a}^{2}(\mathrm{~s}-\mathrm{a})-\mathbf{b}(\mathrm{s}-\mathrm{c})(\mathrm{s}$
(c)
$-\mathbf{a})$ and $\operatorname{cg}_{\mathrm{c}}^{2} \xlongequal{\cong} \mathbf{a}^{2}(\mathrm{~s}-\mathbf{a})+\mathbf{b}^{2}(\mathrm{~s}-\mathrm{b})-\mathbf{c}(\mathrm{s}-\mathbf{a})(\mathrm{s}-\mathbf{b})$

$$
N o w, \frac{g_{a}^{2}}{b c}+\frac{g_{b}^{2}}{c a}+\frac{g_{c}^{2}}{a b}=\frac{\mathbf{a g}_{a}^{2}+\mathbf{b g}_{b}^{2}+\mathbf{c g}_{\mathrm{c}}^{2}}{\mathbf{a b c}}
$$

$\stackrel{(1)}{\stackrel{m}{=}} \frac{\mathbf{b}^{2}(s-b)+\mathbf{c}^{2}(s-\mathbf{c})-\mathbf{a}(\mathbf{s}-\mathbf{b})(s-\mathbf{c})+\mathbf{c}^{2}(s-\mathbf{c})+\mathbf{a}^{2}(s-\mathbf{a})-\mathbf{b}(s-\mathbf{c})(s-\mathbf{a})+\mathbf{a}^{2}(s-\mathbf{a})+\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{c}(\mathbf{s}-\mathbf{a})(s-\mathbf{b})}{\mathbf{a b c}}$ Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathrm{s}-\mathrm{b}=\mathrm{y}$ and $\mathrm{s}-\mathbf{c}=\mathrm{z} \therefore \mathbf{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathrm{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

## Solution 3 by Bogdan Fuştei-Romania

$$
\begin{gathered}
g_{a} \leq A I+r \text { (triangle inequality) and analogs. } \\
\frac{A I}{w_{a}}=\frac{b+c}{2 s} \text { and analogs, then }
\end{gathered}
$$

$$
\begin{aligned}
& =x+y \quad \therefore u \operatorname{sing}(1), \frac{\mathbf{g}_{a}^{2}}{b c}+\frac{\mathbf{g}_{b}^{2}}{\mathbf{c a}}+\frac{\mathbf{g}_{\mathrm{c}}^{2}}{\mathbf{a b}} \\
& =\frac{2 y(z+x)^{2}+2 z(x+y)^{2}+2 x(y+z)^{2}-y z(y+z)-z x(z+x)-x y(x+y)}{(x+y)(y+z)(z+x)} \leq \frac{9}{4} \\
& \Leftrightarrow 9(x+y)(y+z)(z+x) \\
& \geq 8 y(z+x)^{2}+8 z(x+y)^{2}+8 x(y+z)^{2}-4 y z(y+z)-4 z x(z+x) \\
& -4 x y(x+y) \\
& \Leftrightarrow \sum x^{2} y+\sum x^{2} \geq 6 x y z \rightarrow \text { true by } A M-G M \text { inequality } \Rightarrow \frac{\mathbf{g}_{a}^{2}}{b c}+\frac{\mathbf{g}_{b}^{2}}{\mathbf{c a}}+\frac{\mathbf{g}_{\mathrm{c}}^{2}}{\mathrm{ab}} \\
& \leq \frac{9}{4} \text { (Proved) }
\end{aligned}
$$



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$$
\begin{gathered}
\frac{w_{a}}{A I}=\frac{2 s}{b+c}=\frac{a+b+c}{b+c}=1+\frac{a}{b+c} \\
w_{a}=A I+\frac{a}{b+c} \cdot A I ; A I=\frac{r}{\sin \frac{A}{2}} \text { and analogs. } \\
\cos \frac{B-C}{2}=\frac{h_{a}}{w_{a}}=\frac{b+c}{a} \cdot \sin \frac{A}{2} \text { and analogs. } \\
\Rightarrow \frac{w_{a}}{h_{a}}=\frac{a}{b+c} \cdot \frac{1}{\sin \frac{A}{2}}=\frac{a}{b+c} \cdot \frac{A I}{r} \Leftrightarrow \frac{w_{a}}{h_{a}} \cdot r=\frac{a}{b+c} \cdot \text { AI and analogs. }
\end{gathered}
$$

So, we have: $w_{a}=A I+\frac{w_{a}}{h_{a}} \geq A I+r, h_{a} \leq w_{a}$, then

$$
g_{a} \leq A I+r \leq w_{a} \text { and analogs. }
$$

$B u t: w_{a}^{2} \leq s(s-a)=r_{b} r_{c}=\frac{h_{a}\left(r_{b}+r_{c}\right)}{2}$

$$
\text { So, } \frac{g_{a}^{2}}{b c} \leq \frac{h_{a}\left(r_{b}+r_{c}\right)}{2 b c}=\frac{h_{a}\left(r_{b}+r_{c}\right)}{4 R h_{a}}, \text { then we have: } \frac{g_{a}^{2}}{b c} \leq \frac{r_{b}+r_{c}}{4 R}
$$

$$
\begin{aligned}
\frac{g_{a}^{2}}{b c}+\frac{g_{b}^{2}}{c a}+\frac{g_{c}^{2}}{a b} & \leq \frac{2\left(r_{a}+r_{b}+r_{c}\right)}{4 R}=\frac{r_{a}+r_{b}+r_{c}}{2 R}=\frac{4 R+r}{2 R} \\
& =2+\frac{r}{2 R} \stackrel{\text { Euler }}{\leq} 2+\frac{1}{4}=\frac{9}{4} . \text { Proved }
\end{aligned}
$$

## Solution 4 by Tran Hong-Dong Thap-Vietnam

Lemma: In $\triangle A B C: g_{a} \leq \sqrt{s(s-a)} ; g_{b} \leq \sqrt{s(s-b)} ; g_{c} \leq \sqrt{s(s-c)}$

$$
\begin{gathered}
\text { Proof: } g_{a}^{2}=\frac{b^{2}(s-b)+c^{2}(s-c)+a(s-b)(s-c)}{a}= \\
=\frac{2\left[b^{2}(a+c-b)+c^{2}(a+b-c)\right]-a(a+c-b)(a+b-c)}{4 a}
\end{gathered}
$$

We need to prove:

$$
\begin{gathered}
\frac{2\left[b^{2}(a+c-b)+c^{2}(a+b-c)\right]-a(a+c-b)(a+b-c)}{4 a} \leq s-a \\
=\frac{(a+b+c)(b+c-a)}{4} \Leftrightarrow \\
2\left[b^{2}(a+c-b)+c^{2}(a+b-c)\right]-a(a+c-b)(a+b-c) \\
\leq a(a+b+c)(b+c-a) \\
\Leftrightarrow a[(a+c-b)(a+b-c)+(a+b+c)(b+c-a)] \\
-2\left[b^{2}(a+c-b)+c^{2}(a+b-c)\right] \geq 0 \\
\Leftrightarrow 2(b-c)^{2}(b+c-a) \geq 0 \text { true by } 2(b-c)^{2} \geq 0, b+c>a
\end{gathered}
$$



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Now,

$$
\begin{aligned}
L H S & =\frac{g_{a}^{2}}{b c}+\frac{g_{b}^{2}}{c a}+\frac{g_{c}^{2}}{a b} \leq \frac{s(s-a)}{b c}+\frac{s(s-b)}{c a}+\frac{s(s-c)}{a b} \\
& =\frac{s[a(s-a)+b(s-b)+c(s-c)] \stackrel{(*)}{\leq} 9}{a b c} \\
(*) \Leftrightarrow & \frac{s\left[2 s^{2}-\left(2 s^{2}-8 R r-2 r^{2}\right)\right]}{4 R r s} \leq \frac{9}{4} \Leftrightarrow \frac{2 s r(4 R+r)}{4 R r s} \leq \frac{9}{4} \\
& \Leftrightarrow 8 R+2 r \leq 9 R \Leftrightarrow R \geq 2 r \text { (Euler).Proved. }
\end{aligned}
$$

1594. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\frac{\mathbf{g}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}}+\frac{\mathbf{g}_{\mathrm{b}}}{\mathbf{w}_{\mathrm{b}}}+\frac{\mathbf{g}_{\mathrm{c}}}{\mathbf{w}_{\mathrm{c}}} \geq \sqrt{\frac{\mathbf{r}}{2 R}}\left(\frac{\mathbf{r}_{\mathrm{b}}+\mathbf{r}_{\mathrm{c}}}{\mathbf{n}_{\mathrm{a}}}+\frac{\mathbf{r}_{\mathrm{c}}+\mathbf{r}_{\mathrm{a}}}{\mathbf{n}_{\mathrm{b}}}+\frac{\mathbf{r}_{\mathrm{a}}+\mathbf{r}_{\mathrm{b}}}{\mathbf{n}_{\mathrm{c}}}\right)
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{gathered}
r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{sos}^{2} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2} \\
\therefore r_{b}+r_{c} \stackrel{(i)}{\leftrightarrows} 4 R \cos ^{2} \frac{A}{2}
\end{gathered}
$$

Now, $(b+c)^{2} \geq 32 \operatorname{Rrcos}^{2} \frac{A^{\text {by }}}{2} \stackrel{(i)}{\leftrightarrows} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$
$=8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a)$
$\Leftrightarrow(b+c)^{2}+4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow$ true
$\therefore b+c \stackrel{\text { (ii) }}{\geq} 4 \sqrt{2 \operatorname{Rr}} \cos \frac{A}{2}$ and analogs
Stewart's theorem $\Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathrm{b})$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$

$$
\therefore \mathbf{a n}_{\mathbf{a}}^{2} \cdot \mathbf{a g}_{\mathrm{a}}^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
$$

$$
\Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right.
$$

(a)
$-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{\text { ² }}{\geq} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}$
Let $s-a=x, s-b=y$ and $s-c=z \therefore s=x+y+z \Rightarrow a=y+z, b=z+x$ and $c$

$$
=x+y
$$



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Using these substitutions, (a)

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathrm{x})^{2}+\mathbf{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right. \\
& +\mathrm{z})\} \geq \mathrm{x}^{2}(\mathrm{y}+\mathrm{z})^{2}(\mathrm{x}+\mathrm{y}+\mathrm{z})^{2} \\
& \Leftrightarrow x^{2}+x^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow \text { true } \\
& \text { (iii) } \\
& \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \geq s(s-a) \\
& \text { Now, } \frac{n_{a} g_{a}}{w_{a}\left(r_{b}+r_{c}\right)} \stackrel{\text { by (i) and (iii) }}{\geq} \frac{s(s-a)(b+c)}{2 b \cos \frac{A}{2} \cdot 4 R \cos ^{2} \frac{A}{2}} \stackrel{\text { by (ii) }}{\geq} \frac{4 \sqrt{2 R r} \cos \frac{A}{2} \cdot s(s-a)}{2 b c \cos \frac{A}{2} \cdot 4 R\left\{\frac{s(s-a)}{b c}\right\}}=\sqrt{\frac{r}{2 R}} \\
& \Rightarrow \frac{\mathbf{g}_{\mathrm{a}}}{\mathbf{w}_{\mathrm{a}}} \geq \sqrt{\frac{\mathbf{r}}{2 R}}\left(\frac{\mathbf{r}_{\mathrm{b}}+\mathbf{r}_{\mathrm{c}}}{\mathbf{n}_{\mathrm{a}}}\right) \text { and analogs } \\
& \Rightarrow \frac{\mathbf{g}_{a}}{\mathbf{w}_{a}}+\frac{\mathbf{g}_{b}}{\mathbf{w}_{\mathbf{b}}}+\frac{\mathbf{g}_{\mathbf{c}}}{\mathbf{w}_{\mathbf{c}}} \geq \sqrt{\frac{\mathbf{r}}{2 R}}\left(\frac{\mathbf{r}_{\mathrm{b}}+\mathbf{r}_{\mathrm{c}}}{\mathbf{n}_{\mathrm{a}}}+\frac{\mathbf{r}_{\mathbf{c}}+\mathbf{r}_{\mathrm{a}}}{\mathbf{n}_{\mathrm{b}}}+\frac{\mathbf{r}_{\mathrm{a}}+\mathbf{r}_{\mathbf{b}}}{\mathbf{n}_{\mathbf{c}}}\right) \text { (Proved) }
\end{aligned}
$$

1595. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\frac{1}{2} \sum \frac{n_{a}+g_{a}}{w_{a}} \geq \sqrt{\frac{2 r}{R}} \sum \frac{m_{a}}{h_{a}}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathbf{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$

Adding the above two, we get : $\left(b^{2}+c^{2}\right)(2 s-b-c)$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a g}_{\mathrm{a}}{ }^{2}+2 \mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(\mathbf{n}_{\mathrm{a}}{ }^{2}+\mathrm{g}_{\mathrm{a}}{ }^{2}\right)+\mathbf{a}(\mathbf{a}+\mathbf{b}-\mathbf{c})(\mathbf{c}+\mathbf{a}-\mathbf{b}) \Rightarrow \mathbf{2}\left(\mathrm{b}^{2}+\mathbf{c}^{2}\right) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}{ }^{2}\right) \\
& \Rightarrow 4 m_{a}^{2}+(b-c)^{2}+4 r_{b} r_{c}=2\left(n_{a}^{2}+g_{a}^{2}\right)+4 r_{b} r_{c} \Rightarrow 4 m_{a}^{2}+(b-c)^{2}+4 s(s-a) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+4 s(s-a) \\
& \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+4 \mathrm{~m}_{\mathrm{a}}{ }^{2}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2}\right)+4 \mathrm{~s}(\mathrm{~s}-\mathrm{a}) \Rightarrow \mathrm{na}^{2}+\mathrm{ga}^{2} \xlongequal[=]{\cong} \mathbf{m a}_{\mathrm{a}}{ }^{2}-\mathbf{2 s}(\mathrm{s}-\mathrm{a}) \\
& \mathbf{a n}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}_{\mathrm{a}}{ }^{2} \geq \mathbf{a}^{2} \mathbf{s}^{2}(\mathrm{~s}-\mathrm{a})^{2} \\
& \Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right. \\
& \text { (a) } \\
& -\mathbf{b})(\mathbf{s}-\mathbf{c})\}{ }^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}
\end{aligned}
$$



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Let $\mathbf{s}-\mathbf{a}=\mathrm{x}, \mathbf{s}-\mathbf{b}=\mathrm{y}$ and $\mathrm{s}-\mathbf{c}=\mathrm{z} \therefore \mathbf{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathbf{b}=\mathrm{z}+\mathrm{x}$ and $\mathbf{c}$

$$
=x+y
$$

Using these substitutions, (a)
by (1) and (2)

Now, $n_{a}{ }^{2}+g_{a}{ }^{2}+2 n_{a} g_{a} \xrightarrow{\text { m }} \quad 4 m_{a}{ }^{2}-2 s(s-a)+2 s(s-a)=4 m_{a}{ }^{2}$

$$
\Rightarrow\left(n_{a}+g_{a}\right)^{2} \geq 4 m_{a}^{2} \Rightarrow n_{a}+g_{a} \stackrel{(3)}{\geq} 2 m_{a}
$$

$$
\text { Also, } r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{scos} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2}
$$

$$
\therefore r_{b}+r_{c} \stackrel{(i)}{=} 4 \operatorname{Rcos}^{2} \frac{A}{2}
$$

Now, $(b+c)^{2} \geq 32 \operatorname{Rrcos}^{2} \frac{A^{\text {by }}}{2} \stackrel{(i)}{\leftrightarrows} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$

$$
\begin{aligned}
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a) \\
\Leftrightarrow(b+c)^{2}+ & 4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \operatorname{true} \\
\therefore b+c & \stackrel{(4)}{\geq} 4 \sqrt{2 \operatorname{Rr}} \cos \frac{A}{2}
\end{aligned}
$$

$$
\therefore \frac{1}{2}\left(\frac{n_{a}+g_{a}}{m_{a}}\right)\left(\frac{h_{a}}{\mathbf{w}_{a}}\right) \stackrel{\text { by }(3)}{\leq} \frac{1}{2}\left(\frac{2 m_{a}}{m_{a}}\right) \sqrt{\frac{2 r}{R}}=\sqrt{\frac{2 r}{R}}
$$

$$
\Rightarrow \frac{1}{2}\left(\frac{n_{a}+g_{a}}{\mathbf{w}_{a}}\right) \geq \sqrt{\frac{2 r}{R}}\left(\frac{m_{a}}{h_{a}}\right) \text { and analogs } \Rightarrow \frac{1}{2} \sum \frac{n_{a}+g_{a}}{w_{a}} \geq \sqrt{\frac{2 r}{R}} \sum \frac{m_{a}}{h_{a}} \text { (Proved) }
$$

1596. In $\triangle A B C, n_{a}$-Nagel's cevian, $g_{a}$-Gergonne's cevian, the following relationship holds:

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathbf{z}(\mathrm{z}+\mathrm{x})^{2}+\mathbf{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right. \\
& +z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2} \\
& \Leftrightarrow \mathrm{xy}^{2}+\mathrm{xz}^{2}+\mathrm{y}^{3}+\mathrm{z}^{3} \geq 2 \mathrm{xyz}+\mathrm{yz}(\mathrm{y}+\mathrm{z}) \Leftrightarrow \mathrm{x}(\mathrm{y}-\mathrm{z})^{2}+(\mathrm{y}+\mathrm{z})(\mathrm{y}-\mathrm{z})^{2} \geq 0 \rightarrow \text { true } \\
& \text { (2) } \\
& \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \xrightarrow{2} \mathbf{s}(\mathbf{s}-a)
\end{aligned}
$$



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$$
\frac{1}{2} \sum\left(n_{a}+g_{a}\right) \geq \sqrt{\frac{2 r}{R}} \sum \frac{m_{a} \mathbf{w}_{a}}{\mathbf{h}_{\mathrm{a}}}
$$

## Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

Stewart's theorem $\Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})$

$$
\begin{aligned}
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$

Adding the above two, we get : $\left(b^{2}+c^{2}\right)(2 s-b-c)$
$\Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+4 \mathrm{~m}_{\mathrm{a}}{ }^{2}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2}\right)+\mathbf{4 s}(\mathrm{s}-\mathrm{a}) \Rightarrow \mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}^{2} \stackrel{\mathrm{~m}}{=} \mathbf{4 \mathrm { m } _ { \mathrm { a } }}{ }^{2}-\mathbf{2 s}(\mathrm{s}-\mathrm{a})$ $\mathrm{an}_{\mathrm{a}}{ }^{2} \cdot \mathrm{ag}_{\mathrm{a}}{ }^{2} \geq \mathrm{a}^{2} \mathrm{~s}^{2}(\mathrm{~s}-\mathrm{a})^{2}$
$\Leftrightarrow\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b})-\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})\right\}\left\{\mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c})-\mathbf{a}(\mathbf{s}\right.$ (a)
$-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \mathrm{a}^{2} \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}$
Let $\mathbf{s}-\mathbf{a}=\mathbf{x}, \mathbf{s}-\mathbf{b}=\mathbf{y}$ and $\mathbf{s}-\mathbf{c}=\mathbf{z} \therefore \mathbf{s}=\mathbf{x}+\mathbf{y}+\mathbf{z} \Rightarrow \mathbf{a}=\mathbf{y}+\mathbf{z}, \mathbf{b}=\mathbf{z}+\mathbf{x}$ and $\mathbf{c}$

$$
=x+y
$$

Using these substitutions, (a)

Now, $n_{a}{ }^{2}+g_{a}{ }^{2}+2 n_{a} g_{a} \xrightarrow{2} \quad 4 m_{a}{ }^{2}-2 s(s-a)+2 s(s-a)=4 m_{a}{ }^{2}$

$$
\begin{equation*}
\Rightarrow\left(\mathbf{n}_{\mathrm{a}}+\mathbf{g}_{\mathrm{a}}\right)^{2} \geq \mathbf{4} \mathbf{m}_{\mathrm{a}}^{2} \Rightarrow \mathbf{n}_{\mathrm{a}}+\mathbf{g}_{\mathrm{a}} \geq \mathbf{2 m _ { a }} \tag{3}
\end{equation*}
$$

$$
\text { Also, } r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{s\operatorname {cos}^{2}\frac {A}{2}}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2}
$$

$$
\therefore \mathrm{r}_{\mathrm{b}}+\mathrm{r}_{\mathrm{c}} \stackrel{(\mathrm{i})}{\cong} 4 \mathrm{R} \cos ^{2} \frac{\mathrm{~A}}{2}
$$

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathbf{x})^{2}+\mathbf{y}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathbf{x})^{2}+\mathrm{z}(\mathrm{x}+\mathbf{y})^{2}-\mathbf{y z}(\mathrm{y}\right. \\
& +\mathrm{z})\} \geq \mathrm{x}^{2}(\mathrm{y}+\mathrm{z})^{2}(\mathrm{x}+\mathrm{y}+\mathrm{z})^{2} \\
& \Leftrightarrow x y^{2}+x z^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow \text { true } \\
& \text { (2) } \\
& \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \geq \mathbf{s}(s-a) \\
& \text { by (1) and (2) }
\end{aligned}
$$

$$
\begin{align*}
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{2 a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \\
& \Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(\mathbf{n}_{\mathrm{a}}{ }^{2}+\mathrm{g}_{\mathrm{a}}{ }^{2}\right)+\mathbf{a}(\mathbf{a}+\mathbf{b}-\mathbf{c})(\mathbf{c}+\mathbf{a}-\mathbf{b}) \Rightarrow \mathbf{2}\left(\mathrm{b}^{2}+\mathrm{c}^{2}\right) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+a^{2}-(b-c)^{2} \\
& \Rightarrow \mathbf{2}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)-\mathbf{a}^{2}+(\mathrm{b}-\mathrm{c})^{2}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}_{\mathrm{a}}{ }^{2}\right) \Rightarrow 4 \mathrm{~m}_{\mathrm{a}}{ }^{2}+(\mathrm{b}-\mathrm{c})^{2}=\mathbf{2}\left(\mathrm{n}_{\mathrm{a}}{ }^{2}+\mathrm{ga}_{\mathrm{a}}{ }^{2}\right) \\
& \Rightarrow 4 m_{a}{ }^{2}+(b-c)^{2}+4 r_{b} r_{c}=\mathbf{2}\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+4 r_{b} r_{c} \Rightarrow 4 m_{a}{ }^{2}+(b-c)^{2}+4 s(s-a) \\
& =2\left(n_{a}{ }^{2}+g_{a}{ }^{2}\right)+4 s(s-a) \tag{1}
\end{align*}
$$



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$$
\begin{aligned}
& \text { Now, } \begin{aligned}
(b+c)^{2} & \geq 32 \operatorname{Rrcos}^{2} \frac{A^{b y(i)}}{2} \stackrel{\cong}{=} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right) \\
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a) \\
\Leftrightarrow(b+c)^{2}+ & 4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \text { true } \\
& \therefore b+c \geq 4 \sqrt{2 \operatorname{Rr}} \cos \frac{A}{2}
\end{aligned}
\end{aligned}
$$

Now, $\frac{h_{a}}{w_{a}}=\frac{2 r s(b+c)}{a .2 b \cos \frac{A}{2}} \stackrel{\text { by }{ }^{(4)}}{\geq} \frac{2 r s .4 \sqrt{2 R r} \cos \frac{A}{2}}{8 R \operatorname{rscos} \frac{A}{2}}=\sqrt{\frac{2 r}{R}}$

$$
\Rightarrow \frac{1}{2}\left(n_{a}+g_{a}\right) \geq \sqrt{\frac{2 r}{R}}\left(\frac{m_{a} w_{a}}{h_{a}}\right) \text { and analogs } \Rightarrow \frac{1}{2} \sum\left(n_{a}+g_{a}\right)
$$

$$
\geq \sqrt{\frac{2 r}{R}} \sum \frac{m_{a} w_{a}}{h_{a}}(\text { Proved })
$$

1597. In $\triangle A B C, \boldsymbol{n}_{a}$-Nagel's cevian, $\boldsymbol{g}_{\boldsymbol{a}}$-Gergonne's cevian, the following relationship holds:

$$
\sum \cos \left(\frac{A-B}{2}\right) \geq 2 \sqrt{\frac{2 r}{R}} \sum \frac{m_{a}}{n_{a}+g_{a}}
$$

Proposed by Bogdan Fuștei-Romania

## Solution by Soumava Chakraborty-Kolkata-India

$$
\begin{aligned}
& \text { Stewart's theorem } \Rightarrow \mathbf{b}^{2}(\mathbf{s}-\mathbf{c})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{b}) \\
& \\
& =\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c}) \text { and } \mathbf{b}^{2}(\mathbf{s}-\mathbf{b})+\mathbf{c}^{2}(\mathbf{s}-\mathbf{c}) \\
& \\
& =\mathbf{a g}_{\mathrm{a}}{ }^{2}+\mathbf{a}(\mathbf{s}-\mathbf{b})(\mathbf{s}-\mathbf{c})
\end{aligned}
$$

Adding the above two, we get : $\left(b^{2}+c^{2}\right)(2 s-b-c)$

$$
=\mathbf{a n}_{\mathrm{a}}{ }^{2}+\mathrm{ag}_{\mathrm{a}}{ }^{2}+2 \mathbf{a}(\mathrm{~s}-\mathrm{b})(\mathrm{s}-\mathrm{c})
$$

$$
\Rightarrow \mathbf{2 a}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)=\mathbf{2 a}\left(n_{a}^{2}+g_{a}^{2}\right)+\mathbf{a}(\mathbf{a}+\mathbf{b}-\mathbf{c})(\mathbf{c}+\mathbf{a}-\mathbf{b}) \Rightarrow \mathbf{2}\left(\mathbf{b}^{2}+\mathbf{c}^{2}\right)
$$

$$
=2\left(n_{a}^{2}+g_{a}^{2}\right)+a^{2}-(b-c)^{2}
$$

$$
\Rightarrow 2\left(b^{2}+c^{2}\right)-a^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right) \Rightarrow 4 m_{a}^{2}+(b-c)^{2}=2\left(n_{a}^{2}+g_{a}^{2}\right)
$$

$$
\Rightarrow 4 m_{a}^{2}+(b-c)^{2}+4 r_{b} r_{c}=2\left(n_{a}^{2}+g_{a}^{2}\right)+4 r_{b} r_{c} \Rightarrow 4 m_{a}^{2}+(b-c)^{2}+4 s(s-a)
$$

$$
=2\left(n_{a}^{2}+g_{a}^{2}\right)+4 s(s-a)
$$



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(1)
 (a)
$-\mathbf{b})(\mathbf{s}-\mathbf{c})\} \stackrel{\text { m }}{ } \mathbf{a}^{2} \mathbf{s}^{2}(\mathbf{s}-\mathbf{a})^{2}$
Let $\mathbf{s}-\mathbf{a}=\mathbf{x}, \mathbf{s}-\mathbf{b}=\mathbf{y}$ and $\mathbf{s}-\mathbf{c}=\mathrm{z} \therefore \mathbf{s}=\mathrm{x}+\mathrm{y}+\mathrm{z} \Rightarrow \mathbf{a}=\mathrm{y}+\mathrm{z}, \mathrm{b}=\mathrm{z}+\mathrm{x}$ and c

$$
=x+y
$$

Using these substitutions, (a)

Now, $n_{a}{ }^{2}+g_{a}{ }^{2}+2 n_{a} g_{a} \xrightarrow{m} \quad 4 m_{a}{ }^{2}-2 s(s-a)+2 s(s-a)=4 m_{a}{ }^{2}$

$$
\Rightarrow\left(n_{a}+g_{a}\right)^{2} \geq 4 m_{a}^{2} \Rightarrow n_{a}+g_{a} \sum^{(3)} 2 m_{a}
$$

$$
\text { Also, } r_{b}+r_{c}=s\left(\frac{\sin \frac{B}{2}}{\cos \frac{B}{2}}+\frac{\sin \frac{C}{2}}{\cos \frac{C}{2}}\right)=\frac{\sin \left(\frac{B+C}{2}\right) \cos \frac{A}{2}}{\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}}=\frac{\operatorname{scos} \frac{A}{2}}{\left(\frac{s}{4 R}\right)}=4 R \cos ^{2} \frac{A}{2}
$$

$$
\therefore \mathrm{r}_{\mathrm{b}}+\mathrm{r}_{\mathrm{c}} \stackrel{(\mathrm{i})}{=} 4 \operatorname{Reos}^{2} \frac{\mathrm{~A}}{2}
$$

Now, $(b+c)^{2} \geq 32 \operatorname{Rrcos} \frac{A^{2}}{2} \stackrel{\text { by (i) }}{=} 8 r\left(r_{b}+r_{c}\right)=8 r^{2} s\left(\frac{1}{s-b}+\frac{1}{s-c}\right)$

$$
\begin{aligned}
& =8(s-a)(s-b)(s-c) \frac{a}{(s-b)(s-c)}=4 a(b+c-a) \\
\Leftrightarrow(b+c)^{2}+ & 4 a^{2}-4 a(b+c) \geq 0 \Leftrightarrow(b+c-2 a)^{2} \geq 0 \rightarrow \operatorname{true} \therefore b+c \\
& \geq 4 \sqrt{2 R r} \cos \frac{A}{2} \Rightarrow 4 R \cos \frac{A}{2} \cos \left(\frac{B-C}{2}\right) \geq 4 \sqrt{2 R r} \cos \frac{A}{2} \\
\Rightarrow \cos \left(\frac{B-C}{2}\right) & \geq \sqrt{\frac{2 r}{R}} \Rightarrow \sqrt{\frac{2 r}{R}}\left(\frac{2 m_{a}}{n_{a}+g_{a}}\right) \stackrel{b y(3)}{\leq} \cos \left(\frac{B-C}{2}\right) \Rightarrow \cos \left(\frac{B-C}{2}\right) \\
& \geq 2 \sqrt{\frac{2 r}{R}}\left(\frac{m_{a}}{n_{a}+g_{a}}\right) \text { and analogs }
\end{aligned}
$$

$$
\begin{aligned}
& \Leftrightarrow\left\{\mathrm{z}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{y}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}+\mathrm{z})\right\}\left\{\mathbf{y}(\mathrm{z}+\mathrm{x})^{2}+\mathrm{z}(\mathrm{x}+\mathrm{y})^{2}-\mathrm{yz}(\mathrm{y}\right. \\
& +z)\} \geq x^{2}(y+z)^{2}(x+y+z)^{2} \\
& \Leftrightarrow x^{2}+x^{2}+y^{3}+z^{3} \geq 2 x y z+y z(y+z) \Leftrightarrow x(y-z)^{2}+(y+z)(y-z)^{2} \geq 0 \rightarrow \text { true } \\
& \Rightarrow(a) \text { is true } \Rightarrow n_{a} g_{a} \xlongequal[\sim]{n} s(s-a) \\
& \text { by (1) and (2) }
\end{aligned}
$$



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$$
\begin{gathered}
\Rightarrow \sum \cos \left(\frac{B-C}{2}\right) \geq 2 \sqrt{\frac{2 r}{R}} \sum \frac{m_{a}}{n_{a}+g_{a}} \Rightarrow \sum \cos \left(\frac{A-B}{2}\right) \\
\geq 2 \sqrt{\frac{2 r}{R}} \sum \frac{m_{a}}{n_{a}+g_{a}}(\text { Proved })
\end{gathered}
$$

1598. In $\triangle A B C$ the following relationship holds:

$$
a^{3}+b^{3}+c^{3} \geq 8 \sqrt[4]{3 S^{6}}
$$

## Proposed by Daniel Sitaru-Romania

Solution 1 by Florin George Şerban-Romania

$$
\begin{gathered}
\sum_{\text {cyc }} a^{3} \stackrel{\text { Holder }}{\geq} \frac{\left(\sum a\right)^{3}}{3 \cdot 3}=\frac{(2 s)^{3}}{9}=\frac{8 s^{3}}{9} \geq 8 \sqrt[4]{3 S^{6}} \\
\Leftrightarrow s^{3} \geq 9 \sqrt[4]{3 \cdot r^{6} s^{6}} \Leftrightarrow s^{12} \geq 9^{4} \cdot 3 \cdot r^{6} s^{6} \Leftrightarrow s^{6} \geq 3^{9} \cdot r^{6} \\
\Leftrightarrow s^{2} \geq 3^{3} r^{2} \Leftrightarrow s \geq 3 \sqrt{3} r \text { (Mitrinovic) }
\end{gathered}
$$

## Solution 2 by Adrian Popa-Romania

$$
\begin{gather*}
\quad \frac{a^{3}}{1}+\frac{b^{3}}{1}+\frac{c^{3}}{1} \stackrel{\text { Radon }}{\geq} \frac{(a+b+c)^{3}}{9}=\frac{(2 s)^{3}}{9}=\frac{8 s^{3}}{9} \\
\frac{8 s^{3}}{9} \geq 8 \sqrt[4]{3 S^{6}} \Leftrightarrow s^{3} \geq\left. 9 \sqrt[4]{3 S^{6}}\right|^{4} \Leftrightarrow s^{12} \geq 3^{9} S^{6} \Leftrightarrow s^{2} \geq 3 \sqrt{3} S \\
a^{2}+b^{2}+c^{2} \geq 4 \sqrt{3} S \text { (lonescu-Weitzenbock);(1) } \\
a b+b c+c a \geq 4 \sqrt{3} S \Rightarrow 2(a b+b c+c a) \geq 8 \sqrt{3} S ; \text { (2) }  \tag{2}\\
\text { From (1),(2) we get: }(a+b+c)^{2} \geq 12 \sqrt{3} S \Leftrightarrow(2 s)^{2} \geq 12 \sqrt{3} S \\
\qquad \Leftrightarrow 4 s^{2} \geq 12 \sqrt{3} S \Leftrightarrow s^{2} \geq 3 \sqrt{3} S \text {.Proved. }
\end{gather*}
$$



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Solution 3 by Avishek Mitra-West Bengal-India

$$
a^{3}+b^{3}+c^{3} \geq 8 \sqrt[4]{3 S^{6}} \Leftrightarrow \sum a^{3} \stackrel{A m-G m}{\geq} 3 a b c=3 \cdot 4 R S=12 R S
$$

We need to prove: $12 R S \geq 8 \sqrt[4]{3 S^{6}}$

$$
\begin{gathered}
\Leftrightarrow 3 R S \geq 2 \sqrt[4]{3 S^{6}} \Leftrightarrow 81 R^{4} S^{4} \geq 16 \cdot 3 S^{6} \Leftrightarrow 27 R^{4}=16 r^{2} s^{2} \\
\Leftrightarrow(3 \sqrt{3} R)^{2} R^{2} \geq(2 s)^{2}(2 s)^{2} \text { true by } R \geq 2 r \text { (Euler) and } \\
3 \sqrt{3} R \geq 2 s \text { (Mitrinovic).Proved. }
\end{gathered}
$$

## Solution 4 by Mokhtar Khassani-Mostaganem-Algerie

$$
\begin{aligned}
& a^{3}+b^{3}+c^{3}=a \cdot a^{2}+b \cdot b^{2}+c \cdot c^{2} \stackrel{\text { Chebyshev }}{\geq} \frac{(a+b+c)\left(a^{2}+b^{2}+c^{2}\right)}{3} \\
& \geq \frac{(a+b+c)^{3}}{9}=\frac{8 s^{3}}{9}=\frac{8 s \sqrt{s} \cdot s \sqrt{s}}{9} \stackrel{s \geq 3 \sqrt{3} r}{\geq} \frac{8 s \sqrt{s} \cdot 3 \sqrt{3} r \cdot \sqrt{3 \sqrt{3} r}}{9}=8 \sqrt[4]{3 S^{6}}
\end{aligned}
$$

1599. In $\triangle A B C$ the following relationship holds:

$$
4+\sum_{c y c}\left(\frac{a}{m_{a}}\right)^{2} \leq 8 \prod_{c y c} \frac{r_{a}}{h_{a}}
$$

Proposed by Bogdan Fuştei-Romania

## Solution by Tran Hong-Dong Thap-Vietnam

$$
\begin{gathered}
r_{a} r_{b} r_{c}=s^{2} r \\
h_{a} h_{b} h_{c}=\frac{2 s^{2} r^{2}}{R} \\
\frac{r_{a} r_{b} r_{c}}{h_{a} h_{b} h_{c}}=\frac{s^{2} r}{\frac{2 s^{2} r^{2}}{R}}=s^{2} r \cdot \frac{R}{2 s^{2} r^{2}}=\frac{R}{2 r} \Rightarrow R H S=8 \prod_{c y c} \frac{r_{a}}{h_{a}}=\frac{4 R}{r} \\
m_{a} \geq \sqrt{s(s-a)} \Rightarrow m_{a}^{2} \geq s(s-a) \text { and analogs. }
\end{gathered}
$$



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$$
\begin{gathered}
L H S=4+\sum_{c y c}\left(\frac{a}{m_{a}}\right)^{2}=4+\sum_{c y c} \frac{a^{2}}{m_{a}^{2}} \leq 4+\sum_{c y c}\left(\frac{a^{2}}{s(s-a)}\right) \\
=\frac{4 s(s-a)(s-b)(s-c)+\sum_{c y c} a^{2}(s-b)(s-c)}{s(s-a)(s-b)(s-c)}
\end{gathered}
$$

$$
=\frac{4 s^{2} r^{2}+\sum_{c y c} a^{2}\left(s^{2}-(b+c) s+b c\right)}{s^{2} r^{2}}=\frac{4 s^{2} r^{2}+\sum_{c y c} a^{2}\left(s^{2}-(2 s-a) s+b c\right)}{s^{2} r^{2}}
$$

$$
=\frac{4 s^{2} r^{2}+\sum_{c y c} a^{2}\left(a s-s^{2}+b c\right)}{s^{2} r^{2}}
$$

$$
=\frac{4 s^{2} r^{2}+\left(a^{3}+b^{3}+c^{3}\right) s-\left(a^{2}+b^{2}+c^{2}\right) s^{2}+a b c(a+b+c)}{s^{2} r^{2}}
$$

$=\frac{4 s^{2} r^{2}+\left((2 s)^{3}-3(2 s)\left(s^{2}+4 R r+r^{2}\right)+3 \cdot 4 R r s\right) s-\left(2 s^{2}-8 R r-2 r^{2}\right) s^{2}+4 R r s \cdot 2 s}{s^{2} r^{2}}$
$=\frac{4 s^{2} r^{2}+2 s^{2}\left(4 s^{2}-3 s^{2}-12 R r-3 r^{2}+6 R r\right)-\left(2 s^{2}-8 R r-2 r^{2}\right) s^{2}+8 R r s^{2}}{s^{2} r^{2}}$

$$
=\frac{4 s^{2} r^{2}+2\left(s^{2}-6 R r-3 r^{2}\right)-2 s^{2}+8 R r+2 r^{2}+8 R r}{r^{2}}=\frac{4 R r}{r^{2}}=\frac{4 R}{r}=R H S
$$

1600. In $\triangle A B C$ the following relationship holds:

$$
\sum_{c y c} \frac{r_{a}+r}{r_{a}-r} \leq \sqrt{\frac{2 R}{r}} \sum_{c y c} \frac{s_{a}}{w_{a}} \sqrt{\frac{h_{a}}{r_{a}}}
$$

## Proposed by Bogdan Fuştei-Romania

Solution by Tran Hong-Dong Thap-Vietnam

$$
\frac{h_{a}}{w_{a}}=\frac{\frac{2 S}{a}}{\frac{2 b c}{b+c} \cdot \cos \frac{A}{2}}=S \cdot\left(\frac{b+c}{a b \cdot \cos \frac{A}{2}}\right)=\frac{S \cdot(b+c)}{4 R S \cdot \cos \frac{A}{2}}=\frac{b+c}{4 R \cdot \cos \frac{A}{2}}
$$



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$$
\begin{aligned}
& \frac{h_{a}}{r_{a}}=\frac{\frac{2 S}{a}}{s \cdot \tan \frac{A}{2}}=\frac{\frac{2 s r}{4 R \cdot \sin \frac{A}{2} \cos \frac{A}{2}}}{s \cdot \frac{\sin \frac{A}{2}}{\cos \frac{A}{2}}}=\frac{1}{\sin ^{2} \frac{A}{2}} \cdot \frac{r}{2 R} \Rightarrow \sqrt{\frac{h_{a}}{r_{a}}}=\frac{1}{\sin \frac{A}{2}} \cdot \sqrt{\frac{r}{2 R}} \\
& \sum_{c y c} \frac{h_{a}}{w_{a}} \sqrt{\frac{h_{a}}{r_{a}}}=\sqrt{\frac{r}{2 R}} \cdot \sum_{c y c} \frac{b+c}{4 R \sin \frac{A}{2} \cos \frac{A}{2}}=\sqrt{\frac{r}{2 R}} \cdot \sum_{c y c}^{\frac{b+c}{a}}=\Phi \\
& \sum_{c y c} \frac{b+c}{a}=\sum_{c y c} \frac{2 s-a}{a}=2 s \sum_{c y c} \frac{1}{a}-3=2 s \cdot \frac{s^{2}+4 R r+r^{2}}{4 R r s}-3=\frac{s^{2}-2 R r+r^{2}}{2 R r} \\
& \Rightarrow \Phi=\sqrt{\frac{r}{2 R}} \cdot \frac{s^{2}-2 R r+r^{2}}{2 R r} \\
& \Rightarrow R H S=\sqrt{\frac{2 R}{r}} \sum_{c y c} \frac{s_{a}}{w_{a}} \sqrt{\frac{h_{a}}{r_{a}}} \stackrel{s_{a} \geq h_{a}}{\gtrless} \sqrt{\frac{2 R}{r}} \cdot \Phi=\frac{s^{2}-2 R r+r^{2}}{2 R r} \\
& \frac{r_{a}+r}{r_{a}-r}=\frac{s \cdot \tan \frac{A}{2}+(s-a) \cdot \tan \frac{A}{2}}{s \cdot \tan \frac{A}{2}-(s-a) \cdot \tan \frac{A}{2}}=\frac{2 s-a}{a}=2 s \cdot \frac{1}{a}-1 \\
& \Rightarrow L H S=\sum_{c y c} \frac{r_{a}+r}{r_{a}-r}=2 s \sum_{c y c} \frac{1}{a}-3=2 s \cdot \frac{s^{2}+4 R r+r^{2}}{4 R r s}-3=\frac{s^{2}-2 R r+r^{2}}{2 R r} \leq R H S
\end{aligned}
$$



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It's nice to be important but more important it's to be nice. At this paper works a TEAM.

This is RMM TEAM.
To be continued!
Daniel Sitaru

