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REVERSE INEQUALITY OF HUYGENS AND SOME FRAMING WITH MEANS

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In this work we find a reverse inequality of Huygens inequality . Frames with means are obtained for the Huygens product and other derived expressions . There are highlighted certain means that refines the inequalities of classical means .

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A very elegant inequality - but unfortunately, an inequality still little known and used - is *Huygens inequality* , [2.] . It has a simple enunciation :

1. Proposition (*Huygens inequality*)

If $a_1 , a_2 , . . . , a_n$ are positive real numbers , then ,

$$(1+a_1)\cdot(1+a_2)\cdot\ \ldots\ \cdot(1+a_n) \geq \left(1+\sqrt[n]{a_1\cdot a_2\cdot\ \ldots\ \cdot a_n}\right)^n \cdot \qquad \textbf{(1)}$$

Although several demonstrations are known, for the bibliographic independence we reproduce the following short demonstration here.

Proof

Applying twice the *AM-GM inequality* ,

$$\frac{\sum_{k=1}^{n}\frac{1}{1+a_k}}{n} \geq \frac{1}{\sqrt[n]{\prod_{k=1}^{n}(1+a_k)}}, \tag{2}$$

$$\frac{\sum_{k=1}^{n}\frac{a_k}{1+a_k}}{n} \geq \frac{\sqrt[n]{\prod_{k=1}^{n}a_k}}{\sqrt[n]{\prod_{k=1}^{n}(1+a_k)}}, \tag{3}$$

by adding the two relations , we obtain :

$$1 \geq \frac{1+\sqrt[n]{\prod_{k=1}^{n}a_k}}{\sqrt[n]{\prod_{k=1}^{n}(1+a_k)}},$$

equivalent to the inequality in the statement.

Equality occurs if $a_1 = a_2 = \ldots = a_n$.

Looking for a complement to the left of the inequality in the relation (1) , I found an expression as beautiful as like that the direct inequality of *Huygens* .

<u>2</u>. <u>Proposition</u> (*reverse Huygens inequality*)

If $a_1, a_2, \ldots, a_n$ are positive real numbers , then ,

$$\left(1+\frac{a_1+a_2+\ldots+a_n}{n}\right)^n \geq (1+a_1)\cdot(1+a_2)\cdot\ldots\cdot(1+a_n)\ . \tag{4}$$

The ***Proof*** will be revealed later, in a more general context (Theorem 7).

Relative to the vector $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}$, The following classic means of numbers $a_1, a_2, \ldots, a_n$ are well known :

$$\mathrm{A}_n[a] := \frac{1}{n}\cdot\sum_{k=1}^{n} a_k \quad \text{(the \textit{arithmetical mean} of numbers } a_1, a_2, \dots, a_n\text{)}, \qquad \textbf{(5)}$$

$$\mathrm{G}_n[a] := \sqrt[n]{\prod_{k=1}^{n} a_k} \quad \text{(the \textit{geometric mean} of numbers } a_1, a_2, \dots, a_n\text{)}, \qquad \textbf{(6)}$$

$$\mathrm{H}_n[a] := \frac{n}{\sum_{k=1}^{n}\frac{1}{a_k}} \quad \text{(the \textit{harmonic mean} of numbers } a_1, a_2, \dots, a_n\text{)}, \qquad \textbf{(7)}$$

as well as the inequality between them, $\mathrm{A}_n[a] \geq \mathrm{G}_n[a] \geq \mathrm{H}_n[a]$, **(8)**
(where equality occurs if and only if $a_1 = a_2 = \dots = a_n$).

We will also note the product from the inequalities in (1) and (4) , which we can also call ***Huygens product*** ,

$$\mathrm{Huy}_n[a] := (1+a_1)\cdot(1+a_2)\cdot\ \dots\ \cdot(1+a_n)\cdot \qquad \textbf{(9)}$$

With these notations, from *Propositions* 1 and 2 we obtain the following very beautiful framing for the *Huygens product* :

<u>3</u>. <u>Proposition</u> (*of framing with means of Huygens product*)

If $a_1, a_2, \dots, a_n$ are positive real numbers , then ,

$$\left(1+\mathrm{G}_n[a]\right)^n \leq (1+a_1)\cdot(1+a_2)\cdot\ \dots\ \cdot(1+a_n) \leq \left(1+\mathrm{A}_n[a]\right)^n \ . \qquad \textbf{(10)}$$

<u>4</u>. *<u>Corollary</u>*

The expression $\sqrt[n]{\mathrm{Huy}_n[a]} - 1$ is a *mean* of the numbers $a_1, a_2, \dots, a_n$. which *refines* the *inequality* AM - GM .

<u>Proof</u>

From the relation (10) we deduce ,

$$\mathrm{G}_n[a] \leq \sqrt[n]{(1+a_1)\cdot(1+a_2)\cdot\ \dots\ \cdot(1+a_n)} - 1 \leq \mathrm{A}_n[a] \ . \qquad \textbf{(11)}$$

so the conclusion follows .

We now present a framing for another product ,

$$\mathrm{P}_n[x] := \left(1+\frac{1}{x_1}\right)\cdot\left(1+\frac{1}{x_2}\right)\cdot\,...\,\cdot\left(1+\frac{1}{x_n}\right). \qquad \mathbf{(12)}$$

5. Proposition (*of framing with means for* $P_n[x]$)

If $x_1, x_2, ..., x_n$ are positive real numbers , then ,

$$\left(1+\frac{1}{\mathrm{G}_n[x]}\right)^n \le \left(1+\frac{1}{x_1}\right)\cdot\left(1+\frac{1}{x_2}\right)\cdot\,...\,\cdot\left(1+\frac{1}{x_n}\right) \le \left(1+\frac{1}{\mathrm{H}_n[x]}\right)^n . \qquad \mathbf{(13)}$$

Proof

We apply the inequality from *Proposition* 3.

For real numbers $a_1, a_2, ..., a_n > 0$, we will note $x_k = 1/a_k$, $(\forall)\ k=\overline{1,n}$.

Obviously , $x_k > 0$, and $1+a_n = 1+\frac{1}{x_n}$, $(\forall)\ k=\overline{1,n}$. We also have :

$$\mathrm{A}_n[a] := \frac{1}{n}\cdot\sum_{k=1}^{n} a_k = \frac{1}{n}\cdot\sum_{k=1}^{n}\frac{1}{x_k} = \frac{1}{\mathrm{H}_n[x]} , \quad \mathrm{G}_n[a] := \sqrt[n]{\prod_{k=1}^{n} a_k} = \sqrt[n]{\prod_{k=1}^{n}\frac{1}{x_k}} = \frac{1}{\mathrm{G}_n[x]} .$$

By substituting these last three expressions in inequality (10) , the inequality from the statement is obtained .

6. *Corollary*

The expression $\dfrac{1}{\sqrt[n]{\mathrm{P}_n[a]}-1}$ is a *mean* of the numbers $a_1, a_2, ... , a_n$.

which *refines* the *inequality* HM - GM .

Proof

Indeed , from relation (13) we deduce ,

$$\mathrm{H}_n[a] \le \frac{1}{\sqrt[n]{\mathrm{P}_n[a]}-1} \le \mathrm{G}_n[a] . \qquad \mathbf{(14)}$$

so the conclusion follows .

We will now present a generalization of the framing with means (10) for the *generalized Huygens product* ,

$$\mathbf{Huy},\mathbf{g}_n[x] := (\alpha+\beta x_1)\cdot(\alpha+\beta x_2)\cdot \ldots \cdot(\alpha+\beta x_n), \qquad \mathbf{(15)}$$

associated with the vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}$, with $\alpha, \beta > 0$.

7. Theorem **(*of framing with means of generalized Huygens product*)**

Let α, β positive real numbers . For real numbers $x_1, x_2, \ldots, x_n > 0$, there is double inequality ,

$$\left(\alpha+\beta\, \mathbf{G}_n[x]\right)^n \le (\alpha+\beta x_1)\cdot(\alpha+\beta x_2)\cdot \ldots \cdot(\alpha+\beta x_n) \le \left(\alpha+\beta\, \mathbf{A}_n[x]\right)^n . \qquad \mathbf{(16)}$$

with equality iff $x_1 = x_2 = \ldots = x_n$.

Proof

For the inequality on the left, we use the *inequality of Huygens* (1) ,

$$\prod_{k=1}^{n}(\alpha+\beta x_k) = \prod_{k=1}^{n}\alpha\cdot\left(1+\frac{\beta}{\alpha}x_k\right) = \alpha^n\cdot\prod_{k=1}^{n}\left(1+\frac{\beta}{\alpha}x_k\right) \overset{(Huygens)}{\ge}$$

$$\overset{(Huygens)}{\ge} \alpha^n\cdot\left(1+\sqrt[n]{\prod_{k=1}^{n}\frac{\beta}{\alpha}x_k}\right)^n = \alpha^n\cdot\left(1+\frac{\beta}{\alpha}\cdot\sqrt[n]{\prod_{k=1}^{n}x_k}\right)^n =$$

$$= \left(\alpha+\beta\cdot\sqrt[n]{\prod_{k=1}^{n}x_k}\right)^n = \left(\alpha+\beta\,\mathbf{G}_n[x]\right)^n .$$

Equality occurs if $x_1 = x_2 = \ldots = x_n$.

For the right inequality, we have by using the *GM-AM inequality* :

$$\prod_{k=1}^{n}(\alpha+\beta x_k) \le \left(\frac{1}{n}\cdot\sum_{k=1}^{n}(\alpha+\beta x_k)\right)^n = \left(\alpha+\frac{1}{n}\cdot\sum_{k=1}^{n}\beta x_k\right)^n =$$

$$= \left(\alpha+\beta\cdot\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right)^n = \left(\alpha+\beta\,\mathbf{A}_n[x]\right)^n .$$

Equality occurs if :

$$\alpha+\beta x_1 = \alpha+\beta x_2 = \ldots = \alpha+\beta x_n \Leftrightarrow x_1 = x_2 = \ldots = x_n .$$

8. *Remark* For $\alpha = \beta = 1$ the *inequality of Huygens* is obtained , both in the direct version and in the reverse version .

Because of (16) we deduce equivalently ,

$$\mathrm{G}_n[a] \le \frac{\sqrt[n]{\mathrm{Huy,g}_n[x]} - \alpha}{\beta} \le \mathrm{A}_n[a] \ , \qquad \textbf{(17)}$$

so the following

9. *Corollary*

For α , β positive real numbers , the number $\dfrac{\sqrt[n]{\mathrm{Huy,g}_n[x]} - \alpha}{\beta}$ is a *mean* of positive real numbers $x_1, x_2, \ldots , x_n$, which refines the *AM - GM inequality* .

With replacement $x_k \to 1/x_k$, $(\forall)\ k = \overline{1,n}$, in relations (16) , (17) , we get the following:

10. Proposition

***a*)** For α , β ; $x_1 , x_2 ,\ldots, x_n$ strictly positive real numbers , there is double inequality,

$$\left(\alpha + \frac{\beta}{\mathrm{G}_n[x]}\right)^n \le \left(\alpha + \frac{\beta}{x_1}\right)\cdot\left(\alpha + \frac{\beta}{x_2}\right)\cdot \ldots \cdot\left(\alpha + \frac{\beta}{x_n}\right) \le \left(\alpha + \frac{\beta}{\mathrm{H}_n[x]}\right)^n . \qquad \textbf{(18)}$$

***b*)** $$\mathrm{H}_n[a] \le \frac{\beta}{\sqrt[n]{\mathrm{P}_n[a]} - \alpha} \le \mathrm{G}_n[a] \ . \qquad \textbf{(19)}$$

In the relations (10), (13), (16), (18) were highlighted the *framing* for different product - expressions . We conclude with the setting of some framing established for the sum – expressions . Thus in [3], the following two *framing with means* were presented and demonstrated :

11. Proposition

Let $a , b > 0$. For real numbers $x_1 , x_2 ,\ldots, x_n \in (0 , a/b)$ and weights $p_1 , p_2 , \ldots, p_n > 0$, with $\sum_{k=1}^{n} p_k = 1$, there is double inequality ,

$$\frac{1}{a+b\,A_n[x]} \le \sum_{k=1}^{n} \frac{p_k}{a+b\,x_k} \le \frac{1}{a+b\,G_n[x]} . \qquad (20)$$

with equality if $x_1 = x_2 = \ldots = x_n$.

12. Proposition

Let $a, b > 0$. For real numbers $y_1, y_2, \ldots, y_n \in (b/a, \infty)$ and weights $p_1, p_2, \ldots, p_n > 0$, with $\sum_{k=1}^{n} p_k = 1$, there is double inequality ,

$$\frac{H_n[y]}{a\,H_n[y]+b} \le \sum_{k=1}^{n} \frac{p_k y_k}{a\,y_k+b} \le \frac{G_n[y]}{a\,G_n[y]+b} . \qquad (21)$$

with equality if $y_1 = y_2 = \ldots = y_n$.

Two other beautiful framing with means were presented in [5] .
See also [1] and [4] :

13. Proposition

Let $n \ge 2$ a natural number . For real numbers $a_1, a_2, \ldots, a_n \in (0, 2)$, there is double inequality ,

$$\frac{n}{\sqrt{1+A_n[a]}} \le \sum_{k=1}^{n} \frac{1}{\sqrt{1+a_k}} \le \frac{n}{\sqrt{1+G_n[a]}} . \qquad (22)$$

14. Proposition

For $n \ge 2$ a natural number and for real numbers $x_1, x_2, \ldots, x_n \in (1/2, \infty)$ there is double inequality ,

$$n \cdot \sqrt{\frac{H_n[x]}{1+H_n[x]}} \le \sum_{k=1}^{n} \sqrt{\frac{x_k}{1+x_k}} \le n \cdot \sqrt{\frac{G_n[x]}{1+G_n[x]}} . \qquad (23)$$

Equality occurs if and only if $x_1 = x_2 = \ldots = x_n$.

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